3: Arc Length in Space:

Definition (8):

The length of a smooth curve $r(t) = x(t)i + y(t)j + z(t)k, a \leq t \leq b$, that is traced exactly once as to increases from $t = a$ to $t = b, is$

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \; dt \quad \ldots \ldots \ldots \ldots .. \quad (1)$$

$The\ square\ root\ in\ equation\ (1)\ is\ |v(t)|$, the length of a velocity vector $\frac{dr}{dt}$. This enables us to write the formula for length as shorter way:

$$L = \int_a^b |\, v(t)| dt \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (2)$$

Example (7):

A glider is soaring upward along the helix $r(t) = (\cos t)i + (\sin t)j + tk$. $How\ long\ is\ the\ glider's\ path\ from\ t = 0\ \ to\ t = 2\pi$?

Solution:

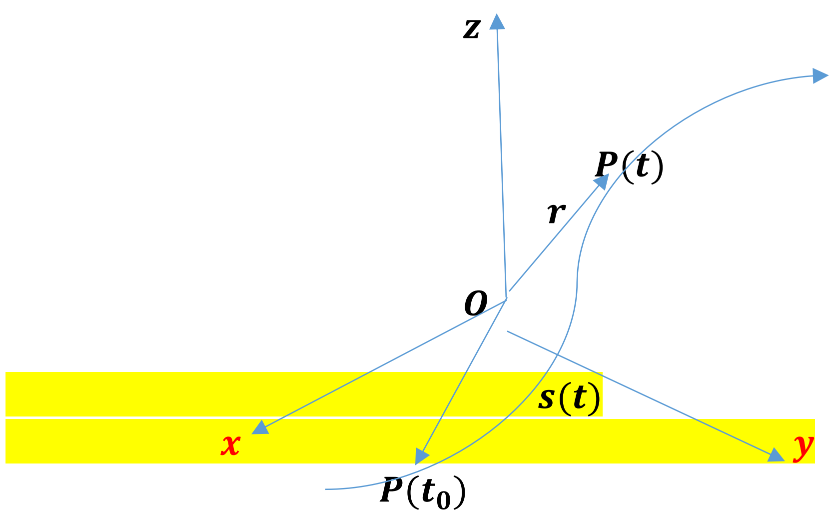
$$L = \int_a^b |\, v(t)| dt = \int_0^{2\pi} \sqrt{sin^2(t) + cos^2(t) + 1}\, dt = \int_0^{2\pi} \sqrt{2}\ dt = 2\sqrt{2}\,\pi \,.$$

Arc Length Parameter for a Curve

If we choose a base point $P(t_0)$ on a smooth curve parameterized by t each value of t determined a point $P(t) = (x(t), y(t), z(t))$ on C and a "directed distance"

$$S(t) = \int_{t_0}^{t} |v(\tau)|\, d\tau \,.$$

$measured\ along\ C\ from\ the\ base\ point.$



If $t > t_0$, $s(t)$ is the distance along the curve from $P(t_0)$ to $P(t)$. $If\ t < t_0$, $s(t)$ $is\ the$ negative of the distance. Each value of s determines a point $on\ C$, and this parameterizes C with $respect\ to\ s$. We call s an arc length parameter for the curve. Thus the arc length parameter with base point $P(t_0)$ is given by

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\ d\tau = \int_{t_0}^{t} |v(\tau)|\ d\tau \ldots\ldots\ldots \quad (3)$$

Note:

If a curve $r(t)$ is already given in terms of some parameter t and $s(t)$ is the arc length function given by equation (3), then we may be able to solve for t as a function of s: $t = t(s)$. $Then$ the curve can be reparametrized in terms of s by $substituting\ for\ t$: $r = r(t(s))$. The new parameterization identifies a point on the curve with its directed distance along the curve from the base point.

Example (8):

Given $r(t) = (\cos t)i + (\sin t)j + tk$. Find the directed distance from $t_0 = 0$ $to\ \ t\ and\ write\ r(t)\ in\ terms\ of\ s$.

