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QUAISI INVARIANT CONHARMONIC TENSOR OF SPECIAL CLASSES OF LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLD

The authors classified a locally conformal almost cosymplectic manifold ($\mathcal{LCAC_S}$-manifold) according to the conharmonic curvature tensor. In particular, they have determined the necessary conditions for a conharmonic curvature tensor on the $\mathcal{LCAC_S}$-manifold of classes $CT_i, i = 1, 2, 3$ to be Φ-quaisi invariant. Moreover, it has been proved that any $\mathcal{LCAC_S}$-manifold of the class CT_1 is conharmonicly Φ-paracontact.

Keywords: locally conformal almost cosymplectic manifold, conharmonic curvature tensor, Φ-quaisi invariant, conharmonicly Φ-paracontact.

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Introduction

The first classification of an almost contact metric structure has been introduced by Chinea and Marrero [9]. Like an almost Hermitian structure, another principle of classification of an almost contact metric structure for differential-geometric invariants of the second order depends on the properties of the Riemannian curvature tensor R. Volkova introduced analogues of these classes in contact geometry CR_1, CR_2 and CR_3 [19].

Our study focuses on the conharmonic tensor of the locally conformal almost cosymplectic manifold of three spacial classes CT_i, $i = 1, 2, 3$ which are related with the classes CR_1, CR_2, CR_3.

A harmonic function is a function whose Laplacian vanishes. It is known that the conformal transformation on the Riemannian manifold preserves the angle between two vectors. Generally, a harmonic function is not invariant. The condition to remain such a function invariant has been studied by Ishi [13]. Specifically, he introduced a conharmonic transformation that preserves the harmonicity of a certain function. On the other hand, Ghosh et al. [11] studied the $N(k)$-contact metric manifolds satisfying curvature conditions on the conharmonic tensor. Dwivedi and Kim [10] obtained certain necessary and sufficient conditions for the K-contact and Sasakian manifolds to be quasi conharmonically flat, ξ-conharmonically flat and Φ-conharmonically flat. Furthermore, Asghari and Taleshian [4] studied the conharmonic curvature tensor on the Kenmotsu manifold. Chanyal and Uperti [8] proved that Φ-conharmonically flat (k, μ)-contact manifolds are η-Einstein manifolds. Taleshian et al. [18] considered LP-Sasakian manifolds admitting a conharmonic curvature tensor. Abood and Al-Hussaini [3] studied the geometrical properties of the conharmonic tensor of a locally conformal almost cosymplectic manifold. In particular, the authors established the necessary and sufficient conditions for the conharmonic tensor to be flat, the aforementioned manifold to be normal and an η-Einstein manifold.

§ 1. Preliminaries

The main purpose of this section is to construct an almost contact metric structure and a locally conformal almost cosymplectic manifold in the adjoined G-structure space.

Definition 1 (see [5]). Let M be $2n+1$ dimensional smooth manifold, η be differential 1-form called a *contact form*, ξ be a vector field called *a characteristic*, Φ be an endomorphism of the module of the vector fields $X(M)$ called a *structure endomorphisim*. The family of tensors $\{\eta, \xi, \Phi\}$ is called *an almost contact structure* if the following conditions hold

1. $\eta(\xi) = 1$;
2. $\Phi(\xi) = 0$;
3. $\eta \circ \Phi = 0$;
4. $\Phi^2 = -id + \eta \otimes \xi$.

In addition, if there is a Riemannian metric $g = \langle .,. \rangle$ on M such that

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \;\; X, Y \in X(M),$$

then the family of tensors $\{\eta, \xi, \Phi, g\}$ is called an almost contact metric structure. In this case, the mentioned manifold M endowed with this structure is called an almost contact metric manifold.

Definition 2 (see [16]). At each point $p \in M^{2n+1}$, there is a frame in a complexification of a tangent space $T_p^c(M)$ of the form $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\pi(e_a)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\pi}(e_a)$, $\hat{a} = a + n$, $\varepsilon_0 = \xi_p$, $\pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$ and $\bar{\pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$. The frame $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ is called an A-frame.

A set of such frames defines G-structure on M with a structure group $1 \times U(n)$. This structure is called a G-adjoined structure.

Lemma 1 (see [15]). *The component matrices of the tensors Φ_p and g_p in A-frame have the following froms respectively:*

$$(\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \qquad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n.

