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Research Article

Hankel and Toeplitz Determinants for q-Starlike Functions Involving a q-Analog Integral Operator and q-Exponential Function

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This article investigates the upper bounds of the second Hankel and Toeplitz determinants for a family of q-starlike functions defined by a q-analog integral operator, which is a more general form of the q-Srivastava-Attiya operator, and the q-exponential function $e_q(z)$. The well-known class of subordination starlike functions associated with the exponential function is used to construct this subordination q-starlike class. The investigation of Hankel determinants $\mathscr{H}_2(1), \mathscr{H}_2(2)$ and the Toeplitz determinants $\mathscr{T}_2(2), \mathscr{T}_3(1)$ results from the analysis of this subordination class. All of these boundaries have been demonstrated to be sharp. Our main findings are inspired by some special cases that are also discussed in this study.

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1. Introduction

Many investigators find the generalization of many classes of analytic functions using quantum calculus (QC) attractive. Jackson [1, 2] first introduced the concept of q-calculus, which includes the q-derivative and the q-integral. The most popular investigations are indicated on the class q-starlike functions. By offering the q-starlike functions' family in a logical sequence, Agrawal and Sahoo [3] expanded on this idea. The relationship between the Janowski functions and various forms of q-starlike functions was examined by Hadid et al. in their study published in [4]. They created and presented Janowski functions, a brand-new subclass of q-starlike functions. Works by Mahmood et al. [5], Ul-Haq et al. [6], and Ibrahim [7] included information on recent research. A subclass of quantum starlike functions containing q-derivative was first presented by Seoudy and Aouf [8]. A recent requirement for q-starlike functions was proposed by Zainab et al. [9] utilizing a unique curve. Hadi et al. [10] developed the derivative operator via q-exponential function. They also investigated in [11] some types of q-analytic functions. Additionally, QC is used to generalize many differential and integral operators (e.g., [12–15]). Ibrahim et al. [16] used the notion of QC coupled with the convolution to propose a symmetric q-derivative operator (see also [17, 18]). Furthermore, the fractional q-calculus is a q-

extension of the traditional fractional calculus (see [19]). Fardi et al. [20] explored certain fractional q-integral and q-derivative operators.

This study's primary objective is to compute the upper bound of the second order of Hankel and Toeplitz determinants for the certain class of q-starlike functions created by a q-analog integral operator coupled to the q-exponential function. Additionally, the examination of this class yields discussion of some special issues. It is interesting to observe that the class $\mathscr{K}\mathcal{S}^*(\tau,\mu;e_q(z))$ is a generalization of the familiar subordination class defined by Mendiratta et al. [21].

Consider the class $\mathscr{A}$ of analytic and univalent functions $f(z)$ in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ normalizing by

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j. \tag{1}$$

Two functions f and $h\in\mathscr{A}$ are convoluted as follows:

$$(f*h)(z)=z+\sum_{j=2}^{\infty}a_jd_jz^j=(h*f)(z), \tag{2}$$

where $h(z)=z+\sum_{j=2}^{\infty}d_jz^j$. Additionally, they are subordinated if they have the following relations:

$$f(z)\prec h(z)\Longleftrightarrow f(0)=h(0)\quad\text{and}\quad f(\mathbb{U})\subset h(\mathbb{U})(z\in\mathbb{U}).$$

Definition 1.1 [1, 2]. For $0<q<1$, the q-derivative operator is defined by

$$\mathfrak{D}_qf(z)=\frac{f(qz)-f(z)}{(q-1)z}\,(z\neq 0).$$

From above, for the function $f(z)$ in Equation (1), we obtain

$$\mathfrak{D}_q\left\{z+\sum_{j=2}^{\infty}a_jz^j\right\}=1+\sum_{j=2}^{\infty}[j]_qa_jz^{j-1},$$

where the q-number $[j]_q$ is given by

$$[j]_q:=\begin{cases}\dfrac{1-q^j}{1-q} & (j\in\mathbb{C})\\ \displaystyle\sum_{j=0}^{n-1}q^j & (j=n\in\mathbb{N}),\end{cases}$$

and $[j]_q!$ denotes the q-factorial, which is defined as follows:

$$[j]_q!=\begin{cases}[j]_q[j-1]_q\cdots[2]_q[1]_{q,} & j=1,2,3,\cdots\\ 1, & j=0.\end{cases}$$

Then,

$$f'(z)=lim_{q\longrightarrow 1^-}\mathfrak{D}_q\left\{1+\sum_{j=2}^{\infty}[j]_qa_jz^j\right\}=1+\sum_{j=2}^{\infty}ja_jz^{j-1}.$$

Definition 1.2 [1, 2]. Let $\varrho\in\mathscr{R}$ and $j\in\mathbb{N}$ be positive integers. The symbol for the q-generalized Pochhammer is given by

$$[\varrho\,;j]_q=[\varrho]_q[\varrho+1]_q[\varrho+2]_{q\ldots}\cdots[\varrho+j-1]_q.$$

Also, the q-gamma function is defined for $\varrho>0$,

$$\Gamma_q(\varrho+1)=[\varrho]_q\Gamma_q(\varrho)\quad\text{and}\quad\Gamma_q(1)=1.$$

In the following inequalities, a function $f(z)$ from class $\mathscr{A}$ is designated as a starlike function ($f(z)\in S^*$) and a q-starlike function ($f(z)\in S^*(q)$):

$$\mathfrak{R}\left[\frac{zf'(z)}{f(z)}\right]>0\quad\text{and}\quad\mathfrak{R}\left[\frac{z\mathfrak{D}_qf(z)}{f(z)}\right]>0,$$

respectively.

Definition 1.3 [22]. The q-exponential function $e_q(z)$ is defined by the power series expansion:

$$e_q(z):=\sum_{j=0}^{\infty}\frac{z^j}{[j]_q!}(z\in\mathbb{U}), \tag{3}$$

then, $e(z)=lim_{q\longrightarrow 1^-}e_q(z)=\sum_{j=0}^{\infty}z^j/j!$.

Definition 1.4 [23]. The normalized function $f(z)$ belongs to the class $\mathcal{S}^*(e_q(z))$ if the subordination condition holds

$$\frac{z\mathfrak{D}_q(f(z))}{f(z)}\prec e_q(z).$$

Remark 1.1. Figure 1 displays the illustration domain $e_q(z)$.

