On Concircular Curvature Tensor and Curvature Identities of $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-Manifolds

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ABSTRACT

In this article, we determine the necessary and sufficient condition that makes $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold classified as almost $C(\kappa)$-manifold. We discover $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold belongs to the class CR_2 and then to the class CR_3 because $CR_2 \subset CR_3$, however, this manifold does not belong to the class CR_1 unless its dimension is 3, or it rounds to $\mathcal{C}_{12}$-manifold. We obtain the basic condition that makes $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold has pointwise constant Φ-holomorphic sectional curvature (pointwise constant ΦHS-curvature). Moreover, we get the components of the concircular curvature tensor on the G-structure adjoined space (GSA-space) of $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold. Also, we conclude that both $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold of class CC_1 and the concircularly flat $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold has scalar curvature $s = -2n(2n+1)\alpha^2$. Finally, we find a specific relationship between an Einstein manifold and a concircularly flat $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold.

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1. Introduction

The study of the concircular curvature tensor is an intriguing area within differential geometry, particularly in the context of almost contact metric manifolds. This tensor serves as a measure of the deviation of a manifold from being a space of constant curvature. It plays a crucial role in understanding the geometric properties and classifications of various types of manifolds. In 1940, Yano [1] proposed and defined the concircular curvature tensor of type $(3{,}1)$ on a Riemannian manifold of dimension n. This tensor is defined in terms of the Riemannian curvature tensor, the scalar curvature, and the metric tensor. The concircular curvature tensor remains invariant under concircular transformations between Riemannian manifolds, where the concircular transformation preserves geodesic circles. Moreover, its importance lies in measuring whether the Riemannian manifold possesses a constant curvature or not. On the other hand, the concircular curvature tensor has physical application in gravity theories, especially in those involving perfect fluids.

On the other hand, Janssens and Vanhecke [2] presented the concept of almost $\mathrm{C}(\kappa)$-manifolds, where κ is a real number. In 2011, Kharitonova [3] established the conditions that make an almost contact metric manifold classified as an almost $\mathrm{C}(\kappa)$-manifold on a G-structure adjoined space. Also, she established the validity of contact analogs of the second and third Gray's curvature identities on almost $\mathrm{C}(\kappa)$-manifolds. In 2013, Akbar and Sarkar [4] characterized specific curvature conditions for conharmonic and concircular curvature tensors on an almost $\mathrm{C}(\kappa)$-manifold. Also, they studied concircularly flat and ξ-concircularly flat on almost $\mathrm{C}(\kappa)$-manifolds.

An important study in the field of almost contact metric manifolds focuses on a specific class of manifolds. This class can be considered as a direct sum of many irreducible classes, which were first defined by Chinea and Gonzalez [5]. In 2017, de Candia and Falcitelli [6] established general properties of the slant submanifolds of $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold that have special almost Hermitian structures. In 2019, de Candia and Falcitelli [7] investigated the curvature properties of $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold. They also described some local classification of $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold that satisfied the $\mathrm{N}(\mathrm{k})$-nullity condition in dimension $2\mathrm{n}+1 \geq 5$. In 2021, de Candia and Falcitelli [8] derive conditions under which the codimension of a slant submanifold of an almost contact metric manifold belonging to the class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$ can be reduced.

In 2024, Rustanov and Kharitonova [9] studied nearly trans-Sasakian manifolds with constant holomorphic sectional curvature. They demonstrated that a harmonic nearly trans-Sasakian Einstein manifold possesses non-positive scalar curvature. Furthermore, in the specific case of zero scalar curvature, such a manifold is locally equivalent to the product of a Ricci-flat nearly Kählerian manifold and the real line.

Therefore, our study mixes the results mentioned above. That is, we give the relationships among the class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$, almost $\mathrm{C}(\kappa) -$ class, and contact analogs of Gray's classes CR_1, CR_2 and CR_3 on GSA-space. Also, we discuss the required condition that makes the $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold has pointwise constant Φ- holomorphic sectional curvature, where these concepts are general formulations of constant curvature. However, we determine the components of the concircular curvature tensor on GSA-space, and determine the conditions that make $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold concircularly flat.