Definition 3 (see [5]). A skew-symmetric tensor $\Omega(X, Y) = g(X, \Phi Y)$ is called a fundamental form of the $\mathcal{AC}$-structure.

Definition 4 (see [12]). An almost contact metric structure $S = (\eta, \xi, \Phi, g)$ is called an almost cosymplectic structure ($\mathcal{AC}_f$-structure) if

1. $d\eta = 0$;
2. $d\Omega = 0$.

Definition 5 (see [17]). A conformal transformation of an $\mathcal{AC}$-structure $S = (\eta, \xi, \Phi, g)$ on a manifold M is a mapping from S to an $\mathcal{AC}$-structure $\widetilde{S} = (\widetilde{\eta}, \widetilde{\xi}, \widetilde{\Phi}, \widetilde{g})$ such that

$$\widetilde{\eta} = e^{-\sigma}\eta, \qquad \widetilde{\xi} = e^{\sigma}\xi, \qquad \widetilde{\Phi} = \Phi, \qquad \widetilde{g} = e^{-2\sigma}g,$$

where σ is the determining function of the conformal transformation. If $\sigma = \text{const}$, then the conformal transformation is said to be trivial.

Definition 6 (see [17]). An $\mathcal{AC}$-structure S on a manifold M is said to be a locally conformal almost cosymplectic ($\mathcal{LCAC_S}$-structure) if the restraction of S on a some neighborhood U of an arbitrary point $p \in M$, admits a conformal transformation of an almost cosymplectic structure. This transformation is called a locally conformal. A manifold M equipped with a $\mathcal{LCAC_S}$-structure is called a $\mathcal{LCAC_S}$-manifold.

Lemma 2 (see [14]). *In the adjoined G-structure space, the structure equations of $\mathcal{LCAC_S}$-manifold have the following forms:*

1. $d\omega^a = -\omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^a_b \omega \wedge \omega^b + B^{ab}\omega \wedge \omega_b$;

2. $d\omega_a = \omega^b_a \wedge \omega_b + B^c_{ab}\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B^b_a \omega \wedge \omega_b + B_{ab}\omega \wedge \omega^b$;

3. $d\omega = C_b \omega \wedge \omega^b + C^b \omega \wedge \omega_b$;

where $A^{ac}_{[bd]} = -2\delta^{[c}_{[b}\sigma^{a]}_{d]} + 2\sigma^{[a}\delta^{e]}_{b}\sigma_{[e}\delta^{c}_{d]} + \dfrac{1}{2}B^{dca}B_{ebd}$*, Here* B^{abc}*,* B_{abc}*;* B^{ab}*,* B_{ab}*;* B^a_b*,* B^b_a*;* C^{ab}*,* C_{ab}*;* C^b*,* C_b*;* A^{acd}_b*,* A^b_{acd}*;* A^{ac}_{bd}*;* A^{ac0}_b*,* A^b_{ac0}*;* B^{abci}*,* B_{abci}*;* D^{abi}*,* D_{abi} *and* σ_{ij} *are smooth functions in the adjoined G-structure space. The tensors* B^{abc} *;* B^{ab} *are called the second and third structure tensors respectively.*

Definition 7 (see [7]). A Ricci tensor is a tensor of type $(2,0)$ which is a contracting of the Riemannian curvature tensor and defined by

$$r_{ij} = -R^k_{ijk}.$$

Lemma 3 (see [1]). *In the adjoined G-structure space, the components of the Ricci tensor of* $\mathcal{LCAC_S}$*-manifold are given by the following forms:*

1. $r_{ab} = 2(-2A^c_{(ab)c} - 4(\sigma^{[c}\delta^{h]}_{[b}B_{c]ha} + \sigma^{[c}\delta^{h]}_{[a}B_{c]hb}) + \sigma_0 B_{a[c}\delta^c_{b]} + \sigma_0 B_{b[c}\delta^c_{a]} + 2\sigma_0 B_{ab} - D_{ab0} - \sigma_{ab} - \sigma_a\sigma_b + 2B_{bah}\sigma^h$;