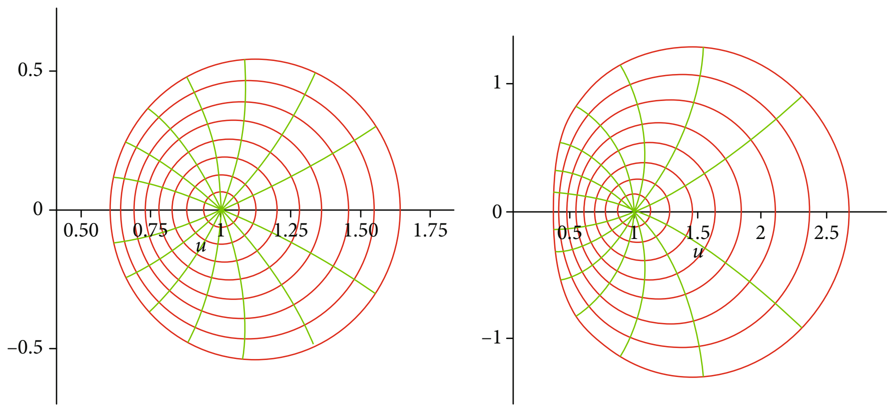
If q approaches to 1^-, the subsequent class is well-defined (see [21]):

$$\mathbb{S}^*_{\exp}=\left\{f\in\mathscr{A}:\frac{zf'(z)}{f(z)}\prec e(z)\right\}. \tag{4}$$

By selecting $f(z)=z+(1/4)z^2$, we may plot the shape of the function class $\mathbb{S}^*_{\exp}$ based on the relation (Equation 4) (refer to Figure 2).

Definition 1.5 [24]. The Ruscheweyh q-derivative operator $\mathscr{F}_q^{\mu+1}(z)$: $\mathscr{A}\longrightarrow\mathscr{A}$ is defined by

$$\mathscr{F}_q^{\mu+1}(z)=z+\sum_{j=2}^{\infty}\frac{[\mu+1\,;j]_q}{[j-1]_q!}z^j,\,(\mu>-1,z\in\mathbb{U}). \tag{5}$$



(a) For $q = 0.5$

(b) For $q = 0.99$

Figure 1: This figure gives the function $e_q(z)$ in various values of q.

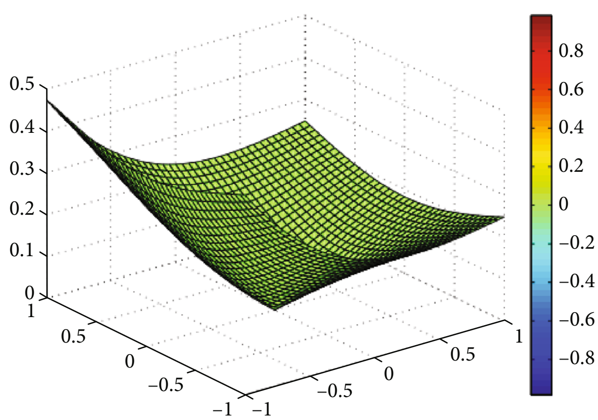


Figure 2: The figure of the class $\mathbb{S}^*_{\exp}$ for $(z) = z + (1/4)z^2$.

In view of the importance of studying the q-calculus, Shah and Noor [25] introduced and investigated the q-Srivastava-Attiya operator (q-SAO), which is defined by the q-Hurwitz–Lerch zeta function $\Phi_q(u, \tau ; z)$ and is given as

$$\mathcal{J}^u_{q,\tau} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{[1+u]_q}{[j+u]_q} \right)^{\tau} a_j z^j, \tag{6}$$

where $u \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\tau \in \mathbb{C}$ when $|z| < 1$ and $R(\tau) > 1$ when $|z| = 1$.

In recent years, some studies have been presented on the concept of the q-SAO (e.g., see [26]).

Hadi et al. [15] defined the q-derivative of the q-SAO $\mathcal{J}^u_{q,\tau} f(z)$ as follows:

$$\mathfrak{D}_q \mathcal{J}^u_{q,\tau} f(z) := \frac{\mathcal{J}^u_{q,\tau} f(qz) - \mathcal{J}^u_{q,\tau} f(z)}{(q-1)z}, (z \in \mathbb{U}),$$

hence,

$$z \mathfrak{D}_q \mathcal{J}^u_{q,\tau} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{[1+u]_q}{[j+u]_q} \right)^{\tau} [j]_q a_j z^j. \tag{7}$$

For $\mu > -1$, also, the authors [15] defined the q-analog integral operator $\mathcal{K}^{\mu}_{q,\tau,u} f(z)$: $\mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\mathcal{K}^{\mu}_{q,\tau,u} f(z) * \mathcal{F}^{\mu+1}_q(z) = z \mathfrak{D}_q \mathcal{J}^u_{q,\tau} f(z),$$

where

$$\mathcal{F}^{\mu+1}_q(z) = z + \sum_{j=2}^{\infty} \frac{[\mu+1 ; j]_q}{[j-1]_q!} z^j, (z \in \mathbb{U}).$$

From the above operator, we conclude that

$$\mathcal{K}^{\mu}_{q,\tau,u} f(z) = z + \sum_{j=2}^{\infty} \chi_j a_j z^j, \tag{8}$$

where

$$\chi_j = \left(\frac{[1+u]_q}{[j+u]_q} \right)^{\tau} \frac{[j]_q!}{[\mu+1 ; j]_q},$$

$$(u \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mu > -1, \tau \in \mathbb{C}, \quad \text{when} \quad |z| < 1, \text{and} \quad R(\tau) > 1 \quad \text{when} \quad |z| = 1).$$

Motivated by the importance of q-analysis in many applications of physics (see free fall of bodies [27]) and mathematics, especially, in analytic functions, the purpose of this paper is to study and investigate a new class $\mathcal{K}\mathcal{S}^*(\tau, \mu ; e_q(z))$ of q-starlike functions, which is defined by a new q-analog integral operator. The normalized function $f(z)$ belongs to $\mathcal{K}\mathcal{S}^*(\tau, \mu ; e_q(z))$ if it satisfies the subordination condition below:

$$\frac{z\mathfrak{D}_q\left(\mathscr{K}^{\mu}_{q,\tau,u}f(z)\right)}{\mathscr{K}^{\mu}_{q,\tau,u}f(z)} \prec e_q(z), (z \in \mathbb{U}),$$

$$(u \in \mathbb{C}\setminus\mathbb{Z}_0^-, \mu > -1, \tau \in \mathbb{C}, \quad \text{when} \quad |z| < 1, \quad \text{and} \quad R(\tau) > 1 \quad \text{when} \quad |z| = 1). \tag{9}$$

By the subordination condition (Equation 9), we note that

1. If $\tau = 0$ and $\mu = 1$, we obtain the class $\mathcal{S}^*(e_q(z))$ established by Srivastava et al. [23].
2. For $q \longrightarrow 1^-$, $\tau = 0$, and $\mu = 1$, we get the class $\mathcal{S}^*(e_z)$ presented by Mediratta et al. [21] (also, Zhang et al. [28, 29]).