2. Preliminaries

We use the symbols $\mathrm{M}^{2\mathrm{n}+1}$, g, and ∇ to refer a $(2\mathrm{n}+1)$-dimensional smooth manifold M, a Riemannian metric, and a Levi-Civita connection (Riemannian connection), respectively. Additionally, let $\mathfrak{X}(\mathrm{M})$ be the module of smooth vector fields over $\mathrm{M}^{2\mathrm{n}+1}$, and **d** is the exterior differentiation operator on $\mathrm{M}^{2\mathrm{n}+1}$.

Definition 2.1. [10] Let $(\mathrm{M}^{2\mathrm{n}+1}, \mathrm{g})$ be stand for a Riemannian manifold. The triple (ξ, η, Φ) with the Riemannian manifold forms an almost contact metric (AC-)manifold, if ξ is a characteristic vector field, η is its dual form, and Φ is an endomorphism of the module $\mathfrak{X}(\mathrm{M})$, and the following items hold:

1. $\Phi(\xi) = 0$,
2. $\eta(\xi) = 1$,
3. $\eta(\Phi \mathrm{U}) = 0$,
4. $\Phi^2(\mathrm{U}) = -\mathrm{U} + \eta(\mathrm{U})\xi$,
5. $\mathrm{g}(\Phi \mathrm{U}, \Phi \mathrm{V}) = \mathrm{g}(\mathrm{U}, \mathrm{V}) - \eta(\mathrm{U})\eta(\mathrm{V})$; for all $\mathrm{U}, \mathrm{V} \in \mathfrak{X}(\mathrm{M})$.

Moreover, the structure $(\mathrm{g}, \xi, \eta, \Phi)$ is called an almost contact metric structure (AC-structure).

For an orthonormal basis $\{\xi, \mathrm{X}_1, \dots, \mathrm{X}_\mathrm{n}, \mathrm{X}_{\hat{1}}, \dots, \mathrm{X}_{\hat{\mathrm{n}}}\}$ of $\mathfrak{X}(\mathrm{M})$, we define an A-frame for $m \in M$, as $(\mathrm{m}; \xi, \epsilon_1, \dots, \epsilon_\mathrm{n}, \epsilon_{\hat{1}}, \dots, \epsilon_{\hat{\mathrm{n}}})$, where $\epsilon_\mathrm{a} = \sqrt{2}\, \zeta(\mathrm{X}_\mathrm{a})$, $\epsilon_{\hat{\mathrm{a}}} = \sqrt{2}\, \overline{\zeta}(\mathrm{X}_\mathrm{a})$, $\zeta = \frac{1}{2}\left(\mathrm{id} - \sqrt{-1}\Phi\right)$;

$\overline{\zeta} = \frac{1}{2}\left(\mathrm{id} + \sqrt{-1}\Phi\right)$. Throughout the whole work we take $i, j, k, l = 0, 1, \ldots, 2n$, and take $a, b, c, d, h = 1, \ldots, n$. Also set $\hat{a} = a + n$, $\hat{0} = 0$, and $\hat{\hat{a}} = a$. The set of all such A-frames above determines an GSA-space on M^{2n+1}, whose structure group is the Lie group $U(n) \times \{1\}$ (see [11]).

On the GSA-space, g and Φ have the following values [12]:

$$\left(g_{ij}\right) = \begin{pmatrix} 1 & \vdots & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 0 & \vdots & I_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & I_n & \vdots & 0 \end{pmatrix}; \qquad \left(\Phi^i_j\right) = \begin{pmatrix} 0 & \vdots & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ 0 & \vdots & \sqrt{-1}I_n & \vdots & 0 \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ 0 & \vdots & 0 & \vdots & -\sqrt{-1}I_n \end{pmatrix}, \tag{1}$$

where I_n is n × n identity matrix.

Kirichenko defined six tensors on AC-manifolds represented by the symbols B, C, D, E, F and G as shown in [11]. In particular, $G = \Phi \circ \nabla_{\xi}(\Phi)\xi$. Kirichenko discovered G has non-zero components defined by the following formulas:

$$C^a = -\sqrt{-1}\Phi^0_{\hat{a},0}; \quad C_a = \sqrt{-1}\Phi^0_{a,0}.$$

The characteristics identity for $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold M^{2n+1}, with AC-structure (g, ξ, η, Φ) was defined by de Candia and Falcitelli [7], as follows:

$$\begin{aligned}(\nabla_U\Phi)V = {} & \alpha\{g(\Phi U, V)\xi - \eta(V)\Phi U\} \\ & -\eta(U)\{\eta(V)\Phi(\nabla_\xi\xi) + g(\nabla_\xi\xi, \Phi V)\xi\}, \quad \forall\, U, V \in \mathfrak{X}(M);\end{aligned} \tag{2}$$

Where $\alpha = -\frac{\delta\eta}{2n}$, δ is the codifferential operator.

