Some special continous distribution

3- Exponential distribution:

The Exponential distribution is special case of Gamma distribution in which (∝=1) , then the random variable X is said to have a Exponential distribution if it's p.d.f is given by:

$$f(x)=\begin{cases}\theta\ e^{-\theta X} & , \quad X>0\\ 0 & , \quad o.w\end{cases}$$

And writes briefly : $X\sim EXP(\theta)$

Since $\theta>0$ **it is clear that:**

1-f(x)≥0 for all x>0

2-$\int_{-\infty}^{\infty} f(x)dx=\int_0^{\infty}\theta\ e^{-\theta X}dx=1$

$$\int_0^{\infty}\theta\ e^{-\theta X}dx=\theta\int_0^{\infty}\ e^{-\theta X}dx=\frac{\theta}{-\theta}\int_0^{\infty}-\theta\ e^{-\theta X}dx=\ -e^{-\theta X}\Big|_0^{\infty}=$$
$$-(e^{-\infty}-e^{-0})=-(0-1)=1$$

The M.G.F. :

$$M_x(t)=E(e^{tx})=\int_a^b e^{tx}f(x)dx$$
$$=\int_0^{\infty}e^{tx}\ \theta\ e^{-\theta X}dx=\theta\int_a^b e^{-X(\theta-t)}\,dx$$
$$=\frac{-\theta}{(\theta-t)}\int_a^b-(\theta-t)e^{-X(\theta-t)}dx=\frac{-\theta}{(\theta-t)}(\,e^{-X(\theta-t)}\Big|_0^{\infty})$$
$$=\frac{-\theta}{(\theta-t)}(0-1)=\frac{\theta}{(\theta-t)}$$

$$M_x(t)=\frac{\theta}{(\theta-t)}$$

The mean :

$$\mu_X=E(X)=M'_{(X)}(0)$$

$$M'_{(X)}(t) = \frac{\theta}{(\theta - t)^2}$$

$$M'_{(X)}(0) = \frac{\theta}{(\theta - 0)^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$\mu_X = \frac{1}{\theta}$$

The variance :

$$E(X^2) = M''_{(X)}(0)$$

$$M''_{(X)}(t) = \frac{2\theta}{(\theta - t)^3}$$

$$E(X^2) = M''_{(X)}(0) = \frac{2\theta}{(\theta - 0)^3} = \frac{2}{\theta^2}$$

$$V(X) = \sigma^2_x = E(X^2) - [E(X)]^2 = \frac{2}{\theta^2} - \left[\frac{1}{\theta}\right]^2 = \frac{1}{\theta^2}$$

$$\sigma^2_x = \frac{1}{\theta^2}$$

The cumulative distribution function of X is given by :

$$F(X) = 1 - e^{-\theta X} \quad , X > 0$$

Example:

If you have a random variable x representing the lifetime of a light bulb ,and if x has the following probability density function:

$$f(x) = \begin{cases} 0.002\, e^{-0.002X} & , \; X \geq 0 \\ 0 & , o.w \end{cases}$$

1-Find The probability that the lamp will stay at least 600 hours

2- Find The probability that the lamp will stay between 400 and 600 hours.

3-Find μ_x and σ_x^2

Solve:

1-

$$P(x \geq 600) = \int_{600}^{\infty} 0.002\, e^{-0.002X}\, dx = -e^{-0,002X}\Big|_{600}^{\infty} =$$
$$-\left(e^{-\infty} - e^{-(0.002)(600)}\right) = -(0 - 0.30) = 0.30$$

2-

$$P(400 \leq x \leq 600) = \int_{400}^{600} 0.002\, e^{-0.002X}\, dx =$$
$$-e^{-0,002X}\Big|_{400}^{600} = -\left(e^{-(0.002)(600)} - e^{-(0.002)(400)}\right) =$$
$$-(0.30 - 0.45) = 0.15$$

3- Since $(\theta = 0.002)$ $then:$

$$\mu_X = \frac{1}{\theta} = \frac{1}{0.002} = 500$$
$$\sigma^2{}_x = \frac{1}{\theta^2} = \frac{1}{0.002^2} = 250000$$

Example:

IF X has an exponential distribution with the parameter θ and $\mu_X = 10$

find P(x>100).

solve :

$$Since \quad \mu_X = \frac{1}{\theta} \to \mu_X = 10 \to 10 = \frac{1}{\theta} \to 10 * \theta = 1 \to \theta = \frac{1}{10} = 0.1$$

$$\theta = 0.1$$

And since X has an exponential distribution then :

$$f(x) = \begin{cases} 0.1\ e^{-0.1X} & , \quad X > 0 \\ 0 & , \quad o.w \end{cases}$$

$$P(X > 100) = \int_{100}^{\infty} 0.1\, e^{-0.1X}\, dx = -e^{-0.1X}\Big|_{100}^{\infty}$$

$$= -\left(e^{-\infty} - e^{-(0.1)(100)}\right) = -(0 - 0.00005) = 0.00005$$

Homework

IF X has an exponential distribution with the parameter θ and

$\mu_X = 2\sigma^2{}_x$ *where* $\theta > 0$ *then find* μ_X , $\sigma^2{}_x$ *and* $P(X > 500)$