The $\mathscr{H}_l(j)$ Hankel determinant for integers $j, l \in \mathbb{N}$ was studied and investigated by Noonan and Thomas [30] in 1976, given by

$$\mathscr{H}_l(j) = \begin{vmatrix} a_j & a_{j+1} & \cdots & a_{j+l-1} \\ a_{j+1} & a_{j+2} & \cdots & a_{j+l} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j+l-1} & a_{j+l} & \cdots & a_{j+2(l-1)} \end{vmatrix}.$$

We find that

$$\mathscr{H}_2(1) = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = a_1a_3 - a_2^2 = a_3 - a_2^2, (a_1 = 1),$$

$$\mathscr{H}_2(2) = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix} = a_2a_4 - a_3^2,$$

and the third-order Henkel determinant (THD) is given as

$$\mathscr{H}_3(1) = a_3\left(a_2a_4 - a_3^2\right) - a_4(a_4 - a_2a_3) + a_5\left(a_3 - a_2^2\right).$$

Many researchers have been interested in studying this determinant; for example, the function $f(z)$ given by Equation (1), the average of growth of $\mathscr{H}_l(j)$ *as* $j \longrightarrow 0$, was determined by Noor [31]. In particular, the second- and third-order Hankel determinants have been investigated by many researchers (see [28, 32–36]).

The Toeplitz determinant for integers $j, l \in \mathbb{N}$ is defined by Ali et al. [37] given by

$$\mathscr{T}_l(j) = \begin{vmatrix} a_j & a_{j+1} & \cdots & a_{j+l-1} \\ a_{j+1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{j+l-1} & \cdots & \cdots & a_j \end{vmatrix}.$$

This means

$$\mathscr{T}_2(2) = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_2 \end{bmatrix}, \mathscr{T}_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \mathscr{T}_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

The above determinants are closely related. Henkel matrix contains constant values along the inverse diagonal, which is the opposite of the Toeplitz matrix which contains constant values along the diameter. The Toeplitz matrix has been studied in some classes by several researchers (see [29, 38–40]).

Few authors have been interested in studying the Hankel and Toeplitz determinants for the subordination class connected with the exponential function.

In this paper, we determine the upper bounds of second order of Hankel $\mathscr{H}_2(2)$ and Toeplitz $\mathscr{T}_2(2)$ determinants for the new subordination class $\mathscr{K}\mathcal{S}^*(\tau, \mu; e_q(z))$ of q-starlike functions connected with the q-exponential function.

2. Main Lemmas

Assume that $\omega(z) = 1 + \sum_{j=1}^{\infty}\omega_j z^j, z \in \mathbb{U}$ is an analytic function in $\mathbb{U}$ such that $\mathscr{R}(\omega(z)) > 0$. The class of function $\omega(z)$ is denoted by $\mathscr{P}$. To obtain our main results, we need the following lemmas:

Lemma 2.1 [41]. *If the function* $\omega(z) \in \mathscr{P}$, *then*

$$\left|\omega_j\right| \le 2, (j \ge 2).$$

Lemma 2.2. [42]. *If the function* $\omega(z) \in \mathscr{P}$, *then*

$$2\omega_2 = \omega_1^2 + \xi\left(4 - \omega_1^2\right),$$
$$4\omega_3 = \omega_1^3 + 2\left(4 - \omega_1^2\right)\omega_1\xi - \left(4 - \omega_1^2\right)\omega_1\xi^2 + 2\left(4 - \omega_1^2\right)\left(1 - |\xi|^2\right)z$$

with $|\xi| \le 1$ *and* $|z| \le 1$, *for some* ξ *and* z.

3. Main Results for Hankel and Toeplitz Determinants

To obtain the upper bound of Hankel and Toeplitz determinants for the class $\mathscr{K}\mathcal{S}^*(\tau, \mu; e_q(z))$, we must begin by solving the first-order Hankel determinant $\mathscr{H}_2(1)$.

Theorem 3.1. *If the function* $f(z) \in \mathscr{K}\mathcal{S}^*(\tau, \mu; e_q(z))$, *then*

$$\mathscr{H}_2(1) = \left|a_3 - a_2^2\right| \le \frac{1}{q(1+q)\chi_3}, \tag{10}$$

where $\chi_3 = ([1+u]_q/[3+u]_q)^{\tau}([3]_q!/[\mu+1;3]_q)$.

The equality holds and may be derived from the following extremal function:

$$f(z)=z\exp\left(\int_0^z \frac{e^\rho-1}{\rho}d_q\rho\right)=z+\frac{1}{[2]_q!}z^2+\frac{1}{[3]_q!}z^3+\cdots,(z\in\mathbb{U}). \tag{11}$$

Proof 3.1. Since $f(z)\in\mathscr{KS}^*(\tau,\mu;e_q(z))$, by the subordination condition (Equation 9), it follows that

$$\frac{z\mathfrak{D}_q\left(\mathscr{K}^\mu_{q,\tau,u}f(z)\right)}{\mathscr{K}^\mu_{q,\tau,u}f(z)}=e_q(v(z)).$$

If $\omega\in\mathscr{P}$, we define the function $\omega(z)$ as follows:

$$\omega(z)=\frac{1+v(z)}{1-v(z)}=1+\omega_1z+\omega_2z^2+\omega_3z^3+\omega_4z^4+\cdots$$

From the value of $\omega(z)$, we show that

$$v(z)=\frac{\omega(z)-1}{\omega(z)+1}=\frac{\omega_1z+\omega_2z^2+\omega_3z^3+\cdots}{2+\omega_1z+\omega_2z^2+\omega_3z^3+\cdots}. \tag{12}$$

Taking the q-exponential function $e_q(z)$ of v defined by the power series expansion, we have

$$e_q(v(z))=1+\frac{v(z)}{[1]_q!}+\frac{(v(z))^2}{[2]_q!}+\frac{(v(z))^3}{[3]_q!}+\frac{(v(z))^4}{[4]_q!}+\cdots$$

Using $v(z)$ in Equation (12), we infer that

$$\begin{aligned}e_q(v(z))&=\sum_{j=0}^{\infty}\frac{(v(z))^j}{[j]_q!}=1+\frac{1}{2}\omega_1z+\left\{\frac{\omega_2}{2}-\frac{q\omega_1^2}{4[2]_q}\right\}z^2\\&\quad+\left\{\frac{\omega_3}{2}-\frac{q\omega_1\omega_2}{2[2]_q}+\frac{q\omega_1^3}{8[3]_q[2]_q}\right\}z^3+\\&=1+\frac{1}{2}\omega_1z+\left[\frac{\omega_2}{2}-\frac{q\omega_1^2}{4(1+q)}\right]z^2\\&\quad+\left[\frac{\omega_3}{2}+\frac{q\omega_1\omega_2}{2(1+q)}+\frac{q^3\omega_1^3}{8(1+q)(1+q+q^2)}\right]z^3+\cdots\end{aligned} \tag{13}$$

