

Solution of Sheet 2

Q1:-

$$(M1) \quad d_1(f, g) = \int_a^b |f(x) - g(x)| dx = 0 \Leftrightarrow f(x) - g(x) = 0, \forall f, g \in C([a, b]).$$

$$(M2) \quad d_1(f, g) = \int_a^b |f(x) - g(x)| dx = \int_a^b |(-1)(g(x) - f(x))| dx \\ = \int_a^b |g(x) - f(x)| dx = d_1(g, f), \forall f, g \in C([a, b]).$$

$$(M3) \quad d_1(f, g) = \int_a^b |f(x) - g(x)| dx = \int_a^b |f(x) - h(x) + h(x) - g(x)| dx \\ \leq \int_a^b |f(x) - h(x)| dx + \int_a^b |h(x) - g(x)| dx = d_1(f, h) + d_1(h, g), \forall f, g \in C([a, b]).$$

We are going to show that $\|\cdot\|_1$ define a norm on $C([a, b])$.

$$(N1) \quad \|f\|_1 = \int_a^b |f(x)| dx = 0 \Leftrightarrow f(x) = 0, \forall f \in C([a, b]).$$

$$(N2) \quad \|\lambda f\|_1 = \int_a^b |\lambda f(x)| dx = |\lambda| \int_a^b |f(x)| dx = |\lambda| \|f\|_1, \quad \forall \lambda \in \mathbb{R} \text{ and } f \in C([a, b]).$$

$$(N3) \quad \|f + g\|_1 = \int_a^b |f(x) + g(x)| dx \leq \int_a^b |f(x)| dx + \int_a^b |g(x)| dx \\ = \|f\|_1 + \|g\|_1, \quad \forall f, g \in C([a, b]).$$

Q2:-(a) We need to verify axioms (N1)-(N3) of the norm. Let $f, g \in C^1([a, b])$ and $\lambda \in \mathbb{R}$.

$$(N1) \quad \|f\|_{\infty,1} = \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |\dot{f}(x)| = 0 \Leftrightarrow f(x) = 0, \forall f \in C^1([a, b]).$$

$$(N2) \quad \|\lambda f\|_{\infty,1} = \max_{x \in [a,b]} |\lambda f(x)| + \max_{x \in [a,b]} |\lambda \dot{f}(x)| = |\lambda| \max_{x \in [a,b]} |f(x)| + |\lambda| \max_{x \in [a,b]} |\dot{f}(x)| \\ = |\lambda| \|f\|_{\infty,1}, \quad \forall \lambda \in \mathbb{R} \text{ and } f \in C^1([a, b]).$$

$$(N3) \quad \|f + g\|_{\infty,1} = \max_{x \in [a,b]} |f(x) + g(x)| + \max_{x \in [a,b]} |\dot{f}(x) + \dot{g}(x)| \\ \leq \max_{x \in [a,b]} (|f(x)| + |g(x)|) + \max_{x \in [a,b]} (|\dot{f}(x)| + |\dot{g}(x)|) \text{ properties of absolute value}$$

$$\leq \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |g(x)| + \max_{x \in [a,b]} |\dot{f}(x)| + \max_{x \in [a,b]} |\dot{g}(x)| \text{ properties of maximum}$$

(or trigonometric inequality on real numbers).

$$= \|f\|_{\infty,1} + \|g\|_{\infty,1}, \quad \forall f, g \in C^1([a, b]).$$

$$(b) \quad \|\sin(\pi x)\|_{\infty,1} = \max_{x \in [0,1]} |\sin(\pi x)| + \max_{x \in [0,1]} |\pi \cos(\pi x)| = 1 + \pi.$$

Q3:- (a) $A = \{x \in \mathbb{R}^2 \mid d_2(x, x_0) \leq 2\}$, where $x_0 \in \mathbb{R}^2$.

$\partial A = \{x \in \mathbb{R}^2 \mid d_2(x, x_0) = 2\}$, Disk circumference.