2. $r_{\hat{a}b} = -4(\delta^{[a}_{[b}\sigma^{c]}_{c]} - \sigma_{[c}\delta^{b}_{h]}\sigma^{[h}\delta^{a]}_{c} - \dfrac{1}{2}\sigma^{[a}\delta^{h]}_{b}\sigma_h + B^{hca}B_{hcb} + B^{bch}B_{cha}) + (B^{cb}B_{ac} - B_{hb}B^{ah}) + A^{cb}_{ac} - \delta^a_b\sigma_{00} - 2n\sigma_0^2 - \sigma^a_b - \sigma^a\sigma_b$;

3. $r_{a0} = -A^c_{ac0} - \sigma^c B_{ac} + n\sigma_0\sigma_a + 2(\sigma_{0[c}\delta^c_{a]} + B^{cb}B_{bca} - 2\sigma^{[c}\delta^{h]}_{[c}B_{a]h})$;

4. $r_{oo} = -2n(\sigma_{00} + \sigma_0^2) - 2B_{hc}B^{ch} - 2(\sigma^c_c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta^{h]}_{c}\sigma_h$.

The remaining components are conjugate to the above-mentioned components.

Definition 8 (see [2]). A $\mathcal{LCAC_S}$-manifold is said to have Φ-invariant Ricci tensor, if $\Phi \circ r = r \circ \Phi$.

Lemma 4 (see [2]). *A* $\mathcal{LCAC_S}$*-manifold has* Φ*-invariant Ricci tensor if and only if, in the adjoined G-structure space, the following condition*

$$r^{\hat{a}}_b = r_{ab} = r^{\hat{a}}_0 = r_{a0} = 0$$

holds.

Definition 9 (see [6]). A pseudo-Riemannian manifold M is known as an η-Einstein of type (α, β) if its Ricci tensor satisfies the following condition:

$$r = \alpha g + \beta \eta \otimes \eta,$$

where α and β are suitable smooth functions. If $\beta = 0$, then M is referred to as an Einstein manifold.

Definition 10 (see [12]). Let M be an $\mathcal{AC}$-manifold of dimension $2n+1$. A tensor T of type $(4,0)$ which is invariant under conharmonic transformation and defined by the form

$$T_{ijkl} = R_{ijkl} - \frac{1}{2n-1}(r_{il}g_{jk} - r_{jl}g_{ik} + r_{jk}g_{il} - r_{ik}g_{jl})$$

is called a conharmonic curvature tensor.

In the work [1], we have calculated all possible components of the mentioned tensor which are listed in the next lemma.

Lemma 5. *In the adjoined G-structure space, the components of the conharmonic curvature tensor of $\mathcal{LCAC_S}$-manifold are given by the following forms:*

1. $T_{abcd} = 2(2B_{[c|ab|d]} - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b})$;
2. $T_{\hat{a}bcd} = 2(A^a_{bcd} + 4\sigma^{[a}\delta^{h]}_{[c}B_{d]hb} - \sigma_0 B_{b[d}\delta^a_{c]}) - \frac{1}{2n-1}(r_{bc}\delta^a_d - r_{bd}\delta^a_c)$;
3. $T_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^a_c\delta^d_b\sigma^2_0 + \frac{1}{2n-1}(r^d_b\delta^a_c + r^a_c\delta^d_b)$;
4. $T_{\hat{a}\hat{b}cd} = 2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} + 2B^{hab}B_{hdc} - \delta^a_{[c}\delta^b_{d]}\sigma^2_0) - \frac{4}{2n-1}(r^{[d}_{[a}\delta^{c]}_{b]})$;
5. $T_{\hat{a}0cd} = 2(\sigma_{0[c}\delta^a_{d]} + B^{ab}B_{bcd} - 2\sigma^{[a}\delta^{h]}_{[c}B_{d]h}) + \frac{1}{2n-1}(r_{0d}\delta^a_c - r_{0c}\delta^a_d)$;
6. $T_{\hat{a}b\hat{c}0} = A^{ac0}_b + \sigma_b B^{ac} - \delta^c_b\sigma_0\sigma^a - \frac{1}{2n-1}(r^a_0\delta^c_b)$;
7. $T_{abc0} = 2B_{cab0} + 2B_{cab}\sigma_0$;
8. $T_{\hat{a}0b0} = -\delta^a_b\sigma_{00} - \delta^a_b\sigma^2_0 - B_{cb}B^{ac} - \sigma^a_b - \sigma^a\sigma_b + 2\sigma^{[a}\delta^{c]}_b\sigma_c + \frac{1}{2n-1}(r_{00}\delta^a_b + r^a_b)$;
9. $T_{\hat{a}0\hat{b}0} = 2\sigma_0 B^{ab} - D^{ab0} - \sigma^{ab} - \sigma^a\sigma^b + 2B^{bac}\sigma_c + \frac{1}{2n-1}(r_{\hat{a}\hat{b}})$.

