Republic of Iraq
Ministry of Higher Education and Scientific Research
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Department of Mathematics
Second Class

Advanced Calculus

Chapter Five
Vector-Valued Functions and Motion in Space

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1: Curves in Space and Their Tangents:

When a particle moves through space during a time interval I the particle's coordinates as functions defined on I.

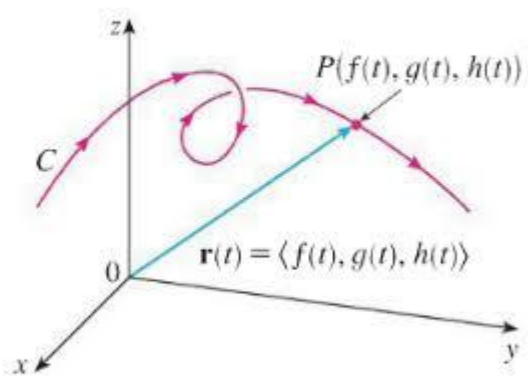
$\therefore x = f(t), y = g(t), z = h(t), t \in I.$ ……………………….(1)

$\therefore$The points $(x, y, z) = (f(t), g(t), h(t)), t \in I$ make up the curve in space that we call the particle's path.

A curve in space can also be represented in vector form. The vector

$r(t) = \overrightarrow{OP} = f(t)i + g(t)j + h(t)k$ ………………………. (2)

From the origin to the particle's position $P(f(t), g(t), h(t))$ at time t is the particle's position vector. The functions $f(t), g(t)$ & $h(t)$ are the component functions of the position vector. Equation (2) defines r as a vector function of real variable t on the interval I.



Definition (1):

A vector-valued function of vector function on a domain D is a rule that assigns a vector in space to each element in D. *The* domains will be intervals of real numbers, and the graph of the function represents a curve in space.

Note:

1: The components of r in Equation (2) are scalar functions of t.

2: The domain of a vector-valued function is the common domain of its components.

Limits and Continuity:

Definition (2):

Let $r(t) = f(t)i + g(t)j + h(t)k$ be a vector function *with domain* D, and let L be a vector. We say that r has limit L as t approaches t_0 and write

$\lim_{t \to t_0} r(t) = L$ *if* for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$, $|r(t) - L| < \epsilon$ whenever $0 < |t - t_0| < \delta$.

Notes:

1: *If* $L = L_1 i + L_2 j + L_3 k$, then $\lim_{t \to t_0} r(t) = L$ when $\lim_{t \to t_0} f(t) = L_1$, $\lim_{t \to t_0} g(t) = L_2$ & $\lim_{t \to t_0} h(t) = L_3$.

2: $\lim_{t \to t_0} \mathrm{r(t)} = \left(\lim_{t \to t_0} f(t)\right) i + \left(\lim_{t \to t_0} g(t)\right) j + \left(\lim_{t \to t_0} h(t)\right) k$ . ... (3)

$t \to t_0$

Example (1):

If $r(t) = (\cos t)i + (\sin t)j + tk$. Find $\lim_{t \to \frac{\pi}{4}} r(t)$.

Solution:

$$\lim_{t \to \frac{\pi}{4}} r(t) = \lim_{t \to \frac{\pi}{4}} (\cos t)i + \lim_{t \to \frac{\pi}{4}} (\sin t)j + \lim_{t \to \frac{\pi}{4}} tk = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j + \frac{\pi}{4} k.$$

Definition (3):

A vector function $r(t)$ is continuous at a point $t = t_0$ in its domain if $\lim_{t \to t_0} r(t) = r(t_0)$. The function is continuous if *its* continuous at every point in its domain.

Example (2):

The vector functions

$r(t) == (\cos t)i + (\sin t)j + (\sin 2t)k$ &

$r(t) = (\sin 3t)(\cos t)i + (\sin 3t)(\sin t)j + tk$.

are continuous in their domains.

While the vector function

$f(t) = (\cos t)i + (\sin t)j + [t]k$ *is discontinuous* at every integer, because the greatest integer function $[t]$ is discontinuous at every integer.

Derivative and Motion

Definition (4):

The vector function $r(t) = f(t)i + g(t)j + h(t)k$ is differentiable at t if f, g, & h *have* derivatives at t.

The derivative is the vector function

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \frac{df}{dt} i + \frac{dg}{dt} j + \frac{dh}{dt} k.$$

Notes:

1: A vector function r is differentiable if it is differentiable at every point of its domain.

2: The curved traced by r is smooth if $\frac{dr}{dt}$ is continuous and never zero, that is $f, g, \& \ h$ *have continuous first* derivatives that are not simultaneously zero.