The identity in equation (2) is equivalent to a system of equations on GSA-space, which is given by [13]:

$$\Phi^i_{j,k} = \alpha(g_{mj}\Phi^m_k\delta^i_0 - \eta_j\Phi^i_k) - \eta_k(\eta_j\Phi^i_l G^l + \Omega_{lj}G^l\delta^i_0),$$

where Ω is a skew-symmetric tensor of type (2, 0) defined by $\Omega(U, V) = g(U, \Phi V)$; $\forall\, U, V \in \mathfrak{X}(M)$, and G^l is the components of the tensor G.

Theorem 2.1. [13] If M^{2n+1} is an AC-manifold of class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$, then its first group of structure equations is formulated by:

1. $\mathbf{d}\omega^a = -\theta^a_b \wedge \omega^b - \alpha\omega^a \wedge \omega$;
2. $\mathbf{d}\omega_a = \theta^b_a \wedge \omega_b - \alpha\omega_a \wedge \omega$;
3. $\mathbf{d}\omega = C_b\omega \wedge \omega^b + C^b\omega \wedge \omega_b$,

where θ is the 1-form of ∇, and $\{\omega, \omega^1, \dots, \omega^{2n}\}$ is the dual A-frame on M^{2n+1}.

Theorem 2.2. [13] If M^{2n+1} is an AC-manifold of class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$, then its second group of structure equations is formulated as follows:

1. $\mathbf{d}C_b = C_h\theta^h_b + C_{bh}\omega^h + C_b{}^h\omega_h + C_{b0}\omega$;
2. $\mathbf{d}C^b = -C^h\theta^b_h + C^{bh}\omega_h + C^b{}_h\omega^h + C^{b0}\omega$;
3. $\mathbf{d}\theta^a_b = -\theta^a_h \wedge \theta^h_b + A^{ad}_{bh}\omega^h \wedge \omega_d + A^a_{bh0}\omega^h \wedge \omega + A^{ah0}_b\omega_h \wedge \omega$;
4. $\mathbf{d}\alpha = \alpha_d\omega^d + \alpha^d\omega_d + \alpha_0\omega$,

where $A^{ad}_{[bh]} = A^{[ad]}_{bh} = A^a_{[bh]0} = A^{[ah]0}_b = C_{[bh]} = C^{[bh]} = 0$. Also $C_b{}^h = C^h{}_b$ and for $n > 1$, we have $\alpha^d = \alpha C^d$ and $\alpha_d = \alpha C_d$. Any index has a range from 1 to n.

Theorem 2.3. [13] On the GSA-space, the Riemannian curvature tensor has the following components on $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifolds:

1. $R^a_{0c0} = C^a{}_c - C^aC_c - \delta^a_c\alpha_0 - \delta^a_c\alpha^2$;
2. $R^a_{0c\hat{d}} = -\delta^a_c\alpha^d$;
3. $R^a_{0\hat{c}0} = C^{ac} - C^aC^c$;
4. $R^a_{0cd} = -2\delta^a_{[c}\alpha_{d]}$;
5. $R^a_{bc0} = A^a_{bc0} - \alpha\delta^a_cC_b$;
6. $R^a_{b\hat{c}0} = A^{ac0}_b + \alpha\delta^c_bC^a$;
7. $R^a_{bc\hat{d}} = A^{ad}_{bc} - \alpha^2\delta^a_c\delta^d_b$.
8. $R^a_{\hat{b}cd} = -2\alpha^2\delta^a_{[c}\delta^b_{d]}$;
9. $R^a_{\hat{b}c0} = 2\alpha C^{[a}\delta^{b]}_c$;

the remaining components are either zero, determined by symmetry, or can be found using the conjugates of the given components.