Also, from the left-hand side, we get

$$\begin{aligned}\frac{z\mathfrak{D}_q\left(\mathscr{K}^\mu_{q,\tau,u}f(z)\right)}{\mathscr{K}^\mu_{q,\tau,u}f(z)}&=1+q\chi_2a_2z+q((1+q)\chi_3a_3-\chi_2^2a_2^2)z^2\\&\quad+(q(1+q+q^2)\chi_4a_4-q(2+q)\chi_2\chi_3a_2a_3+q\chi_2^3a_2^3)z^3+\cdots\end{aligned} \tag{14}$$

When the coefficients of $z^n(n=1,2,3,\cdots)$ in Equations (13) and (14) are compared, the result is

$$a_2=\frac{1}{2q\chi_2}\omega_1, \tag{15}$$

$$a_3=\frac{\omega_2}{2q(1+q)\chi_3}+\frac{1+q-q^2}{4q^2(1+q)^2\chi_3}\omega_1^2, \tag{16}$$

and

$$\begin{aligned}a_4&=\frac{\omega_3}{2q(1+q+q^2)\chi_4}+\frac{2-2q^2+q}{4q^2(1+q)(1+q+q^2)\chi_4}\omega_1\omega_2\\&\quad-\frac{(-1-2q+2q^3+3q^4-q^6)}{8q^3(1+q)^2(1+q+q^2)^2\chi_4}\omega_1^3.\end{aligned} \tag{17}$$

Putting the values of a_2 and a_3 from above in the functional $|a_3-a_2^2|$, we obtain

$$|a_3-a_2^2|=\left|\frac{\omega_2}{2q(1+q)\chi_3}+\left[\frac{1+q-q^2}{4q^2(1+q)^2\chi_3}-\frac{1}{4q^2\chi_2^2}\right]\omega_1^2\right|,$$

$$|a_3-a_2^2|=\frac{1}{2q(1+q)\chi_3}\left|\omega_2-\left(\frac{(1+q)^2\chi_3-(1+q-q^2)\chi_2^2}{2q(1+q)\chi_2^2}\right)\omega_1^2\right|.$$

Making use of Lemma 2.2 with $|\omega_1|\le 2$, we show that

$$|a_3-a_2^2|=\frac{1}{2q(1+q)\chi_3}\left|\frac{\xi(4-\omega_1^2)}{2}-\left(\frac{(1+q)^2\chi_3-(1+2q)\chi_2^2}{2q(1+q)\chi_2^2}\right)\omega_1^2\right|.$$

Since $0\le\omega\le 2$, if we take $\omega_1=\omega$ with $|\xi|=\delta$, we have

$$\begin{aligned}|a_3-a_2^2|&\le\frac{1}{2q(1+q)\chi_3}\left(\frac{\delta(4-\omega^2)}{2}+\left(\frac{(1+q)^2\chi_3-(1+2q)\chi_2^2}{2q(1+q)\chi_2^2}\right)\omega^2\right)\\&=F_q(\omega,\delta).\end{aligned} \tag{18}$$

Taking the partially differentiating of the function $F_q(\omega,\delta)$ in Equation (18) with respect to δ, we note that

$$\frac{\partial F_q(\omega,\delta)}{\partial\delta}>0.$$

This leads to the fact that the function $F_q(\omega,\delta)$ is a rising function of δ, when $\delta\in[0,1]$. That means the maximum value of $F_q(\omega,\delta)$ at $\delta=1$, then

$$\max\{F_q(\omega,\delta)\}=F_q(\omega,1)=H_q(\omega),$$

where

$$H_q(\omega) := \frac{1}{2q(1+q)\chi_3}\left(2 + \left(\frac{(1+q)^2\chi_3 - (1+3q+q^2)\chi_2^2}{2q(1+q)\chi_2^2}\right)\omega^2\right).$$

Eventually, $H_q(\omega)$ has a maximum record at $\omega = 0$; hence,

$$|a_3 - a_2^2| \le H_q(\omega) := \frac{1}{q(1+q)\chi_3}.$$

□

Corollary 3.1. *If the function $f(z) \in \mathscr{K}\mathcal{S}^*(\tau, \mu; e_q(z))$, then*

$$|a_2| \le \frac{1}{q\chi_2},$$
$$|a_3| \le \frac{1+2q}{q^2(1+q)^2\chi_3},$$

When $\tau = 0$ and $\mu = 1$, Theorem 3.1 gives the following outcome:

Corollary 3.2 [23]. *If $f(z) \in \mathcal{S}^*(e_q(z))$, then*

$$|a_3 - a_2^2| \le \frac{1}{q(1+q)}.$$

When $q \longrightarrow 1^-$, $\tau = 0$, and $\mu = 1$, it becomes the following outcomes:

Corollary 3.3 [28]. *If $f(z) \in \mathcal{S}^*(e(z))$, then $|a_2| \le 1$, $|a_3| \le 3/4$, and $|a_4| \le 17/36$.*

Corollary 3.4 [28]. *If $f(z) \in \mathcal{S}^*(e(z))$, then*

$$|a_3 - a_2^2| \le \frac{1}{2}.$$

The boundaries a_2 and a_3 are sharp and may be derived from the function provided by (see [35])

$$f_1(z) = z\ \exp\left(\int_0^z \frac{e^\rho - 1}{\rho}\, d\rho\right) = z + z^2 + \frac{3}{4}z^3 + \cdots, (z \in \mathbb{U}).$$

