Jordan Journal of Mathematics and Statistics (JJMS), 15(3A), 2022, pp 497 - 506
DOI: https://doi.org/10.47013/15.3.7

ON THE PSEUDO - PROJECTIVE TENSOR OF NEARLY COSYMPLECTIC MANIFOLD

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ABSTRACT. The authors focused on the geometry of the pseudo projectively tensor of nearly cosymplectic manifold. In particular, it has established that the scalar curvature tensor of the aforementioned manifold is constant. Moreover, under the flatness property, the necessary condition for the nearly cosymplectic manifold to be an Einstein space, has been determined.

1. INTRODUCTION

One of the significant curvature tensors is a pseudo - projective tensor. Prasad [18], studied the aforementioned tensor on a Riemannian manifold of dimension n greater than 2. On the other hand, Bagewadi et al. [3], [4], obtained certain condition of vanishing pseudo - projective on LP - Sasakian manifolds. Moreover, Venkatosha and Bagewadi [21], studied the recurrent property of pseudo - projective tensor on Kenmotsu manifold. Nagaraja and Somashekhara [16], showed that every pseudo - projectively vanishing and pseudo - projective semi symmetric Sasakian manifolds are locally isomorphic to the unit sphere. Abood and Nawaf [1], [2], [15] , established some results on a projective curvature tensor in nearly cosymplectic manifold. Furthermore, Maralabhavi and Shivaprasanna [14], proved that $R(X,Y)\xi = 0$ if the LP-Sasakian manifold is ξ-pseudo projectively flat. Guler and Demirbag [7], considered the pseudo - projective flatness on generalized quasi - Einstein manifold. Finally, Shivamurthy et al. [20], established some results on pseudo - projective tensor in a

2000 *Mathematics Subject Classification.* 53C55; 53B35.

Key words and phrases. Nearly cosymplectic manifold, pseudo - projective tensor, Einstein space.

Received: Jan. 23, 2021 Accepted: Nov. 3, 2021 .

Kenmotsu manifold.

2. Preliminaries

Suppose that M is a smooth manifold of odd dimension greater than 3, η is a differential one form, ξ is a smooth vector field, Φ is an endomorphism of $X(M)$ and $g = \langle .,. \rangle$ is the *Riemannian metric* on M.

The set (M, η, ξ, Φ, g) is called an almost contact manifold (AC-manifold), if the following conditions hold:

$$\eta(\xi) = 1, \Phi(\xi) = 0, \eta \circ \Phi = 0, \Phi^2 = -id + \eta \otimes \xi,$$

and $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y)$; $X, Y \in X(M)$[5]. Further, AC-manifold is said to be a nearly cosymplectic manifold (NC-manifold) if the equality

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0,$$

supports for some $X, Y \in X(M)$ [6].

The frame $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ is called an A-frame, where $p \in M$, $\varepsilon_a = \sqrt{2}\sigma_p(e_p)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_p)$, $\varepsilon_0 = \xi_p$ and $2n+1$ is the dimension of manifold [12].

The matrices of the endomorphism Φ_p and the Riemannian metric g_p in A-frame of a tangent space of M at a point p are given by the following forms respectively:

$$(\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} \tag{2.1}$$

where I_n is the identity matrix of order n, and $i, j = 0, 1, ..., 2n+1$ [9].

A tensor of type $(4,0)$ which is defined by $\Re(X, Y, Z, W) = g(\Re(Z, W)Y, X)$, where $\Re(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$, and has the following properties:

(1) $\Re(X, Y, Z, W) = -\Re(Y, X, Z, W)$;
(2) $\Re(X, Y, Z, W) = -\Re(X, Y, W, Z)$;
(3) $\Re(X, Y, Z, W) = \Re(Z, W, X, Y)$;
(4) $\Re(X, Y, Z, W) + \Re(X, Z, W, Y) + \Re(X, W, Y, Z) = 0$,

is called a Riemann-Christoffel tensor of a smooth manifold M [13].