Definition 2.2. [3] An AC-manifold M^{2n+1} with AC-structure (g, ξ, η, Φ) is called an almost $C(\kappa)$-manifold, if the following relation holds:

$$\begin{aligned} g(R(Z,W)V,U) = \; & g(R(\Phi Z,\Phi W)V,U) - \kappa\{g(U,W)g(V,Z) - g(U,Z)g(V,W) \\ & -g(U,\Phi W)g(V,\Phi Z) + g(U,\Phi Z)g(V,\Phi W)\}, \end{aligned}$$

where $U, V, Z, W \in \mathfrak{X}(M)$, and κ is a real number.

Theorem 2.4. [3] An AC-manifold M^{2n+1} is almost $C(\kappa)$-manifold if and only if the following relations hold:

$R^a_{\hat{b}cd} = \kappa\delta^{ab}_{cd}$, $R^a_{0b0} = \kappa\delta^a_b$, $R^a_{bc\hat{d}}$ is arbitrary. Moreover, the Binachi identity given by:

$R^{a}_{bc\hat{d}} - R^{a}_{cb\hat{d}} = -\kappa\delta^{ad}_{bc}$, where $\delta^{ab}_{cd} = \delta^{a}_{c}\delta^{b}_{d} - \delta^{a}_{d}\delta^{b}_{c}$, and other Riemannian curvature tensor components are zero.

Definition 2.3. [14] An AC-manifold M^{2n+1} with AC-structure (g, ξ, η, Φ) and the Riemannian curvature tensor R is said to be

1. of class CR_1 if $g(R(\Phi U, \Phi V)\Phi W, \Phi Z) = g(R(\Phi^2 U, \Phi^2 V)\Phi W, \Phi Z)$;
2. of class CR_2 if
$$g(R(\Phi V, \Phi W)\Phi Z, \Phi U) = g(R(\Phi^2 V, \Phi^2 W)\Phi Z, \Phi U) + g(R(\Phi^2 V, \Phi W)\Phi^2 Z, \Phi U) + g(R(\Phi^2 V, \Phi W)\Phi Z, \Phi^2 U);$$
3. of class CR_3 if $g(R(\Phi U, \Phi V)\Phi W, \Phi Z) = g(R(\Phi^2 U, \Phi^2 V)\Phi^2 W, \Phi^2 Z)$,

for every $U, V, Z, W \in \mathfrak{X}(M)$.

Lemma 2.1. [14] The equivalent conditions of Definition 2.3 on GSA-space are formulated in the following:

$$CR_1 \Leftrightarrow R_{\hat{a}bcd} = R_{abcd} = R_{\hat{a}\hat{b}cd} = 0;$$
$$CR_2 \Leftrightarrow R_{\hat{a}bcd} = R_{abcd} = 0;$$
$$CR_3 \Leftrightarrow R_{\hat{a}bcd} = 0.$$

Definition 2.4. [15] Let M^{2n+1} be an AC-manifold with AC-structure (g, ξ, η, Φ). A Φ-holomorphic sectional (ΦHS-)curvature H of M in the direction of U $\left(U \neq 0;\ U \in \ker(\eta)\right)$ is defined by:

$$H(U) = \frac{g(R(U,\Phi U)U,\Phi U)}{g(U,U)^2},$$

where R is the Riemannian curvature tensor of M^{2n+1}. Moreover, it say that M^{2n+1} has a pointwise constant ΦHS-curvature if $H(U) = \gamma$, where γ is a smooth function on M and does not depend on U.

Theorem 2.5. [15] An AC-manifold M^{2n+1} has a pointwise constant ΦHS-curvature if and only if, on GSA-space, the following relation holds:

$$R^{(a\ \ d)}_{(bc)} = \frac{\gamma}{2}\tilde{\delta}^{ad}_{bc},$$

where $\tilde{\delta}^{ad}_{bc} = \delta^{a}_{b}\delta^{d}_{c} + \delta^{a}_{c}\delta^{d}_{b}$ and (..) represents the symmetric operator involving the specified indices, that is $T_{(ab)} = \frac{1}{2}\{T_{ab} + T_{ba}\}$.

Definition 2.5. [16] A (4, 0)-concircular curvature tensor on an AC-manifold M^{2n+1} is defined by:

$$L(U, V, Z, W) = R(U, V, Z, W) - \frac{s}{2n(2n+1)}\{g(U, Z)g(V, W) - g(U, W)g(V, Z)\},$$

for all $U, V, Z, W \in \mathfrak{X}(M)$, where s is the scalar curvature.