Theorem 3.2. *If $f(z) \in \mathscr{K}\mathcal{S}^*(\tau, \mu; e_q(z))$, then the equality bound is*

$$|a_2a_3 - a_4| \le \frac{Y_2(q) + Y_3(q)\sqrt{Y_1(q)}}{\mathcal{O}(q)},$$

where

$$\begin{aligned} Y_1(q) = {} & q(1+2q+2q^2+q^3)\big((-6-18q-9q^2+19q^3+34q^4 \\ & + 19q^5 - 8q^6 - 2q^7 + 4q^8)\chi_2^2\chi_3^2 + 3(3+9q+8q^2-8q^4-7q^5 \\ & - q^6 + q^7 + q^8)\chi_2\chi_3\chi_4 + 3(-1-q+q^2)(1+q+q^2)^2\chi_4^2\big), \end{aligned} \tag{19}$$

$$\begin{aligned} Y_2(q) = {} & q^2(1+q)^2(-1-q-q^2)(-1-4q-6q^2-5q^3-q^4)\chi_2^2\chi_3^2 \\ & + q^2(1+q)^2(-1-q-q^2) \\ & \cdot (1+3q+4q^2+5q^3+4q^4+3q^5+q^6)\chi_2\chi_3\chi_4, \end{aligned} \tag{20}$$

$$\begin{aligned} Y_3(q) = {} & (-1-4q-6q^2-5q^3-q^4)\chi_2\chi_3 \\ & + (1+3q+4q^2+5q^3+4q^4+3q^5+q^6)\chi_4, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \mathcal{O}(q) = {} & 27q^3(1+q)^2(1+q+q^2)^2 \\ & \cdot ((1+2q-2q^3-3q^4+q^6)\chi_2^2\chi_3^2\chi_4 + (-1-2q-q^2+q^4)\chi_3\chi_4\chi_4^2), \end{aligned} \tag{22}$$

with χ_j are well-determined in Equation (8) for $j = 2, 3, 4$.

Proof 3.2. From Equations (15)–(17) of Theorem 3.1 and simplification, we conclude that

$$\begin{aligned} |a_2a_3 - a_4| = {} & \left|\frac{((1+q+q^2)\chi_4 - (2-2q^2+q)\chi_2\chi_3)\omega_1\omega_2}{4q^2(1+q)(1+q+q^2)\chi_2\chi_3\chi_4} - \frac{\omega_3}{2q(1+q+q^2)\chi_4}\right. \\ & \left. + \frac{\left((1+q+q^2)^2(1+q-q^2)\chi_3 + (3q^4+2q^3-2q-q^6-1)\chi_2\chi_3\right)\omega_1^3}{8q^3(1+q)^2(1+q+q^2)2\chi_2\chi_3\chi_4}\right|. \end{aligned}$$

Making use of Lemma 2.2 upon substituting for ω_2 and ω_3, we obtain

$$\begin{aligned} |a_2a_3 - a_4| = {} & \left|\frac{(4-\omega_1^2)\omega_1\xi^2}{8q(1+q+q^2)\chi_4} - \frac{(4-\omega_1^2)(1-|\xi|^2)z}{4q(1+q+q^2)\chi_4}\right. \\ & - \frac{\varkappa_1(q)(4-\omega_1^2)\omega_1\xi}{8q^2(1+q)(1+q+q^2)\chi_2\chi_3\chi_4} \\ & \left. + \frac{\varkappa_2(q)\omega_1^3}{8q^3(1+q)^2(1+q+q^2)^2\chi_2\chi_3\chi_4}\right|, \end{aligned} \tag{23}$$

where $\varkappa_1(q) := (1+q+q^2)\chi_4 - (2+3q)\chi_2\chi_3$ and

$$\begin{aligned} \varkappa_2(q) := {} & (1+3q+4q^2+5q^3+4q^4+3q^5+q^6)\chi_4 \\ & - (1+4q+6q^2+5q^3+q^4)\chi_2\chi_3. \end{aligned}$$

Suppose that $\omega_1 = \omega \in [0, 2]$ and $|\xi| = \delta \in [0, 1]$. Then, by applying the trigonometric inequality to Equation (23) yields

$$\begin{aligned} |a_2a_3 - a_4| \le K_q(\omega, \delta) := {} & \frac{(4-\omega^2)\omega\delta^2}{8q(1+q+q^2)\chi_4} + \frac{(4-\omega^2)}{4q(1+q+q^2)\chi_4} \\ & + \frac{\varkappa_1(q)(4-\omega^2)\omega\delta}{8q^2(1+q)(1+q+q^2)\chi_2\chi_3\chi_4} \\ & + \frac{\varkappa_2(q)\omega_1^3}{8q^3(1+q)^2(1+q+q^2)^2\chi_2\chi_3\chi_4}. \end{aligned}$$

Now, by taking the partially differentiating of the function $K_q(\omega,\delta)$ with respect to δ, we obtain

$$\frac{\partial K_q(\omega,\delta)}{\partial\delta} := \frac{(4-\omega^2)\omega\delta}{4q(1+q+q^2)\chi_4} + \frac{\varkappa_1(q)(4-\omega^2)\omega}{8q^2(1+q)(1+q+q^2)\chi_2\chi_3\chi_4} > 0.$$

This leads to the fact that the function $K_q(\omega,\delta)$ is a rising function of δ, when $\delta\in[0,1]$. That means the maximum value of $K_q(\omega,\delta)$ at $\delta=1$, then

$$K_q(\omega,\delta)\le K_q(\omega,1), (0\le\delta\le1),$$

that is,

$$\begin{aligned}K_q(\omega,1) := {}& \frac{(4-\omega^2)\omega}{8q(1+q+q^2)\chi_4} + \frac{(4-\omega^2)}{4q(1+q+q^2)\chi_4}\\ &+ \frac{\varkappa_1(q)(4-\omega^2)\omega}{8q^2(1+q)(1+q+q^2)\chi_2\chi_3\chi_4}\\ &+ \frac{\varkappa_2(q)\omega^3}{8q^3(1+q)^2(1+q+q^2)^2\chi_2\chi_3\chi_4}.\end{aligned}$$

Upon setting $K_q(\omega,1)=Q_q(\omega)$, we conclude that

$$\begin{aligned}Q_q'(\omega) := {}& \frac{\varkappa_3(q)(4-\omega^2)}{8q^2(1+q)(1+q+q^2)\chi_2\chi_3\chi_4}\\ &- \frac{\varkappa_3(q)\omega^2}{4q^2(1+q)(1+q+q^2)\chi_2\chi_3\chi_4}\\ &- \frac{\omega}{2q(1+q+q^2)\chi_4} + \frac{3\varkappa_2(q)\omega^2}{8q^3(1+q)^2(1+q+q^2)^2\chi_2\chi_3\chi_4},\end{aligned}$$

where $\varkappa_3(q) := (1+q+q^2)\chi_4 + (-2-2q+q^2)\chi_2\chi_3$.

If we take $Q_q'(\omega)=0$, then, we obtain the root ω as follows:

$$\omega = r = \frac{2\left(-q^2(1+q)^2(1+q+q^2)\chi_2\chi_3 + \sqrt{Y_1(q)}\right)}{3((1+2q-2q^3-3q^4+q^6)\chi_2\chi_3 + (-1-2q-q^2+q^4)\chi_4)},$$

where $Y_1(q)$ is defined by Equation (19).