The remaining components are conjugate to the above-mentioned components or can be obtained by the property of symmetry for T or identically equal to zero.

The next lemma gives analogues to the Gray's identities in the adjoined G-structure space.

Lemma 6 (see [19]). *An $\mathcal{AC}$-manifold is called a manifold of class*

1. CR_1 *if and only if,* $R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0$;
2. CR_2 *if and only if,* $R_{abcd} = R_{\hat{a}bcd} = 0$;
3. CR_3 *if and only if,* $R_{\hat{a}bcd} = 0$.

§ 2. The main results

In this section, we introduce analogues to the Gray's identities for the conharmonic curvature tensor of a $\mathcal{LCAC_S}$-manifold.

Definition 11. In the adjoined G-structure space, a $\mathcal{LCAC_S}$-manifold is called a manifold of class

1. CT_1 if and only if, $T_{abcd} = T_{\hat{a}bcd} = T_{\hat{a}\hat{b}cd} = 0$;
2. CT_2 if and only if, $T_{abcd} = T_{\hat{a}bcd} = 0$;

3. CT_3 if and only if, $T_{\hat{a}bcd} = 0$.

Definition 12. The conharmonic tensor is called Φ-quasi invariant , if

$$\begin{aligned}\langle T(\Phi X, \Phi Y)\Phi X, \Phi Y\rangle - \langle T(\Phi^2 X, \Phi^2 Y)\Phi^2 X, \Phi^2 Y\rangle = \langle T(\Phi^2 X, \Phi Y)\Phi^2 X, \Phi Y\rangle \\ - \langle T(\Phi X, \Phi^2 Y)\Phi X, \Phi^2 Y\rangle\end{aligned}$$

holds for each $X, Y, Z \in X(M)$.

Definition 13. A $\mathcal{LCAC_S}$-manifold is called a conharmoniclly Φ-paracontact, if its conharmonic tensor satisfies the identity

$$\begin{aligned}\langle T(\Phi^2 X, \Phi^2 Y)\Phi^2 Z, \Phi^2 W\rangle =& \langle T(\Phi^2 X, \Phi^2 Y)\Phi Z, \Phi W\rangle - \langle T(\Phi X, \Phi^2 Y)\Phi Z, \Phi^2 W\rangle \\ &- \langle T(\Phi X, \Phi^2 Y)\Phi^2 Z, \Phi W\rangle,\end{aligned}$$

where $X, Y, \in X(M)$.

Theorem 1. *A conharmonic tensor on $\mathcal{LCAC_S}$-manifold of class CT_i, $i = 1, 2, 3$, is Φ-quasi invariant.*

P r o o f. Consider the component $T_{\hat{a}bab} = 0$. Write this equation in A-frame, we have

$$\langle T(\varepsilon_{\hat{a}}, \varepsilon_b)\varepsilon_a, \varepsilon_b\rangle = 0.$$

Then we get

$$\langle T(\bar{\sigma} X, \sigma Y)\sigma X, \sigma Y\rangle = 0, \quad X, Y \in X(M)$$

We compensate for the value of the projections σ and $\overline{\sigma}$ and, using the linear property of the tensor, we can rewrite the previous expression in the form

$$\begin{aligned}&\{\langle T(X, Y)X, Y\rangle + \langle T(\Phi X, \Phi Y)X, Y\rangle + \langle T(\Phi X, Y)\Phi X, Y\rangle + \langle T(\Phi X, Y)X, \Phi Y\rangle - \\ &\langle T(X, \Phi Y)\Phi X, Y\rangle - \langle T(X, Y)\Phi X, \Phi Y\rangle - \langle T(X, \Phi Y)X, \Phi Y\rangle - \langle T(\Phi X, \Phi Y)\Phi X, \Phi Y\rangle\} + \\ &i\{\langle T(X, \Phi Y)\Phi X, \Phi Y\rangle - \langle T(X, \Phi Y)X, Y\rangle - \langle T(X, Y)\Phi X, Y\rangle - \langle T(X, Y)X, \Phi Y\rangle + \\ &\langle T(\Phi X, Y)X, Y\rangle - \langle T(\Phi X, \Phi Y)\Phi X, Y\rangle - \langle T(\Phi X, \Phi Y)X, \Phi Y\rangle - \langle T(\Phi X, Y)\Phi X, \Phi Y\rangle\} = 0.\end{aligned}$$