$\text{int}(A) = \{x \in \mathbb{R}^2 \mid d_2(x, x_0) < 2\} = B_2(x_0)$, Ball.

$\text{ext}(A) = \{x \in \mathbb{R}^2 \mid d_2(x, x_0) > 2\}$, Ball exterior. A is closed set

(b) $B = \mathbb{R} \times [a, b)$, where $a, b \in \mathbb{R}$, $a < b$.

$\partial B = \mathbb{R} \times \{a, b\}$, $\text{int}(B) = \mathbb{R} \times (a, b)$, $\text{ext}(B) = \mathbb{R} \times (-\infty, a) \cup (b, \infty)$.

B is nether open nor closed.

(c) $C = (a, b)^2 = (a, b) \times (a, b)$, where $a, b \in \mathbb{R}$, $a < b$.

$\partial C = (\{a, b\} \times [a, b]) \cup ([a, b] \times \{a, b\})$, $\text{int}(C) = C = (a, b) \times (a, b)$,

$\text{ext}(C) = \mathbb{R}^2 \setminus ([a, b] \times [a, b])$. C is open.

(d) $D = \{a\} \times [b, c)$, where $a, b, c \in \mathbb{R}$, $b < c$.

$\partial D = \{a\} \times [b, c]$, $\text{int}(D) = \emptyset$, $\text{ext}(D) = \mathbb{R}^2 \setminus (\{a\} \times [b, c])$.

D is nether open nor closed.

(e) $E = \{a\} \times \{b, c\}$, where $a, b, c \in \mathbb{R}$, $b \neq c$.

Note that $E = \{(a, b), (a, c)\}$

$\partial E = E = \{(a, b), (a, c)\}$, $\text{int}(E) = \emptyset$, $\text{ext}(E) = \mathbb{R}^2 \setminus \{(a, b), (a, c)\}$.

E is closed.

Q4:- $A = \bigcap_{n=1}^{\infty} \left[-1 - \frac{1}{n}, 2 + \frac{1}{n}\right] \times \left[-\frac{1}{n^2}, \frac{1}{n^2}\right] \times [0, e^{-n}]$.

Note that $A_1 \supset A_2 \supset A_3 \supset \dots$, we have

$\lim_{n \rightarrow \infty} \left(-1 - \frac{1}{n}\right) = -1$, $\lim_{n \rightarrow \infty} \left(2 + \frac{1}{n}\right) = 2$, $\lim_{n \rightarrow \infty} \left(-\frac{1}{n^2}\right) = 0$, $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) = 0$, $\lim_{n \rightarrow \infty} (e^{-n}) = 0$

$$A = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} [-1, 2] \times \{0\} \times \{0\}.$$

The set A is closed, since $[-1, 2] \times \{0\} \times \{0\} = A$. To prove rigorously that A is closed we can observe that each set A_n is closed as a product of closed intervals. Hence A is closed as intersection of closed intervals.

Q5:-

(a) $B_r(x_0) = \{x \in X \mid d(x, x_0) < r\} = \{x_0\}$

(b) $B_r(x_0) = \{x \in X \mid d(x, x_0) < r\} = X$

(c) $B_r(x_0) = \{x \in X \mid d(x, x_0) < r\} = X \setminus \{x_0\}$

Q6:-

1. $B_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ closed for each $n = 1, 2, \dots$. However, $\bigcup_{n=1}^{\infty} B_n = (0, 1)$ is open.
2. $C_n = \left[\frac{1}{n}, 1\right]$ closed for each $n = 1, 2, \dots$. However, $\bigcup_{n=1}^{\infty} C_n = (0, 1]$ neither open nor closed.
3. $D_n = [-n, n]$ closed for each $n = 1, 2, \dots$. However, $\bigcup_{n=1}^{\infty} D_n = (-\infty, \infty)$ open and closed

Which is mean you can say nothing about the union of infinite family of closed sets.