Then, the function $Q_q(\omega)$ at the above value of ω that means the maximum value of $Q_q(\omega)$ is given by

$$|a_2a_3-a_4| \le Q_q(\omega) := \frac{Y_2(q)+Y_3(q)\sqrt{Y_1(q)}}{\mathcal{O}(q)},$$

with $Y_2(q)$, $Y_3(q)$, and $\mathcal{O}(q)$ defined by Equations (20), (21), and (22), respectively. Here completes the proof of Theorem 3.2. □

If $\tau=0$ and $\mu=1$, we obtain the result as follows:

Corollary 3.5 [23]. *If $f(z)\in\mathcal{S}^*(e_q(z))$, then*

$$|a_2a_3-a_4| \le \frac{\psi_{1,q}+\psi_{2,q}+192\sqrt{\psi_{3,q}}}{27(1+2q+2q^2+q^3)(1+3q+2q^2+q^3)^2},$$

where the functions $\psi_{1,q}, \psi_{2,q}$, and $\psi_{3,q}$ are well-defined in [23].

If $q\longrightarrow1^-$, $\tau=0$, and $\mu=1$, we obtain the following corollary:

Corollary 3.6 [28]. *If $f(z)\in\mathcal{S}^*(e(z))$, then*

$$|a_2a_3-a_4| \le \frac{896\sqrt{2}+385}{3087}.$$

The equality holds.

Theorem 3.3. *Let $f(z)\in\mathcal{KS}^*(\tau,\mu;e_q(z))$, where $f(z)$ is in the form (Equation 1). Then,*

$$\mathcal{H}_2(2) = |a_2a_4-a_3^2| \le \frac{(1+q+q^2)\chi_2\chi_4 + (1+q)^2\chi_3^2}{q^2(1+q)^2(1+q+q^2)\chi_2\chi_3^2\chi_4}.$$

The value is sharp.

Proof 3.3. From the values a_2, a_3, and a_4 in Theorem 3.1, we get

$$\begin{aligned}|a_2a_4-a_3^2| = {}& \left|\frac{\omega_1\omega_3}{4q^2(1+q+q^2)\chi_2\chi_4}\right.\\ &+ \left(\frac{2-2q^2+q}{8q^3(1+q)(1+q+q^2)\chi_2\chi_4} - \frac{1+q-q^2}{4q^3(1+q)^3\chi_3^2}\right)\omega_1^2\omega_2\\ &- \left(\frac{1}{4q^2(1+q)^2\chi_3^2}\right)\omega_2^2 - \left(\frac{(-1-2q+2q^3+3q^4-q^6)}{16q^4(1+q)^2(1+q+q^2)^2\chi_2\chi_4}\right.\\ &\left.\left.+ \frac{(1+q-q^2)^2}{16q^4(1+q)^4\chi_3^2}\right)\omega_1^4\right|.\end{aligned}$$

By Lemma 2.2, with $|\omega_1|\le2$, it follows that

$$\begin{aligned}|a_2a_4-a_3^2| = {}& \left|\frac{\varphi_{1,q}(4-\omega_1^2)\omega_1^2\xi}{16q^3(1+q)^3(1+q+q^2)\chi_2\chi_3^2\chi_4} - \frac{(4-\omega_1^2)^2\xi^2}{16q^2(1+q)^2\chi_3^2}\right.\\ &+ \frac{\omega_1(4-\omega_1^2)\left(1-|\xi|^2\right)z}{8q^2(1+q+q^2)\chi_2\chi_4} - \frac{(4-\omega_1^2)\omega_1^2\xi^2}{16q^2(1+q+q^2)\chi_2\chi_4}\\ &\left.- \frac{\varphi_{2,q}\omega_1^4}{16q^4(1+q)^4(1+q+q^2)^2\chi_2\chi_3^2\chi_4}\right|,\end{aligned}$$

where $\varphi_{1,q} := (1+q)^2(2+3q)\chi_3^2 - 2(1+3q+3q^2+2q^3)\chi_2\chi_4$ and

$$\varphi_{2,q} := \left|(1+q)^2(1+4q+6q^2+5q^3+q^4)\chi_3^2 - (1+3q+3q^2+2q^3)^2\chi_2\chi_4\right|.$$

If we take $\omega_1 = \omega$, with $\omega \in [0, 2]$ and using the triangle inequality with $|\xi| = \delta$, we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \le L_q(\delta) &:= \frac{\varphi_{1,q}\left(4-\omega^2\right)\omega^2\delta}{16q^3(1+q)^3(1+q+q^2)\chi_2\chi_3^2\chi_4} \\ &+ \frac{\left(4-\omega^2\right)^2\delta^2}{16q^2(1+q)^2\chi_3^2} + \frac{\left(4-\omega^2\right)}{4q^2(1+q+q^2)\chi_2\chi_4} \\ &+ \frac{\left(4-\omega^2\right)\omega^2\delta^2}{16q^2(1+q+q^2)\chi_2\chi_4} \\ &+ \frac{\varphi_{2,q}\omega^4}{16q^4(1+q)^4(1+q+q^2)^2\chi_2\chi_3^2\chi_4}. \end{aligned} \tag{24}$$

Now, by the partially differentiating of the function $L_q(\delta)$ in Equation (24) with respect to δ, then

$$\begin{aligned} \frac{\partial L_q(\delta)}{\partial\delta} &= \frac{\varphi_{1,q}\left(4-\omega^2\right)\omega^2}{16q^3(1+q)^3(1+q+q^2)\chi_2\chi_3^2\chi_4} + \frac{\left(4-\omega^2\right)^2\delta}{8q^2(1+q)^2\chi_3^2} \\ &+ \frac{\left(4-\omega^2\right)\omega^2\delta}{8q^2(1+q+q^2)\chi_2\chi_4} > 0. \end{aligned}$$

This leads to the fact that the function $L_q(\delta)$ is a rising function of $\delta (\delta \in [0, 1])$. Then, the maximum value of function $L_q(\delta)$ occurs at $\delta = 1$, which yields

$$\max\left\{L_q(\delta)\right\} = L_q(1) = B_q(\omega),$$

where

$$\begin{aligned} B_q(\omega) &:= \frac{\varphi_{3,q}\left(4-\omega^2\right)\omega^2}{16q^3(1+q)^3(1+q+q^2)\chi_2\chi_3^2\chi_4} \\ &+ \frac{\left(4-\omega^2\right)^2}{16q^2(1+q)^2\chi_3^2} + \frac{\left(4-\omega^2\right)}{4q^2(1+q+q^2)\chi_2\chi_4} \\ &+ \frac{\varphi_{2,q}\omega^4}{16q^4(1+q)^4(1+q+q^2)^2\chi_2\chi_3^2\chi_4}, \end{aligned}$$

and $\varphi_{3,q} := (1+q)^2(2+4q+q^2)\chi_3^2 - 2(1+3q+3q^2+2q^3)\chi_2\chi_4$.