Definition (5):

1: Velocity Vector:

$$v(t) = \frac{dr}{dt}$$

2: $Speed = |v(t)|$

3: Acceleration Vector:

$$a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2}$$

4: The unit vector $\frac{v}{|v|}$ is the direction of motion at time t .

Example (3):

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $r(t) = (2\cos t)i + (2\sin t)j + \left(5cos^2(t)\right)k$.

Solution:

1: Velocity Vector:

$$v(t) = \frac{dr}{dt} = (-2\sin t)i + (2\cos t)j - (10\cos t\ \sin t)k$$

$$= (-2\sin t)i + (2\cos t)j - (5\sin 2t)k\ .$$

2: Speed:

$$Speed = |v(t)| = \sqrt{4sin^2t + 4cos^2t + 100cos^2t\ sin^2t} =$$

$$\sqrt{4 + 100cos^2t\ sin^2t}$$

3: Acceleration Vector:

$$a(t) = \frac{dv}{dt} = (-2\cos t)i - (2\sin t)j - (10\ \cos 2t)k.$$

Note:

We can express the velocity of moving particle as the product of its speed and direction. That is

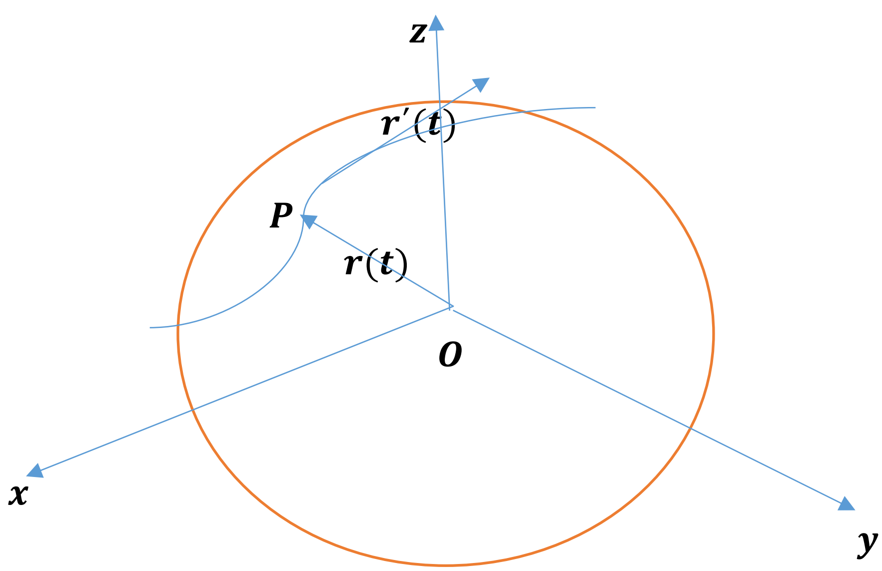
$$Velocity = |v|\left(\frac{v}{|v|}\right) = (Speed)(Direction).$$

Differentiation Rules for Vector Functions:

Let u & v be differentiable vector $functions\ of\ t$, C is a constant vector, α is any scalar, and f is any differentiable scalar function. Then

1) $\frac{d}{dt}(C) = O$.
2) $\frac{d}{dt}[\alpha u(t)] = \alpha u'(t)$.
3) $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$.
4) $\frac{d}{dt}[u(t) \mp v(t)] = u'(t) \mp v'(t)$.
5) $\frac{d}{dt}[u(t) \cdot v(t)] = u(t) \cdot v'(t) + v(t)u'(t)$.
6) $\frac{d}{dt}[u(t) \times v(t)] = u(t) \times v'(t) + v(t) \times u'(t)$.
7)) $\frac{d}{dt}\left[u\left(f(t)\right)\right] = u'\left(f(t)\right)f'(t)$.

Vector Functions of Constant Length



When we track a particle moving on a sphere centered at the origin, the position vector has a constant length equal to the radius of the sphere. The velocity vector $r'(t)$ tangent to the path of motion is tangent to the sphere and hence perpendicular to $r. This$ is always the case for a differentiable vector function of constant length. The vector and its first derivative are orthogonal. We can prove this as follows:

$$r(t)\cdot r(t)=|\,r(t)|^2\ \to \frac{d}{dt}\,[\,r(t)\cdot r(t)]=2r(t)\cdot r'(t)=0\to r(t)\cdot r'(t)=0.$$

Thus

If $r(t)$ is a differentiable vector function of t and the length of $r(t)$ is constant, then:

$$r(t)\cdot r'(t)=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \quad (4)$$

Note:

The converse of Equation (4) is also true.