Note that the real part equals to zero, and regarding the symmetrical properties of the coharmonic curvature tensor, we have

$$\langle T(X, Y)X, Y\rangle + \langle T(\Phi X, Y)\Phi X, Y\rangle - \langle T(X, \Phi Y)X, \Phi Y\rangle - \langle T(\Phi X, \Phi Y)\Phi X, \Phi Y\rangle = 0. \quad (1)$$

Then by substituting $X \to -\Phi^2 X; Y \to -\Phi^2 Y$ in equation (1) and using the fact $\Phi^3(X) = = -\Phi(X)$, we obtain the assertion of the theorem. □

Theorem 2. *A conharmonic curvature tensor on $\mathcal{LCAC_S}$-manifold of class CR_3 with Φ-invariant Ricci tensor is Φ-quasi invariant.*

P r o o f. Since the $\mathcal{LCAC_S}$-manifold M has Φ-ivariant Ricci tensor, then CT_3 and CR_3 are coincide. Regarding the Theorem 1, we get that the conharmonic curvature tensor is Φ-quasi invariant. □

Theorem 3. *Any $\mathcal{LCAC_S}$-manifold M of class CT_1 is conharmoniclly Φ-paracontact.*

P r o o f. On the G-adjoined structure space, we have $T_{\hat{a}\hat{b}cd} = 0$. In A-frame, we get

$$\langle T(\varepsilon_{\hat{a}}, \varepsilon_{\hat{b}})\varepsilon_c, \varepsilon_d\rangle = 0.$$

Consequently, we have

$$\langle T(\bar{\sigma}X, \sigma Y)\sigma X, \sigma Y\rangle = 0, \quad X, Y, Z, W \in X(M).$$

We make up for the value of the projections σ and $\overline{\sigma}$ and, using the linear property of the tensor, we can rewrite this expression in the form

$$\langle T(X,Y)Z,W\rangle - \langle T(\Phi X,\Phi Y)Z,W\rangle + \langle T(X,\Phi Y)\Phi Z,W\rangle + \langle T(X,\Phi Y)Z,\Phi W\rangle - \langle T(X, Y)\Phi Z,\Phi W\rangle + \langle T(\Phi X,Y)\Phi Z,W\rangle + \langle T(\Phi X,Y)Z,\Phi W\rangle + \langle T(\Phi X,\Phi Y)\Phi Z,\Phi W\rangle = 0. \quad (2)$$

According to the Definition 11, we have $T_{\hat{a}bcd} = 0$. Using the same technique, consequantly we obtain

$$\langle T(X,Y)Z,W\rangle + \langle T(\Phi X,\Phi Y)Z,W\rangle - \langle T(X,\Phi Y)\Phi Z,W\rangle - \langle T(X,\Phi Y)Z,\Phi W\rangle - \langle T(X, Y)\Phi Z,\Phi W\rangle + \langle T(\Phi X,Y)\Phi Z,W\rangle + \langle T(\Phi X,Y)Z,\Phi W\rangle - \langle T(\Phi X,\Phi Y)\Phi Z,\Phi W\rangle = 0. \quad (3)$$

From the combination of the equations (2) and (3), it follows that

$$\langle T(X,Y)Z,W\rangle = \langle T(X,Y)\Phi Z,\Phi W\rangle - \langle T(\Phi X,Y)\Phi Z,W\rangle - \langle T(\Phi X,Y)Z,\Phi W\rangle. \quad (4)$$

Then by substituting $X \to -\Phi^2 X; Y \to -\Phi^2 Y; Z \to -\Phi^2 Z, W \to -\Phi^2 W$ in the equation (4) and using the fact $\Phi^3(X) = -\Phi(X)$, we get the result. □

Theorem 4. *Let M be $\mathcal{LCAC}_\mathcal{S}$-manifold of class CR_2, then the first structure tensor is parallel in the first canonical connection.*

P r o o f. Suppose that M is $\mathcal{LCAC}_\mathcal{S}$-manifold of the class CR_2. So we have

$$2((B_{cabd} - B_{dabc}) - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b}) = 0. \quad (5)$$

