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New Numerical Methods for Solving the Initial Value Problem Based on a Symmetrical Quadrature Integration Formula Using Hybrid Functions

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Abstract: In this study, we construct new numerical methods for solving the initial value problem (IVP) in ordinary differential equations based on a symmetrical quadrature integration formula using hybrid functions. The proposed methods are designed to provide an efficient and accurate solution to IVP and are more suitable for problems with non-smooth solutions. The key idea behind the proposed methods is to combine the advantages of traditional numerical methods, such as Runge–Kutta and Taylor's series methods, with the strengths of modern hybrid functions. Furthermore, we discuss the accuracy and stability analysis of these methods. The resulting methods can handle a wide range of problems, including those with singularities, discontinuities, and other non-smooth features. Finally, to demonstrate the validity of the proposed methods, we provide several numerical examples to illustrate the efficiency and accuracy of these methods.

Keywords: initial value problem; numerical method; hybrid function; local truncation errors; stability analysis

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1. Introduction

Differential equations are a fundamental tool in many fields of pure and applied science and are used to model a wide range of real-world phenomena [1,2]. While analytic methods exist for solving differential equations, many of the equations encountered in practice are too complex for a closed-form solution. Even when a solution formula is available, it may involve integrals that can only be approximated numerically. In such cases, numerical methods provide an alternative tool for solving differential equations under specified initial conditions. Initial value problems, which take the form of ordinary differential equations [3], are commonly encountered in science and engineering, and can be written in the form:

$$y' = f(x, y(x)), y(x_0) = y_0 \tag{1}$$

To solve the problem (1), various numerical methods with varying orders of convergence have been described and developed (see references [4–10]). The Runge–Kutta method is one of the most commonly used numerical methods for this purpose among the existing methods and has seen a growing interest in its development in recent years. The general m-stage Runge–Kutta method is given as follows:

$$y_{n+1} = y_n + h\varnothing(x_n, y_n; h) \tag{2}$$

where

$$\varnothing(x_n, y_n; h) = \sum_{i=1}^{m} w_i k_i,$$
$$k_1 = f(x_n, y_n),$$
$$k_i = f\left(x_n + c_i h, y_n + \sum_{j=1}^{i-1} a_{ij} k_j\right),\ i = 2, 3, \ldots, m,$$