The concircular tensor in Definition 2.5, has components for any AC-manifold as follows [17]:

$$L_{ijkl} = R_{ijkl} - \frac{s}{2n(2n+1)}\{g_{ik}g_{jl} - g_{il}g_{jk}\}. \tag{3}$$

Definition 2.6. [17] An AC-manifold M^{2n+1} is said to be concircularly flat if $L_{ijkl} = 0$; for all $i, j, k, l = 0,1,\dots,2n$.

Theorem 2.6. [13] The components of the Ricci tensor r for the class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$ on GSA-space are given by:

1. $r_{00} = 2(C^a{}_a - C^aC_a) - 2n(\alpha_0 + \alpha^2)$;
2. $r_{a0} = A^b_{ab0} - n\alpha C_a - (n-1)\alpha_a$;
3. $r_{ab} = C_{ba} - C_bC_a$;
4. $r_{\hat{a}b} = C^a{}_b - C^aC_b - \alpha_0\delta^a_b + A^{ac}_{cb} - 2n\alpha^2\delta^a_b$;

the remaining components are determined by symmetry, or can be found using the conjugates of the given components.

Definition 2.7. [18] The AC-manifold is called an Einstein manifold if

$$r(U,V) = \lambda\, g(U,V); \qquad \forall \quad U,V \in \mathfrak{X}(M),$$

where λ is a scalar function.

Remark 2.1. [19] According to Lemma 2.1, the classes CR_1, CR_2, CR_3 satisfy the following:

$$CR_1 \subset CR_2 \subset CR_3.$$

3. The Main Results

Theorem 3.1. A $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold M^{2n+1} is an almost $C(\kappa)$-manifold if and only if the following hold:

$$\begin{cases} \kappa = -\alpha^2;\ \alpha_0 = \alpha_d = \alpha^d = 0;\ C^a{}_b = C^aC_b;\ C^{ab} = C^aC^b; & \text{for all } n \\ A^1_{110} = \alpha C_1;\ A^{110}_1 = -\alpha C^1; & \text{if } n = 1 \\ A^a_{bc0} = A^{ac0}_b = 0. & \text{if } n > 1 \end{cases}$$

Proof. Suppose that M^{2n+1} is an almost $C(\kappa)$-manifold then from Theorem 2.4 and Theorem 2.3, we get the following: $R^a_{\hat{b}cd} = \kappa\delta^{ab}_{cd} \Leftrightarrow \kappa = -\alpha^2$, so α is scalar, then $d\alpha = 0$, that is $\alpha_0 = \alpha_d = \alpha^d = 0$. Also, $R^a_{0b0} = \kappa\delta^a_b \Leftrightarrow C^a{}_b = C^aC_b$. Moreover, $R^a_{bc\hat{d}} - R^a_{cb\hat{d}} = -\kappa\delta^{ad}_{bc} \Leftrightarrow A^{ad}_{[bc]} = 0$, (which we already have from Theorem 2.2). Furthermore, since all other components of

Riemannian curvature tensor are zeros, so $C^{ab} = C^aC^b$, and $A^a_{bc0} = \alpha C_b\delta^a_c$, so we get for $n > 1$, $A^a_{bc0} = 0$, since $\alpha C_b = \alpha_b = 0$. Takes $R^a_{\hat{b}c0} = 0$, gives the same result.

Theorem 3.2. Any $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold belongs to the classes CR_2 and CR_3. However, $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold M belongs to the class CR_1 if and only if either M of dimension 3, or M reduced to C_{12}-manifold.

Proof. Suppose that M^{2n+1} is an AC-manifold of class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$. Since $R^i_{jkl} = R_{\hat{i}jkl}$, then from Theorem 2.3 and Lemma 2.1, we have

$$R_{\hat{a}bcd} = R_{abcd} = 0 \Longrightarrow M^{2n+1} \in CR_2,$$

$$R_{\hat{a}bcd} = 0 \Longrightarrow M^{2n+1} \in CR_3,$$

Now, suppose that M^{2n+1} of class CR_1, then from Lemma 2.1, we get $R_{\hat{a}\hat{b}cd} = 0$, and according to Theorem 2.3; item 8, we obtain $-2\alpha^2\delta^a_{[c}\delta^b_{d]} = 0 \Longrightarrow \alpha^2(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) = 0$, contracting a and c, we get $\alpha^2\delta^b_d(n-1) = 0 \Longrightarrow n = 1$ or $\alpha = 0$, which gives the result.