A tensor of type (2,0) which is defined by $r_{ij} = \Re^k_{ijk} = g^{kl}\Re_{kijl}$, is called a Ricci tensor of a smooth manifold M, where $i, j, k, l = 0, 1, ..., 2n + 1$ if M has dimension $2n + 1$, while g^{kl} is the dual tensor of Riemannian metric g on M [19].

In [10], Kirichenko and Kusova established the structure equations of NC-manifold. Moreover, the components of Riemann-Christoffel and Ricci tensors of NC-manifold are calculated and listed in the next lemmas.

Lemma 2.1. [10] *In the G-adjoined space, the structure equations of NC-manifold are given as the following forms:*

(1) $d\omega^a = \omega^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + \frac{3}{2}C^{ab}\omega_b \wedge \omega$;

(2) $d\omega_a = -\omega^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + \frac{3}{2}C_{ab}\omega^b \wedge \omega$;

(3) $d\omega = C^{bc}\omega_b \wedge \omega_c + C_{bc}\omega^b \wedge \omega^c$;

(4) $d\omega^a_b = \omega^a_c \wedge \omega^c_b + [A^{ad}_{bc} - 2B^{adh}B_{hbc} + \frac{3}{2}C^{ad}C_{bc}]\omega^c \wedge \omega_d$,

where $B^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}$, $C^{ab} = \sqrt{-1}\Phi^a_{0,\hat{b}}$, $C_{ab} = -\sqrt{-1}\Phi^{\hat{a}}_{b,0}$ *and* $B_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,c}$.
The tensors B, C and A are called the first, second and third structure tensors respectively. Whereas, $a, b, c, d, h = 1, 2, ..., n$ *and* $\hat{a} = a + n$.

Lemma 2.2. [10] *In the G-adjoined space, the components of Riemann-Christoffel tensor of NC-manifold have the following forms:*

(1) $\Re_{\hat{a}bcd} = 0$;

(2) $\Re_{abcd} = -2B_{ab[cd]}$;

(3) $\Re_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd}$;

(4) $\Re_{\hat{a}0b0} = C^{ac}C_{bc}$;

(5) $\Re_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}$.

The remaining components of the Riemann-Christoffel tensor $\Re$ *can be obtain by the classical symmetrical properties for* $\Re$ *or identical equal to zero. In addition,* $a, b, c, d, h = 1, 2, ..., n$ *and* $\hat{a} = a + n$.

Lemma 2.3. [10] *In the G-adjoined space, the components of the Ricci tensor of NC-manifold are given by the following forms:*

(1) $r_{ab} = 0$;

(2) $r_{a\hat{b}} = -A^{cb}_{ac} + 3B^{cbh}B_{hac} + \frac{2}{3}C^{bc}C_{ac}$;

(3) $r_{a0} = 0$;

(4) $r_{oo} = -2C^{cd}C_{cd}$.

The remaining components of the Ricci tensor can be calculated by taking the conjugate operation to the previous components and those identical equal to zero. Moreover, $a, b, c, d, h = 1, 2, ..., n$ *and* $\hat{b} = b + n$.

Definition 2.1. [8] A tensor of type $(4, 0)$ which is defined by

$$P_{ijkl} = \Re_{ijkl} - \frac{1}{2n}[r_{ik}g_{jl} - r_{jk}g_{il}],$$

is called a projective tensor, where i, j, k, l have range as previous.

Definition 2.2. [17] A Riemannian manifold is called an Einstein space, if the Ricci tensor satisfies the Einstein field equation $r_{ij} = eg_{ij}$, where e is an cosmological Einstein constant and i, j have range as previous.

Definition 2.3. [1] An NC-manifold has Φ-invariant Ricci tensor, if

$$\Phi \circ r = r \circ \Phi.$$

Lemma 2.4. [1] *An NC-manifold has Φ-invariant Ricci tensor if and only if, in the G-adjoined space the following condition*

$$r^{\hat{a}}_{b} = r_{ab} = 0$$

holds, where a, b and $\hat{a}$ have range as previous.

Definition 2.4. [10] Suppose that M is NC-manifold, then M is called a closed cosymplectic structure if $d\eta = 0$.

The following lemma establishes the relation between a significant class of almost Hermitian manifold with the cosymplectic manifold.

