2: Integrals of Vector Functions

Definition (6):

The indefinite integral of r with respect to t is the set of all antiderivatives of r, denoted by $\int r(t)dt$. *If R is any antiderivative of r*, then $\int r(t)dt = R(t) + C$.

Definition (7):

If the components of $r(t) = f(t)i + g(t)j)h(t)k$ are *integrable over* $[a, b]$, then *so is r*, and the definite integral of r from a *to* b is:

$$\int_a^b r(t)dt = (\int_a^b f(t)dt)i + (\int_a^b g(t)dt)j + (\int_a^b h(t)dt)k.$$

Example (4):

Integrate vector function $r(t) = (\cos t)i + j - 2tk$.

Solution:

$$\int r(t)dt = (\int \cos t\, dt)i) + (\int dt)j + (\int -2tdt)k$$

$$= (\sin t)i + tj - t^2\, k + C\, , where\ C = c_1 i + c_2 j + c_3\, k\, .$$

Example (5):

Calculate $\int_0^{\pi}[(\cos t)i + j - 2tk]dt$.

Solution:

$$\int_0^{\pi} [(\cos t)i + j - 2tk]dt = [\sin t]_0^{\pi}\, i + [t]_0^{\pi}\, j - [t^2]_0^{\pi} = \pi j - \pi^2 k.$$

Note:

The Fundamental Theorem of Calculus for Continuous Vector Functions Says that:

$\int_a^b r(t)dt = [\, R(t)]_a^b = R(b) - R(a)$, *where R is* any antiderivative of r, so that $R'(t) = r(t)$.

Notes:

1: An antiderivative of a vector function is also a vector function.

2: A definite integral of a vector function is a single constant vector.

Example (6):

Given acceleration vector $a(t) = (-3\cos t)i - (3\sin t)j + 2k$. Find the position vector $r(t)$ such that $v(0) = 3j \ and \ r(0) = 4i$.

Solution:

$$a(t) = \frac{dv}{dt} = (-3\cos t)i - (3\sin t)j + 2k \rightarrow$$

$$v(t) = \int [(-3\cos t)i - (3\sin t)j + 2k\,]dt$$

$$= -3(\sin t)i + 3(\cos t)j + 2tk + C_1$$

Since $v(0) = 3j \rightarrow 3j = 3j + C_1 \rightarrow C_1 = 0$.

$\therefore v(t) == -3(\sin t)i + 3(\cos t)j + 2tk$

$\therefore \frac{dr}{dt} = -3(\sin t)i + 3(\cos t)j + 2tk$

$\therefore r(t) = \int[-3(\sin t)i + 3(\cos t)j + 2tk]dt \rightarrow$

$r(t) = 3(\cos t)i + 3(\sin t)j + t^2k + C_2$

Since $r(0) = 4i \rightarrow 4i = 3i + C_2 \rightarrow C_2 = i$.

∴The position vector is:

$r(t) = (1 + 3\cos t)i + (3\sin t)j + t^2k$.