Corollary 3.1. Let $M^{2n+1} \in (\mathcal{C}_5 \oplus \mathcal{C}_{12}) \cap CR_1$ and $n > 1$. Then M^{2n+1} is almost $C(0)$-manifold if and only if ${C^a}_b = C^aC_b; C^{ab} = C^aC^b; A^a_{bc0} = A^{ac0}_b = 0$.

Proof. Combining Theorem 3.1 and Theorem 3.2, we get the result.

Corollary 3.2. An AC-manifold M^{2n+1} of class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$ has a pointwise constant ΦHS-curvature if and only if $A^{ad}_{bc} = \frac{1}{2}\tilde{\delta}^{ad}_{bc}(\alpha^2 + \gamma)$.

Proof. From Theorem 2.3; item 7, we obtain

$$R^{(a\ d)}_{(bc)} = A^{(ad)}_{(bc)} - \delta^{(a}_{(c}\delta^{d)}_{b)}\alpha^2 = A^{ad}_{bc} - \frac{1}{4}\left(\delta^a_c\delta^d_b + \delta^d_c\delta^a_b + \delta^a_b\delta^d_c + \delta^d_b\delta^a_c\right)\alpha^2 = A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\alpha^2,$$

so, according to Theorem 2.5, we get the result.

Corollary 3.3. On GSA-space, the concircular curvature tensor has the following components on $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold:

1. $L_{\hat{a}0h0} = {C^a}_h - C^aC_h - \delta^a_h\alpha_0 - \delta^a_h\alpha^2 - \delta^a_h\frac{s}{2n(2n+1)}$;
2. $L_{\hat{a}0\hat{h}0} = C^{ah} - C^aC^h$;
3. $L_{\hat{a}0hd} = -2\delta^a_{[h}\alpha_{d]}$;
4. $L_{\hat{a}0h\hat{d}} = -\delta^a_h\alpha^d$;
5. $L_{\hat{a}bh0} = A^a_{bh0} - \alpha\delta^a_hC_b$;
6. $L_{\hat{a}b\hat{h}0} = A^{ah0}_b + \alpha\delta^h_bC^a$;
7. $L_{\hat{a}bh\hat{d}} = A^{ad}_{bh} - \alpha^2\delta^a_h\delta^d_b - \delta^a_h\delta^d_b\frac{s}{2n(2n+1)}$;

8. $L_{\hat{a}\hat{b}hd} = -2\alpha^2\delta^a_{[h}\delta^b_{d]} - \delta^a_{[h}\delta^b_{d]}\frac{s}{n(2n+1)}$;

9. $L_{\hat{a}\hat{b}h0} = 2\alpha C^{[a}\delta^{b]}_h$;

the remaining components are either zero, determined by symmetry, or can be found using the conjugates of the given components.

Proof. By taking $i = \hat{a}, j = 0, k = h$ and $l = 0$ in equation (3) and use the values of R^i_{jkl} and g_{ij} that were introduced in Theorem 2.3 and equation (1), respectively, with the fact $R^i_{jkl} = R_{\hat{\imath}jkl}$, we obtain the first result, and similarly for the other results.

Definition 3.1. An AC-manifold M^{2n+1} is said to be

1. of class CC_1 if $g(L(\Phi U, \Phi X)\Phi Y, \Phi Z) = g(L(\Phi^2 U, \Phi^2 X)\Phi Y, \Phi Z)$;
2. of class CC_2 if

$$g(L(\Phi X, \Phi Y)\Phi Z, \Phi U) = g(L(\Phi^2 X, \Phi^2 Y)\Phi Z, \Phi U) + g(L(\Phi^2 X, \Phi Y)\Phi^2 Z, \Phi U)$$
$$+g(L(\Phi^2 X, \Phi Y)\Phi Z, \Phi^2 U);$$

3. of class CC_3 if $g(L(\Phi U, \Phi X)\Phi Y, \Phi Z) = g(L(\Phi^2 U, \Phi^2 X)\Phi^2 Y, \Phi^2 Z)$;

for every $X, U, Y, Z \in \mathfrak{X}(M)$.