Lemma 2.5. [11] *Suppose that M is closed cosymplectic manifold, then M is Locally equivalent to the direct product of a nearly Kähler manifold by a real line.*

Definition 2.5. [18] A pseudo - projective tensor Pp in a Riemannian manifold is defined as follows:

$$Pp(X,Y,Z,W) = \alpha\Re(X,Y,Z,W) + \beta[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)] -$$
$$\frac{K}{n}(\frac{\alpha}{n-1} + \beta)[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)],$$

where α, β are constants such that $\alpha, \beta \neq 0$, $\Re$ is a Riemann-Christoffel tensor, S is a Ricci tensor and K is a scalar curvature tensor.

Remark 1. [18] *The pseudo-projective tensor satisfies the following properties:*

(1) $Pp(X,Y,Z,W) = -Pp(Y,X,Z,W)$;
(2) $Pp(X,Y,Z,W) \neq \mp Pp(X,Y,Z,W)$, *where* $X,Y,Z,W \in T_p(M)$

Now, redefine the pseudo - projective tensor on $AC-$manifold as the following components formula:

$$Pp_{ijkl} = \alpha\Re_{ijkl} + \beta[r_{ik}g_{jl} - r_{jk}g_{il}] - \frac{K}{(2n+1)}(\frac{\alpha}{2n} + \beta)[g_{ik}g_{jl} - g_{jk}g_{il}]. \tag{2.2}$$

Remark 2. *If* $\alpha = 1$ *and* $\beta = \frac{-1}{2n}$, *then the equation (3.1) becomes*

$$Pp_{ijkl} = P_{ijkl} = \Re_{ijkl} - \frac{1}{2n}[r_{ik}g_{jl} - r_{jk}g_{il}].$$

Which gives the classical projective tensor as a special case of the pseudo - projective tensor. Also, take into account all indices in this section and next section have the same range mentioned previously.

3. The Pseudo-projective Curvature Tensor

In order to study the geometric properties of the pseudo - projective tensor, we start by calculation the components of the mentioned tensor which are embodied in the following theorem.

Theorem 3.1. *In the G-adjoined space, the components of the pseudo - projective tensor of NC-manifold are given by the following forms:*

(1) $Pp_{abcd} = -2\alpha B_{ab[cd]}$;
(2) $Pp_{\hat{a}bcd} = -2\alpha B^{abh}B_{hcd} + \beta[r^a_c\delta^b_d - r^b_c\delta^a_d] - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta][\delta^a_c\delta^b_d - \delta^b_c\delta^a_d]$;
(3) $Pp_{\hat{a}bc\hat{d}} = \alpha[A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}] + \beta r^a_c\delta^d_b - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta]\delta^a_c\delta^b_d$;

(4) $Pp_{\hat{a}0b0} = \alpha C^{ac}C_{bc} + \beta r^a_b - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta]\delta^a_b$.

The remaining are conjugate to those given above or identical equal to zero.

Proof. By using the Lemma 2.2 and Definition 2.1, we compute the components of pseudo - projective tensor as follows:
1) Put $i = a, j = b, k = c$ and $l = d$, we have

$$Pp_{abcd} = \alpha\Re_{abcd} + \beta[r_{ac}g_{bd} - r_{bc}g_{ad}] - \frac{K}{(2n+1)}(\frac{\alpha}{2n} + \beta)[g_{ac}g_{bd} - g_{bc}g_{ad}]$$

Taking into account the matrices (2.1), Lemmas (2.2) and (2.3) , then we have

$$Pp_{abcd} = -2\alpha B_{ab[cd]}$$

By the same manner, we can get the other components. □

Theorem 3.2. *Let M be NC-manifold. If M is an Einstein space with vanishing pseudo - projective tensor, then the scalar curvature tensor is constant.*

Proof. Suppose that M is Einstein space with vanishing pseudo - projective tensor. According to the Theorem 3.1, we have

$$-2\alpha B^{abh}B_{hcd} + \beta[r^a_c\delta^b_d - r^b_c\delta^a_d] - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta][\delta^a_c\delta^b_d - \delta^b_c\delta^a_d] = 0 \tag{3.1}$$