Symmetrizing (5) by the indices (c,d) and then by the indices (c,b), we get

$$B_{cabd} = 0.$$

Regarding the fundamental theorem of tensor analysis, we have

$$\nabla B_{abc} = dB_{abc} + B_{dbc}\omega_a^d + B_{adc}\omega_b^d + B_{abd}\omega_c^d = B_{abcd}\omega^d.$$

So, we get

$$\nabla B_{abc} = B_{abcd}\omega^d.$$

It follows that

$$\nabla B_{abc} = 0.$$

Therefore, the tensor B_{abc} is parallel. □

Theorem 5. *Let M be $\mathcal{LCAC}_\mathcal{S}$-manifold of class CR_1, then its second and third structure tensors identically vanish.*

P r o o f. Let M be $\mathcal{LCAC_S}$-manifold of the class CR_1. According to the component of Riemannian curvature tensor, we have

$$2(2B[c|ab|d] - 2\sigma_{[a}B_{b]cd}B_{a[c}B_{d]b}) = 0. \tag{6}$$

Symmetrizing and then antisymmetrizing (6) by the indices (a, b), we deduce

$$B_{ac}B_{db} - B_{ad}B_{cb} = 0. \tag{7}$$

Antisymmetrizing (7) by the indices (a, d), we get

$$B_{ac}B_{db} = 0. \tag{8}$$

Contracting (8) by the indices (a, d) and then by (c, b), we get $B^2_{ac} = 0$, then

$$B_{ac} = 0.$$

From the condition $R_{\hat{a}\hat{b}cd} = 0$, we obtain

$$B^{abc} = 0.$$

□

In the rest of this section, we establish a relation between the classes CT_1 and CR_1.

Theorem 6. *Suppose that M is $\mathcal{LCAC_S}$-manifold of class CT_1 with Φ-invariant Ricci tensor, then M is a manifold of class CR_1 if and only if M has flat Ricci tensor.*

P r o o f. Suppose that M has flat Ricci tensor. Since M is a $\mathcal{LCAC_S}$-manifold of class CT_1, then according to the Definition 11, we have

$$T_{abcd} = T_{\hat{a}bcd} = T_{\hat{a}\hat{b}cd} = 0.$$

The condition $T_{abcd} = 0$, imples that $R_{abcd} = 0$; while the condition $T_{\hat{a}bcd} = 0$ with Φ-invariant Ricci tensor, implies that $R_{\hat{a}bcd} = 0$. In addition, $T_{\hat{a}\hat{b}cd} = 0$ with flat Ricci tensor, indicates $R_{\hat{a}\hat{b}cd} = 0$. Then M is a manifold of class CR_1. Conversly, according to the condition $T_{\hat{a}\hat{b}cd} = 0$, we have

$$2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} + 2B^{hab}B_{hdc} - \delta^a_{[c}\delta^b_{d]}\sigma^2_0) - \frac{4}{2n-1}(r^{[a}_{[c}\delta^{b]}_{d]}) = 0.$$

Since M is a manifold of class CR_1, we get

$$-\frac{4}{2n-1}(r^{[a}_{[c}\delta^{b]}_{d]}) = 0. \tag{9}$$

Taking the contraction operation for (9) by the indices (b, d), we deduce

$$r^a_c\delta^b_b - r^a_b\delta^b_c - r^b_c\delta^a_b + r^b_b\delta^a_c = 0.$$

Consequentaly, we obtain

$$(n-2)r^a_c + r^b_b\delta^a_c = 0. \tag{10}$$

Taking the symmetrization and antisymmetrization operations for (10) by the indices (b, a), we conclude

$$r^a_c = 0. \tag{11}$$

Making use of the Theorems 3 and 5, we have

$$-4(\delta^{[a}_{[b}\sigma^{c]}_{c]} - \sigma_{[c}\delta^{c}_{h]}\sigma^{[h}\delta^{a]}_{c} - \frac{1}{2}\sigma^{[a}\delta^{h]}_{b}\sigma_h) + A^{cb}_{ac} - \delta^a_b\sigma_{00} - 2n\delta^a_b\sigma^2_0 - \sigma^a_b - \sigma^a\sigma_b = 0. \tag{12}$$