Also, the differentiating of the function $B_q(\omega)$ with respect to ω yields

$$\begin{aligned} B_q'(\omega) &= \frac{\varphi_{3,q}\left(4-\omega^2\right)\omega}{8q^3(1+q)^3(1+q+q^2)\chi_2\chi_3^2\chi_4} \\ &- \frac{\varphi_{3,q}\omega^3}{8q^3(1+q)^3(1+q+q^2)\chi_2\chi_3^2\chi_4} - \frac{\left(4-\omega^2\right)\omega}{4q^2(1+q)^2\chi_3^2} \\ &- \frac{\omega}{2q^2(1+q+q^2)\chi_2\chi_4} + \frac{\varphi_{2,q}\omega^3}{4q^4(1+q)^4(1+q+q^2)^2\chi_2\chi_3^2\chi_4}. \end{aligned}$$

If we let $B_q'(\omega)$, hence, $\omega = 0$ is a root of this equation. By differentiating the equation $B_q'(\omega)$ with respect to ω, the result is

$$B_q''(\omega) \le 0.$$

That means the maximum value of $B_q(\omega)$ at $\omega = 0$; therefore,

$$\left|a_2a_4 - a_3^2\right| \le B_q(0) := \frac{(1+q+q^2)\chi_2\chi_4 + (1+q)^2\chi_3^2}{q^2(1+q)^2(1+q+q^2)\chi_2\chi_3^2\chi_4}.$$

□

If $\tau = 0$ and $\mu = 1$, we find the following outcome:

Corollary 3.7 [23]. *Let $f(z) \in \mathcal{S}^*(e_q(z))$. Then,*

$$\left|a_2a_4 - a_3^2\right| \le \frac{(q+3q^2+q^4)+2}{q^2(1+q)^2(1+q+q^2)}.$$

When $\tau = 0, q \longrightarrow 1^-$, and $\mu = 1$, we deduce the following outcome:

Corollary 3.8 [28]. *Suppose that $f(z) \in \mathcal{S}^*(e(z))$, then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{7}{12}.$$

The equality holds. In the next outcomes, we obtain some estimates for the Toeplitz determinants $\mathcal{T}_l(j)$ for the values of l and j, when f is q-starlike related to the q-exponential function.

Theorem 3.4. *If $f(z) \in \mathcal{K}\mathcal{S}^*(\tau, \mu; e_q(z))$, then*

$$\mathcal{T}_2(2) = \left|a_3^2 - a_2^2\right| \le \frac{1}{q^2(1+q)^2\chi_3^2}\left(\frac{(1+2q)^2}{q^2(1+q)^2} + \frac{(1+q)^2\chi_3^2}{\chi_2^2}\right). \tag{25}$$

Proof 3.4. From the values a_2 and a_3 in Theorem 3.1, we have

$$\left|a_3^2 - a_2^2\right| = \left|\left(\frac{\omega_2}{2q(1+q)\chi_3} + \frac{1+q-q^2}{4q^2(1+q)^2\chi_3}\omega_1^2\right)^2 - \frac{1}{4q^2\chi_2^2}\omega_1^2\right|,$$

$$\begin{aligned} \left|a_3^2 - a_2^2\right| &= \frac{1}{4q^2(1+q)^2\chi_3^2}\left|\omega_2^2 + \frac{\left(1+q-q^2\right)^2}{4q^2(1+q)^2}\omega_1^4\right. \\ &\left.+ \left(\frac{(1+q-q^2)\chi_2^2\omega_2 - q(1+q)^3\chi_3^2}{q(1+q)\chi_2^2}\right)\omega_1^2\right|. \end{aligned}$$

By Lemma 2.2 with $|\omega_1| \le 2$, we show that

$$\left|a_3^2 - a_2^2\right| = \frac{1}{4q^2(1+q)^2\chi_3^2}\left|\frac{1}{4}\xi^2X^2 + \frac{(1+2q)}{4q(1+q)}\omega_1^2\xi X + \frac{(1+2q)^2}{4q^2(1+q)^2}\omega_1^4 - \frac{(1+q)^2\chi_3^2}{\chi_2^2}\omega_1^2\right|,$$

where $X = (4-\omega_1^2)$.

Since $0 \le \omega \le 2$, if we let $\omega_1 = \omega$ and the moduli $|\xi| = \delta$, then by the triangle inequality, we find that

$$\left|a_3^2 - a_2^2\right| \le \phi_q(\delta) := \frac{1}{4q^2(1+q)^2\chi_3^2} \cdot \left(\left|\frac{(1+2q)^2}{4q^2(1+q)^2}\omega^4 - \frac{(1+q)^2\chi_3^2}{\chi_2^2}\omega^2\right| + \frac{1}{4}\delta^2X^2 + \frac{(1+2q)}{2q(1+q)}\omega^2\delta X\right), \tag{26}$$

with $X = 4-\omega^2$.

Now, if we take the partially differentiating of $\phi_q(\delta)$ with respect to δ when $\delta \in [0,1]$, we obtain

$$\frac{\partial\phi_q}{\partial\delta} = \frac{1}{4q^2(1+q)^2\chi_3^2}\left(\frac{1}{2}\delta X^2 + \frac{(1+2q)}{2q(1+q)}\omega^2 X\right) > 0.$$

This leads to the truth that the function $\phi_q(\delta)$ increases, and the maximum value at $\delta = 1$

$$\max\left\{\phi_q(\delta)\right\} = \phi_q(1),$$

where

$$\left|a_3^2 - a_2^2\right| \le \frac{1}{4q^2(1+q)^2\chi_3^2} \cdot \left(\left|\frac{(1+2q)^2}{4q^2(1+q)^2}\omega^4 - \frac{(1+q)^2\chi_3^2}{\chi_2^2}\omega^2\right| + \frac{1}{4}X^2 + \frac{(1+2q)}{2q(1+q)}\omega^2 X\right).$$

Since $0 \le \omega \le 2$, the maximum point is $\omega = 2$.