So on GSA- space, we have:

$$\begin{aligned} CC_1 &\Leftrightarrow L_{\hat{a}bhd} = L_{abhd} = L_{\hat{a}\hat{b}hd} = 0; \\ CC_2 &\Leftrightarrow L_{\hat{a}bhd} = L_{abhd} = 0; \\ CC_3 &\Leftrightarrow L_{\hat{a}bhd} = 0. \end{aligned} \tag{4}$$

Remark 3.1. According to equation (4), the classes CC_1 , CC_2 , CC_3 satisfy the following:

$$CC_1 \subset CC_2 \subset CC_3.$$

Theorem 3.3. Any $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold belongs to the class CC_2. But $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold belongs to the class CC_1 if and only if, $s = -2n(2n+1)\alpha^2 < 0$ (Hyperbolic).

Proof. Suppose that M^{2n+1} is an AC-manifold of class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$. Then, from Corollary 3.3 and Definition 3.1, we have

$$L_{\hat{a}bhd} = L_{abhd} = 0 \Longrightarrow M^{2n+1} \in CC_2.$$

If $L_{\hat{a}\hat{b}hd} = -2\alpha^2\delta^a_{[h}\delta^b_{d]} - \delta^a_{[h}\delta^b_{d]}\frac{s}{n(2n+1)} = 0$, then $s = -2n(2n+1)\alpha^2$. so, $M^{2n+1} \in CC_1$ if the scalar curvature is given as previously.

Remark 3.2. According to Remark 3.1, $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold is also belong to the class CC_3.

Corollary 3.4. An AC-manifold M^{2n+1} of class $\mathcal{C}_5 \oplus \mathcal{C}_{12}$ is concircularly flat if and only if the following conditions hold:

$$C^{a}{}_{h} - C^{a}C_{h} - \delta^{a}_{h}\alpha_{0} = 0,\quad C^{ah} = C^{a}C^{h},\quad \alpha_{d} = \alpha^{d} = 0,\quad A^{a}_{bh0} - \alpha\delta^{a}_{h}C_{b} = 0,$$

$$A^{ah0}_{b} + \alpha\delta^{h}_{b}C^{a} = 0,\quad A^{ad}_{bh} = 0,\quad s = -2n(2n+1)\alpha^{2}.$$

Also, if $n > 1$, then $A^{a}_{bh0} = A^{ah0}_{b} = 0$

Proof. According to Corollary 3.3 and Definition 2.6, we have the following:

$L_{\hat{a}\hat{b}hd} = 0 \Longrightarrow -2\alpha^{2}\delta^{a}_{[h}\delta^{b}_{d]} - \delta^{a}_{[h}\delta^{b}_{d]}\frac{s}{n(2n+1)} = 0 \Longrightarrow s = -2n(2n+1)\alpha^{2}$, so $L_{\hat{a}0h0} = 0$ and $L_{\hat{a}bh\hat{d}} = 0$, give us $C^{a}{}_{h} - C^{a}C_{h} - \delta^{a}_{h}\alpha_{0} = 0$ and $A^{ad}_{bh} = 0$. Also $L_{\hat{a}0\hat{h}0} = 0 \Longrightarrow C^{ah} = C^{a}C^{h}$. Furthermore, $L_{\hat{a}0h\hat{d}} = 0 \Longrightarrow \alpha^{d} = \alpha_{d} = 0$ and $L_{\hat{a}bh0} = 0 \Longrightarrow A^{a}_{bh0} - \alpha\delta^{a}_{h}C_{b} = 0$. Additionally, $L_{\hat{a}b\hat{h}0} = 0 \Longrightarrow A^{ah0}_{b} + \alpha\delta^{h}_{b}C^{a} = 0$.

Now, if $n > 1$, then $\alpha C_{b} = \alpha_{b} = 0$ and $\alpha C^{a} = \alpha^{a} = 0$, which yields the result.

Corollary 3.5. If $M^{2n+1} \in \mathcal{C}_{5} \oplus \mathcal{C}_{12}$ and it is concircularly flat, then M^{2n+1} has a pointwise constant ΦHS-curvature with $\gamma = -\alpha^{2}$.

