

Solution of Sheet 1

Q1:- (i) We have

$$(M1) d(x, y) = 0 \Leftrightarrow x = y$$

$$(M2) d(x, y) = d(y, x) \quad \forall x, y \in X$$

$$(M3) 5 = d(1,3) \leq d(1,2) + d(2,3) = 2 + 4 = 6$$

$$2 = d(1,2) \leq d(1,3) + d(3,2) = 5 + 4 = 9$$

$$4 = d(2,3) \leq d(2,1) + d(1,3) = 2 + 5 = 7$$

Thus d is metric on X .

(ii) We have

$$(M1) d(x, y) = 0 \Leftrightarrow x = y$$

$$(M2) d(x, y) = d(y, x) \quad \forall x, y \in X$$

$$\text{However, } 7 = d(1,3) > d(1,2) + d(2,3) = 2 + 4 = 6$$

Thus d is not a metric on X .

$$Q2:- (a) d_1(x, y) = \sum_{i=1}^3 |x_i - y_i| = |1 - 3| + |5 - 8| + |-3 + 9| = 11$$

$$(b) d_2(x, y) = \sqrt{\sum_{i=1}^3 |x_i - y_i|^2} = \sqrt{|1 - 3|^2 + |5 - 8|^2 + |-3 + 9|^2} = \sqrt{4 + 9 + 36} =$$

$$(c) d_\infty(x, y) = \max_{1 \leq i \leq 3} \{|x_i - y_i|\} = \max\{2, 3, 6\} = 6.$$

Q3:- (A) $f(x) = x^2$ and $g(x) = x^3$, (i) $d_\infty(f, g)$ in $C([0,1])$.

$$d_\infty(f, g) = \max_{x \in [0,1]} \{|x^2 - x^3|\} = \max_{x \in [0,1]} \{(x^2 - x^3)\}$$

Note that, $|x^2 - x^3| = (x^2 - x^3)$ because $x^2 \geq x^3$ for all $x \in [0,1]$. To find the maximum value we find the critical points of the function $x^2 - x^3$.

$$(x^2 - x^3)' = 2x - 3x^2 = 0 \implies x(2 - 3x) = 0 \implies x = 0 \text{ or } x = \frac{2}{3}.$$

It's clear that the maximum attained at $x = \frac{2}{3}$

$$\max_{x \in [0,1]} \{(x^2 - x^3)\} = \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 = \frac{4}{9} - \frac{8}{27} = \frac{12 - 8}{27} = \frac{4}{27}$$

(ii) $d_\infty(f, g)$ in $C([-1,1])$.

If we sketch the graph of the functions $|x^2 - x^3|$ on the interval $[-1,1]$ we will see that the maximum is attained at the left end point of the interval and

$$d_\infty(f, g) = \max_{x \in [-1,1]} \{|x^2 - x^3|\} = |(-1)^2 - (-1)^3| = 2.$$

B. $f(x) = x^2$ and $g(x) = x^4$. Find (i) $d_\infty(f, g)$ in $C([0,1])$.

$$d_\infty(f, g) = \max_{x \in [0,1]} \{|x^2 - x^4|\} = \max_{x \in [0,1]} \{(x^2 - x^4)\}$$

$$(x^2 - x^4)' = 2x - 4x^3 = 0 \Rightarrow x(2 - 4x^2) = 0 \Rightarrow x = 0 \text{ or } x = \frac{1}{\sqrt{2}}.$$

$$d_\infty(f, g) = \max_{x \in [0,1]} \{|x^2 - x^4|\} = \left| \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^4 \right| = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

(ii) $d_\infty(f, g)$ in $C([0,2])$.

$$d_\infty(f, g) = \max_{x \in [0,2]} \{|x^2 - x^4|\} = |2^2 - 2^4| = 12.$$

Q4:

$$(N1) \|x\|_2 = \sqrt{\sum_{i=1}^N |x_i|^2} = 0 \text{ if and only if } x_i = 0 \quad \forall i = 1, \dots, N.$$

$$(N2) \|\lambda x\|_2 = \sqrt{\sum_{i=1}^N |\lambda x_i|^2} = |\lambda| \sqrt{\sum_{i=1}^N |x_i|^2} = |\lambda| \|x\|_2 \quad \forall \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^N.$$

$$(N3) \|x + y\|_2^2 = \sum_{i=1}^N |x_i + y_i|^2 = \sum_{i=1}^N |x_i|^2 + 2 \sum_{i=1}^N x_i y_i + \sum_{i=1}^N |y_i|^2$$

$$= \sum_{i=1}^N |x_i|^2 + 2|x \cdot y| + \sum_{i=1}^N |y_i|^2$$

$$= \|x\|_2^2 + 2\|x\|_2 \|y\|_2 + \|y\|_2^2 \text{ by Cauchy inequality}$$

$$= (\|x\|_2 + \|y\|_2)^2$$

Q5: (1)

$$(M1) d_1(x, y) = \sum_{i=1}^N |x_i - y_i| = 0 \Leftrightarrow x_i = y_i \quad \forall x, y \in \mathbb{R}^N.$$

$$(M2) d_1(x, y) = \sum_{i=1}^N |x_i - y_i| = \sum_{i=1}^N |-1(y_i - x_i)| = \sum_{i=1}^N |(y_i - x_i)| \\ = d_1(y, x) \quad \forall x, y \in \mathbb{R}^N.$$