Solution:

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\ d\tau = \int_{t_0}^{t} |v(\tau)|\ d\tau$$

$$= \int_{0}^{t} \sqrt{cos^2(\tau) + sin^2(\tau) + 1}\ d\tau = \int_{0}^{t} \sqrt{2}\ d\tau = \sqrt{2}\ t\ .$$

$$\therefore t = \frac{1}{\sqrt{2}}\ s.$$

$$\therefore r(t) = r\left(\frac{1}{\sqrt{2}}\ s\right) = \left[cos\left(\frac{1}{\sqrt{2}}\ s\right)\right] i + \left[sin\left(\frac{1}{\sqrt{2}}\ s\right)\right] j + \frac{1}{\sqrt{2}}\ s\ k\ .$$

Speed on a Smooth Curve

Since $s(t) = \int_{t_0}^{t} |v(\tau)|\ d\tau \ \rightarrow \ \frac{ds}{dt} = |v(t)| \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$

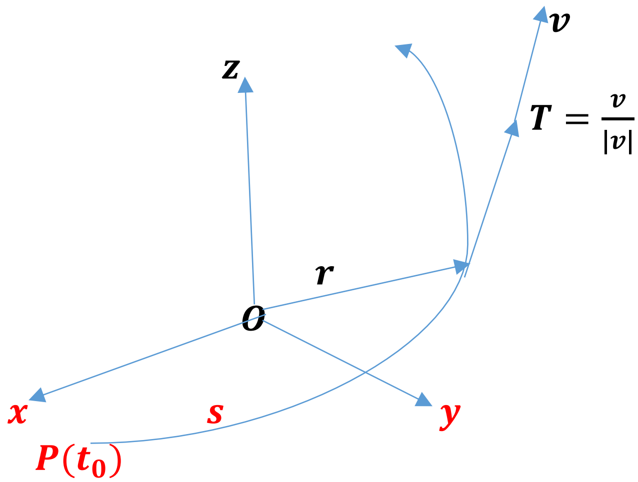
Equation (4), says that the speed with which a particle moves along its path is the magnitude of v .

Note:

$s\ is\ an\ increasing\ function\ of\ t, because\ \frac{ds}{dt} > 0$.

Unit Tangent Vector:

We already know the velocity vector $v = \frac{dr}{dt}$ is tangent to the curve $r(t)$ and that the vector $T = \frac{v}{|v|}$ is therefore a unit vector tangent to the curve, called the unit tangent vector.
The unit tangent vector T is a differentiable function of t whenever v is a differentiable function of t .



Example (9):

Find the unit tangent vector of the curve $r(t) = (1 + 3\cos t)i + (3\sin t)j + t^2\, k$.

Solution:

$$T = \frac{v}{|v|}$$

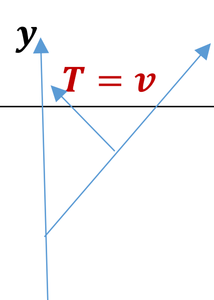
$$v = r'(t) = (-3\sin t)i + (3\cos t)j + 2tk \rightarrow |v| = \sqrt{9sin^2 t + 9cos^2 t + 4t^2}$$

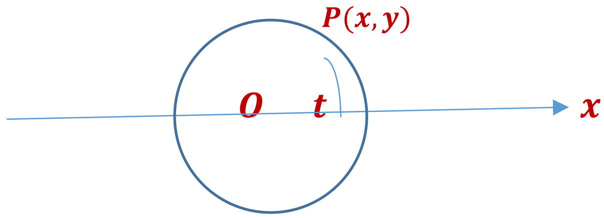
$$\therefore |v| = \sqrt{9 + 4t^2}\ .$$

$$\therefore\ T = \left[\frac{-3\sin t}{\sqrt{9+4t^2}}\right] i + \left[\frac{3\cos t}{\sqrt{9+4t^2}}\right] j + \left[\frac{2t}{\sqrt{9+4t^2}}\right] k\ .$$

Note:

For counterclockwise motion $r(t) = (\cos t)i + (\sin t)j$ around the unit circle, we see that $v = (-\sin t)i + (\cos t)j$ is already unit vector, so $T = v$ and $T\ is\ orthogonal\ to\ r$, because $T \cdot v = -\sin t \cos t + \sin t \cos t = 0$.





Note:

$$\frac{d\boldsymbol{r}}{ds} = \frac{d\boldsymbol{r}}{dt} \cdot \frac{dt}{ds} = \boldsymbol{v}\frac{1}{|\boldsymbol{v}|} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \boldsymbol{T} \;.\dots\dots\dots\dots\dots\dots\dots\dots. \qquad (5)$$

This equation says that $\frac{dr}{ds}$ is the unit tangent vector in the direction of the velocity vector.