Symmetrizing (12) by the indices (c, h), we obtain

$$-\sigma_{[c}\delta^{c}_{h]}\sigma^{[h}\delta^{a]}_{c} = 0. \tag{13}$$

Contracting (13) by the indices (a, c), we get

$$\sigma_a\delta^a_h\sigma^h\delta^a_a - \sigma_a\delta^a_h\sigma^a\delta^h_a - \sigma_h\delta^a_a\sigma^h\delta^a_a + \sigma_h\delta^a_a\sigma^a\delta^h_a = 0.$$

Hence

$$-n(n-1)(\sigma_h\sigma^h) = 0.$$

Or equivalently,

$$\sigma_h\overline{\sigma}_h = 0 \Leftrightarrow \sum_h |\sigma_h|^2 = 0 \Leftrightarrow \sigma_h = 0. \tag{14}$$

Once again, by using the equation (12), we have

$$A^{cb}_{ac} - \delta^a_b\sigma_{00} - 2n\delta^a_b\sigma^2_0 = 0. \tag{15}$$

By virtue of the Theorems 2, 5 and equation (14), the tensors $A^{ac}_{[bd]} = A^{[ac]}_{bd} = 0$ become symmetric regarding the lower and upper indices. Antisymmetrizing (15) by the indices (c, b), we obtain

$$-\delta^a_b\sigma_{00} - 2n\delta^a_b\sigma^2_0 = 0. \tag{16}$$

Contracting (16) by the indices (a, b) , consequently, we get

$$r_{00} = 0. \tag{17}$$

Accordig to the equations (11), (17) and Φ-invariant Ricci property, we get that M has flat Ricci tensor. □

Finally, we established an application by identifying the necessary and sufficient condition for the locally conformal almost contact manifold to be η-Einsten manifold

Theorem 7. *Suppose that M is $\mathcal{LCAC_S}$-manifold of class CT_1 with Φ-invariant Ricci tensor. Then the necessary and sufficient condition for M to be η-Einstein manifold is $\sigma^a_d = E\delta^a_d$, where $E = \frac{2(1-n)\sigma^2_0}{n-2} - \frac{\alpha}{2n-1}$.*

P r o o f. Suppose that M is $\mathcal{LCAC_S}$-manifold of class CT_1. According to the Definition 11 and Lemma 5, we have

$$2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} - \delta^a_{[c}\delta^b_{d]}\sigma^2_0) - \frac{4}{2n-1}(r^{[a}_{[d}\delta^{b]}_{c]}) = 0. \tag{18}$$

Contracting (18) by indices (b, c), we get

$$(n-2)\sigma^a_d + \delta^a_d\sigma^b_b - 2(1-n)\sigma^2_0\delta^a_d - \frac{1}{2n-1}((n-2)r^a_d + \delta^a_d r^b_b) = 0.$$

Then, we deduce

$$\sigma^a_d = \frac{2(1-n)\sigma_0^2\delta^a_d}{n-2} - \frac{r^a_d}{2n-1}. \tag{19}$$

Let M be η-Einstein manifold. We use the Definition9, so the equation (19) becomes

$$\sigma^a_d = E\delta^a_d. \tag{20}$$

Conversely, from the equations (19) and (20), we have

$$r^a_d = \alpha\delta^a_d$$

where α is the cosmological constant. Also

$$r_{00} = \alpha + \beta$$

where $\alpha = -2n(\sigma_{00} + \sigma_0^2) - 2B_{hc}B^{ch} - 2(\sigma^c_c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta^{h]}_c\sigma_h - \alpha$
According to Φ-invariant of Ricci tensor, we obtain that M is η-Einstein manifold. □

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Квазиинвариантный конгармонический тензор специальных классов локально конформного почти косимплектического многообразия

Ключевые слова: локально конформное почти косимплектическое многообразие, тензор конгармонической кривизны, Ф-квазиинвариант, конгармонический Ф-параконтакт.

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В работе описывается классификация локально конформного почти косимплектического многообразия ($\mathcal{LCAC_S}$-многообразия) в соответствии с тензором конгармонической кривизны. В частности, были получены необходимые условия Ф инвариантности тензора конгармонической кривизны на $\mathcal{LCAC_S}$-многообразии классов CT_i, $i = 1, 2, 3$. Кроме того, доказано, что любое $\mathcal{LCAC_S}$-многообразие класса CT_1 оказывается конгармоничным и Ф-параконтактным.

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