(M3)

$$d_1(x, y) = \sum_{i=1}^N |x_i - y_i| = \sum_{i=1}^N |x_i - z_i + z_i - y_i| \\ \leq \sum_{i=1}^N |x_i - z_i| + \sum_{i=1}^N |z_i - y_i| \\ = d_1(x, z) + d_1(z, y) \quad \forall x, y, z \in \mathbb{R}^N.$$

(2)

$$(M1) d_2(x, y) = \left(\sum_{i=1}^N |x_i - y_i|^2 \right)^{\frac{1}{2}} = 0 \Leftrightarrow x_i = y_i \quad \forall x, y \in \mathbb{R}^N$$

$$(M2) d_2(x, y) = \left(\sum_{i=1}^N |x_i - y_i|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^N |y_i - x_i|^2 \right)^{\frac{1}{2}} = d_2(y, x) \quad \forall x, y \in \mathbb{R}^N$$

(M3) We want to show that

$$d_2(x, y) = \left(\sum_{i=1}^N |x_i - y_i|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^N |x_i - z_i|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^N |z_i - y_i|^2 \right)^{\frac{1}{2}} \\ = d_1(x, z) + d_1(z, y)$$

We have

$$(d_2(x, y))^2 = \left(\sum_{i=1}^N |x_i - y_i|^2 \right) = \sum_{i=1}^N |x_i - z_i + z_i - y_i|^2 \\ = \sum_{i=1}^N |x_i - z_i|^2 + 2 \sum_{i=1}^N (x_i - z_i)(z_i - y_i) + \sum_{i=1}^N |z_i - y_i|^2 \\ \leq \sum_{i=1}^N |x_i - z_i|^2 + 2 \left(\sum_{i=1}^N (x_i - z_i)^2 \right)^{\frac{1}{2}} \times \left(\sum_{i=1}^N (z_i - y_i)^2 \right)^{\frac{1}{2}} \\ + \sum_{i=1}^N |z_i - y_i|^2 \text{ by Cauchy Schwarz}$$

$$= \left[\left(\sum_{i=1}^N (x_i - z_i)^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^N (z_i - y_i)^2 \right)^{\frac{1}{2}} \right]^2.$$

(3)

$$(M1) \quad d_{\infty}(x, y) = \max_{1 \leq i \leq N} \{|x_i - y_i|\} = 0 \Leftrightarrow x_i = y_i \quad \forall x, y \in \mathbb{R}^N.$$

$$(M2) \quad d_{\infty}(x, y) = \max_{1 \leq i \leq N} \{|x_i - y_i|\} = \max_{1 \leq i \leq N} \{|y_i - x_i|\} = d_{\infty}(y, x) \quad \forall x, y \in \mathbb{R}^N.$$

$$(M3) \quad d_{\infty}(x, y) = \max_{1 \leq i \leq N} \{|x_i - y_i|\} = \max_{1 \leq i \leq N} \{|x_i - z_i + z_i - y_i|\} \\ \leq \max_{1 \leq i \leq N} \{|x_i - z_i| + |z_i - y_i|\} \\ \leq \max_{1 \leq i \leq N} \{|x_i - z_i|\} + \max_{1 \leq i \leq N} \{|z_i - y_i|\} \\ = d_{\infty}(x, z) + d_{\infty}(z, y) \quad \forall x, y, z \in \mathbb{R}^N.$$

Q6:- (1)

$$(M1) \quad Kd(x, y) = 0 \Leftrightarrow x = y, \forall K \in \mathbb{R}.$$

$$(M2) \quad Kd(x, y) = Kd(y, x) \quad \forall x, y \in X \text{ and } \forall K \in \mathbb{R}.$$

$$(M3) \quad Kd(x, y) \leq Kd(x, z) + Kd(z, y) \quad \forall x, y, z \in X \text{ and } \forall K \in \mathbb{R}.$$

But from (M1') we have $Kd(x, y) \geq 0$. So $K > 0$.

(2)

$$(M1) \quad d(x, y) + K = 0 \Leftrightarrow x = y \text{ and } K = 0.$$

$$(M2) \quad d(x, y) + K = d(y, x) + K \quad \forall x, y \in X \text{ and } \forall K \in \mathbb{R}.$$

$$(M3) \quad d(x, y) + K \leq d(x, z) + K + d(z, y) + K \quad \forall x, y, z \in X \text{ and } K = 0.$$

So $K = 0$.

Q7:-

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| \Rightarrow \|x\| - \|y\| \leq \|x - y\|$$

$$\|y\| = \|(y - x) + x\| \leq \|y - x\| + \|x\| \Rightarrow \|y\| - \|x\| \leq \|x - y\|$$

Hence

$$|\|x\| - \|y\|| \leq \|x - y\|.$$