In short proof, we have

$$\left|a_3^2 - a_2^2\right| \le \left|a_3^2\right| + \left|a_2^2\right| \le \frac{1}{q^2(1+q)^2\chi_3^2}\left(\frac{(1+2q)^2}{q^2(1+q)^2} + \frac{(1+q)^2\chi_3^2}{\chi_2^2}\right).$$

Then, we get the intended result. □

If $q \longrightarrow 1^-, \tau = 0$, and $\mu = 1$, it generates the following result:

Corollary 3.9 [29]. *If $f(z) \in \mathcal{KS}^*(e(z))$, then*

$$\mathcal{T}_2(2) = \left|a_3^2 - a_2^2\right| \le \frac{25}{16}.$$

Theorem 3.5. *If the function $f(z) \in \mathcal{KS}^*(\tau, \mu; e_q(z))$, then*

$$\mathcal{T}_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix} \le 1 + \Psi(q) + \frac{2}{q^2\chi_2^2},$$

where $\Psi(q) = (1+2q)(2(1+q)^2\chi_3 - (1+2q)\chi_2^2)/q^4(1+q)^4 \chi_2^2\chi_3^2$.

Proof 3.5. Making use of the values a_2 and a_3 in Theorem 3.1, we obtain

$$\begin{aligned}\mathcal{T}_3(1) &= \left|1 + 2a_2^2(a_3-1) - a_3^2\right| \\ &= \left|1 + \Psi_{1,q}\omega_1^2\omega_2 + \Psi_{2,q}\omega_1^4 - \frac{1}{2q^2\chi_2^2}\omega_1^2 - \frac{1}{4q^2(1+q)^2\chi_3^2}\omega_2^2\right|,\end{aligned}$$

where

$$\Psi_{1,q} = \frac{(1+q)^2\chi_3 - \left(1+q-q^2\right)\chi_2^2}{4q^3(1+q)^3\chi_2^2\chi_3^2},$$

$$\Psi_{2,q} = \frac{2(1+q)^2\left(1+q-q^2\right)\chi_3 - \left(1+q-q^2\right)^2\chi_2^2}{16q^4(1+q)^4\chi_2^2\chi_3^2}.$$

By Lemma 2.2, we obtain

$$\mathcal{T}_3(1) = \left|1 + \Psi_{4,q}\omega_1^4 - \frac{1}{2q^2\chi_2^2}\omega_1^2 - \frac{1}{16q^2(1+q)^2\chi_3^2}\xi^2X^2 + \Psi_{3,q}\omega_1^2\xi X\right|,$$

where $X = (4-\omega_1^2)$ with

$$\Psi_{3,q} = \frac{(1+q)^2\chi_3 - (1+2q)\chi_2^2}{8q^3(1+q)^3\chi_2^2\chi_3^2},$$

$$\Psi_{4,q} = \frac{(1+2q)\left(2(1+q)^2\chi_3 - (1+2q)\chi_2^2\right)}{16q^4(1+q)^4\chi_2^2\chi_3^2}.$$

If we assume that $\omega_1 = \omega (0 \le \omega \le 2)$ and the moduli $|\xi| = \delta$, then by the triangle inequality with the fact $|\xi| \le 1$, we conclude that

$$\mathcal{T}_3(1) \le \Phi_q(\delta) = \left|1 + \Psi_{4,q}\omega^4 - \frac{1}{2q^2\chi_2^2}\omega^2\right| + \frac{1}{16q^2(1+q)^2\chi_3^2}\delta^2X^2 + \Psi_{3,q}\omega^2\delta X,$$

where $X = (4-\omega^2)$.

Now, if we take the partially differentiating of $\Phi_q(\delta)$ with respect to δ, we obtain

$$\frac{\partial\Phi_q}{\partial\delta} > 0.$$

This leads to the truth that the function $\Phi_q(\delta)$ increases, and the maximum value at $\delta = 1$

$$\max\left\{\Phi_q(\delta)\right\} = \Phi_q(1),$$

where

$$\mathscr{T}_3(1) \le \left|1 + \Psi_{4,q}\omega^4 - \frac{1}{2q^2\chi_2^2}\omega^2\right| + \frac{1}{16q^2(1+q)^2\chi_3^2}X^2 + \Psi_{3,q}\omega^2 X.$$

Since $0 \le \omega \le 2$, this means that the maximum point is $\omega = 2$.

The determinant can be calculated directly by

$$\mathscr{T}_3(1) = \left|1 + 2a_2^2(a_3 - 1)z - a_3^2\right| \le 1 + 2\left|a_2^2\right| + |a_3|\left|a_3 - 2a_2^2\right|.$$

Then, we find the intended result. □

4. Substantive Discussion

This part is an introduction aimed at making an analytical comparison between the existing classes and the proposed new class. Srivastave et al. [23] developed a family of starlike functions that include the q-exponential function. In a parallel vein, Khan and colleagues [43] introduced a distinctive class of q-convex functions, which are related to the q-exponential function. Inequalities of the coefficients of q-analytic functions are of great importance in various areas of mathematics. They shed light on the properties of these functions when compared to the q-analysis of classical functions, such as the q-exponential function. This analysis contributes to a deeper theoretical understanding of q-analytical functions. To compare our results, we can take the following function:

$$f(z) = e_q(z) - 1 = \sum_{j=0}^{\infty} \frac{z^j}{[j]_q!} \in \mathscr{K}\mathscr{S}^*\left(\tau, \mu\,; e_q(z)\right).$$

5. Conclusion

Using the QC so-called q-calculus, we studied and investigated the upper bounds of Hankel and Toeplitz determinants for the subordination class of q-starlike functions $\mathscr{K}\mathscr{S}^*(\tau, \mu\,; e_q(z))$, which is created by a q-analog integral operator $\mathscr{K}^{\mu}_{q,\tau,u}f(z)$ connected with the q-exponential function. For this class of q-starlike functions, we have derived and determined the second order of Hankel and Toeplitz determinants. Theorems 3.1 to 3.5 in this article explained and proved our main findings. The purpose of these results is to determine special cases and consequences, some of which are discussed in this study. This investigation will help direct future research and shed light on new ideas in the field of geometric function theory, especially those mostly concerned with Hankel and Toeplitz determinants, as well as the subordination and superordination properties. Moreover, the subordination classes of symmetric analytic functions employing the operator (Equation 8) will be an open problem for investigation.

Data Availability Statement

The authors have nothing to report.

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The authors have nothing to report.

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The authors have nothing to report.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

Conceptualization: S.H.H. and M.D.; methodology: S.H.H., M.D., and R.W.I.; software: R.W.I.; validation: S.H.H., M.D., and R.W.I.; formal analysis: S.H.H. and R.W.I; investigation: S.H.H.; resources: S.H.H. and M.D.; writing—original draft preparation: S.H.H.; writing—review and editing: M.D. and R.W.I.; visualization: S.H.H. and R.W.I.; supervision: M.D.; project administration: M.D. All authors have read and agreed to the published version of the manuscript.

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