Since M is Einstein manifold, then the equation (3.1), becomes

$$-2\alpha B^{abh}B_{hcd} - (\beta e - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta])[\delta^a_c\delta^b_d - \delta^b_c\delta^a_d] = 0 \tag{3.2}$$

Contracting the equation (3.2) by the indices (b, c) and then by (a, d), we obtain

$$-2\alpha B^{abh}B_{hba} - (\beta e - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta])n(1 - n) = 0 \tag{3.3}$$

By symmetrizating and then antisymmetrizing the equation (3.3) by the indices (b,h), we get

$$K = \frac{\beta e(2n+1)}{[\frac{\alpha}{2n} + \beta]}.$$

□

Theorem 3.3. *Suppose that M is NC-manifold with vanishing pseudo - projective tensor and Φ-invariant Ricci tensor, then M is an Einstein space.*

Proof. Let M be NC-manifold with vanishing pseudo - projective tensor. According to the Theorem 3.1, we have

$$-2\alpha B^{abh}B_{hcd} + \beta[r^a_c\delta^b_d - r^b_c\delta^a_d] - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta][\delta^a_c\delta^b_d - \delta^b_c\delta^a_d] = 0 \tag{3.4}$$

Taking the symmetrization and antisymmetrization operations of the equation (3.4) by the indices (b, h), we get

$$\beta[r^a_c\delta^b_d - r^b_c\delta^a_d] - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta][\delta^a_c\delta^b_d - \delta^b_c\delta^a_d] = 0 \tag{3.5}$$

Contracting the equation (3.5) by the indices (a, d), we get

$$r^b_c = e\delta^b_c,$$

where

$$e = \frac{K[\frac{\alpha}{2n} + \beta]}{\beta(2n+1)}.$$

Since M has Φ-invariant Ricci tensor. Consequently, by taking into account the Definition 2.2, we get that M is an Einstein space. □

Finally, we have found a theoretical application for vanishing NC- manifold. In particular, we have determined the necessary and sufficient condition for the mentioned manifold to be locally equivalent to the direct product of a nearly Kähler manifold by a real line as shown in the theorem below.

Theorem 3.4. *Suppose that M is NC-manifold with vanishing pseudo-projective tensor and Φ-invariant Ricci tensor, then M is an Einstein manifold if and only if, M is locally equivalent to the direct product of a nearly Kähler manifold by a real line.*

Proof. Let M be NC-manifold with vanishing pseudo - projective tensor. According to the Theorem 3.1, we have

$$\alpha C^{ac}C_{bc} + \beta r^a_b - \frac{K}{(2n+1)}[\frac{\alpha}{2n} + \beta]\delta^a_b = 0 \tag{3.6}$$

Suppose that M is locally equivalent to the direct product of a nearly Kähler manifold by a real line, then the equation (3.6), becomes

$$r^a_b = e\delta^a_b,$$

where $e = \frac{K}{(2n+1)\beta}[\frac{\alpha}{2n} + \beta]$.
According to the Φ-invariant Ricci tensor and the Definition 2.2, we deduce that M is Einstein space.
Conversely,
Suppose that M is an Einstein space, then the equation (3.6), becomes

$$\alpha C^{ac} C_{bc} = 0 \tag{3.7}$$

in view of $\alpha \neq 0$, we have

$$C^{ac} C_{bc} = 0 \tag{3.8}$$

contracting the equation (3.8) by the indices (a, b), we obtain

$$C^{ac} C_{ac} = 0 \tag{3.9}$$

since C^{ac} and C_{ac} are antisymmetric tensors, then we get

$$\sum_{a,c} |C^{ac}|^2 = 0$$

consequently, we deduce

$$C^{ac} = 0.$$

Suppose that π is a natural projection of a principle fiber bundle space, then

$$dw^0 = dw = \pi^* d\eta = 0$$

thus we get $d\eta = 0$.
From the Definition 2.4, M is closed cosymplectic.
According to the Lemma 2.5, we get that M is locally equivalent to the direct product of a nearly Kähler manifold by a real line. □

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