Proof. Suppose that M^{2n+1} is concircularly flat of class $\mathcal{C}_{5} \oplus \mathcal{C}_{12}$, then, according to Corollary 3.4, we get $A^{ad}_{bh} = 0$. So based on Corollary 3.2, we get $\frac{1}{2}\tilde{\delta}^{ad}_{bc}(\alpha^{2} + \gamma) = 0 \Longrightarrow \gamma = -\alpha^{2}$.

Corollary 3.6. Let $M^{2n+1} \in \mathcal{C}_{5} \oplus \mathcal{C}_{12}$ and it is concircularly flat. Then M^{2n+1} is an Einstein manifold if and only if $\lambda = -2n\alpha^{2}$.

Proof. According to Definition 2.7, any AC-manifold is an Einstein manifold if $r_{00} = \lambda$, $r_{a0} = r_{ab} = 0$ and $r_{\hat{a}b} = \lambda\delta^{a}_{b}$. Suppose that $M^{2n+1} \in \mathcal{C}_{5} \oplus \mathcal{C}_{12}$ and it is concircularly flat. hence, by Corollary 3.4 and Theorem 2.6, we obtain $r_{a0} = r_{ab} = 0$, $r_{00} = -2n\alpha^{2} = \lambda$ and $r_{\hat{a}b} = -2n\alpha^{2}\delta^{a}_{b} = \lambda\delta^{a}_{b}$. Then the desired result is obtained.

4. Conclusions

We conclude the following:

i. The $\mathcal{C}_{5} \oplus \mathcal{C}_{12}$ – manifold is an almost $C(\kappa)$-manifold of $\kappa \leq 0$, where the equality holds ($\kappa = 0$) if $\mathcal{C}_{5} \oplus \mathcal{C}_{12}$ – manifold belongs to the class CR_{1} and it has a dimension greater than 3.

ii. The concircularly flat $\mathcal{C}_{5} \oplus \mathcal{C}_{12}$ – manifold has non–positive scalar curvature and non–positive pointwise constant ΦHS-curvature.

iii. The $\mathcal{C}_{5} \oplus \mathcal{C}_{12}$ – manifold of non–positive scalar curvature (hyperbolic manifold) fits into the class CC_{1}.

iv. The concircularly flat $\mathcal{C}_{5} \oplus \mathcal{C}_{12}$ – manifold is an Einstein manifold with a non–positive cosmological constant.

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حول تنسر الانحناء الدائري ومتطابقات الانحناء للمانيفولدات من الفئة $C_5 \oplus C_{12}$

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المستخلص

في هذه المقالة، حددنا الشرط الضروري والكافي الذي يجعل المانيفولد من الفئة $C_5 \oplus C_{12}$ يصنف كمانيفولد شبه C(κ). أكتشفنا أن المانيفولد من الفئة $C_5 \oplus C_{12}$ ينتمي إلى الفئة CR_2 وبالتالي ينتمي للفئة CR_3 ، ولكن هذا المانيفولد لا ينتمي إلى الفئة CR_1 مالم يكون من البعد 3 أو يدور الى مانيفولد من الفئة C_{12} . حصلنا على الشرط الاساسي الذي يجعل المانيفولد من الفئة $C_5 \oplus C_{12}$ يمتلك انحناء ΦHS ثابتًا نقطيًا. علاوة على ذلك، حصلنا على مركبات تنسر الانحناء الدائري على الفضاء الملحق بهيكل G للمانيفولد من الفئة $C_5 \oplus C_{12}$. أيضًا، استنتجنا أن كلاً من المانيفولد ذو الفئة $C_5 \oplus C_{12}$ والفئة CC_1 والمانيفولد المسطح الدائري من الفئة $C_5 \oplus C_{12}$ يمتلك انحناء قياسي مقداره $s = -2n(2n+1)\alpha$. أخيرًا، وجدنا علاقة خاصة بين مانيفولد أينشتاين والمانيفولد المسطح الدائري من الفئة $C_5 \oplus C_{12}$.

الكلمات المفتاحية: مانيفولد التماس التقريبي، تنسر انحناء ريمان، انحناء مقطعي هولومورفي – Φ ، مانيفولدات شبه - C(κ).