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NEARLY COSYMPLECTIC MANIFOLD OF HOLOMORPHIC SECTIONAL CURVATURE TENSOR

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Abstract

This paper is devoted to the study of geometry of nearly cosymplectic manifold of holomorphic sectional curvature tensor. In particular, the necessary conditions in which a nearly cosymplectic manifold is a manifold of point constant holomorphic sectional curvature tensor have been found.

1. Introduction

One of the interesting benefits in the study of Kahlel manifold is to present the concept of nearly Kahler manifold. Nearly cosymplectic manifold (*NC*-manifold) was defined by the same way from coKahler or also called *cosymplectic manifold*.

The notion of *NC*-manifold was introduced by Blair and Showers [5, 6]. Endo and Fueki [8-10] studied certain curvature tensors of *NC*-manifold of Φ-sectional curvature. Concerning nearly cosymplectic manifold, Nicola et al. [18] proved that *NC*-manifold (non-symplectic manifold) of dimension

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$2n+1>5$ is locally isometric to one of the Riemannian products; $R \times N^{2n}$ and $M^5 \times N^{2n-4}$, where N^{2n} is a nearly Kahler (non-Kahler) manifold, N^{2n-4} is a nearly Kahler manifold and M^5 is a nearly cosymplectic (non-cosymplectic) manifold. Kirichenko and Kusova [15] studied the geometry of *NC*-manifold. They found its structure equations and the components of Riemann-Christoffel tensor in the *G*-adjoined structure space.

The present paper deals with the study of (*NC*-manifold) holomorphic sectional curvature tensor related with the projective tensor. Many researchers studied the geometric properties of projective tensor on some kinds of almost Hermitian manifolds and almost contact manifolds. For more details, we refer to [1-3] and [11].

2. Preliminaries

In this section, we demonstrate many concepts and facts of almost contact manifold and nearly cosymplectic manifold, in particular, the structure equations of these manifolds have been explained.

Definition 2.1 [4]**.** Let *M* be an $2n+1$-dimensional smooth manifold. The set of tensors (η, ξ, Φ, g) is called an *almost contact metric structure* if such that: $\eta(\xi)=1, \quad \Phi(\xi)=0, \quad \eta \circ \Phi = 0$ and $\Phi^2 = -id + \eta \otimes \xi$, where η is differential 1-form called *contact form*, ξ is a vector field called the *characteristic*, Φ endomorphism of $X(M)$ called the *structure endomorphism* and $g = \langle .,. \rangle$ is a Riemannian structure on *M* such that $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in X(M)$. Then *M* with almost contact metric structure is called an *almost contact metric manifold* (*AC-manifold*).

Definition 2.2 [7]**.** An almost contact manifold is called a *nearly cosymplectic manifold* (*NC*-manifold) if the equality

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0, \quad X, Y \in X(M)$$

holds.

Definition 2.3 [12]**.** Suppose that (M, η, ξ, Φ, g) is an *AC*-manifold. In the module $X^c(M)$, define two endomorphisms σ and $\overline{\sigma}$ as follows: $\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi)$ and $\overline{\sigma} = -\frac{1}{2}(id + \sqrt{-1}\Phi)$ and we can define two projections as follows:

$$\prod = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 - \sqrt{-1}\Phi) \quad \text{and} \quad \overline{\prod} = \overline{\sigma} \circ \ell = \frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi),$$

where $\sigma \circ \Phi = \Phi \circ \sigma = i\sigma$ and $\overline{\sigma} \circ \Phi = \Phi \circ \overline{\sigma} = -i\overline{\sigma}$. Therefore, if we denote $Im \prod = D_\Phi^{\sqrt{-1}}$ and $Im \overline{\prod} = D_\Phi^{-\sqrt{-1}}$, then

$$X^c(M) = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0,$$

where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{-\sqrt{-1}}$ and D_Φ^0 are proper submodules of endomorphism Φ with proper values $\sqrt{-1}, -\sqrt{-1}$ and 0, respectively.

Definition 2.4 [16]**.** At each point $p \in M^{2n+1}$, we can construct a frame in $T_p(M)^c$ by the form $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\sigma_p(e_p)$, $\varepsilon_{\hat{a}} = \sqrt{2}\overline{\sigma}(e_p)$ and $\varepsilon_0 = \xi_p$. The frame $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ is called an *A-frame*. The principle fiber bundle of all *A*-frames with structure group $\{1\} \times U(n)$ is called a *G-adjoined structure space*.

Lemma 2.1 [14]**.** *For an AC-manifold, the matrices of the AC-structure* Φ_p *and Riemann metric* g_p *in A-frame are given by the following forms*:

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n *is the identity matrix of order n.*

The following lemma describes the structure equations of *NC*-manifold in the *G*-adjoined structure space.

Lemma 2.2 [15]**.** *The structure equations of NC-manifold in G-adjoined structure are given by the following forms*:

(1) $d\omega^a = \omega^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + \frac{3}{2}C^{ab}\omega_b \wedge \omega;$

(2) $d\omega_a = -\omega^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + \frac{3}{2}C_{ab}\omega^b \wedge \omega;$

(3) $d\omega = C^{bc}\omega_b \wedge \omega_c + C_{bc}\omega^b \wedge \omega^c;$

(4) $d\omega^a_b = \omega^a_c \wedge \omega^c_b + \left[A^{ad}_{bc} - 2B^{adh}B_{hbc} + \frac{3}{2}C^{ad}C_{bc}\right]\omega^c \wedge \omega_d,$

where $B^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}$, $C^{ab} = \sqrt{-1}\Phi^a_{0,\hat{b}}$, $C_{ab} = -\sqrt{-1}\Phi^{\hat{a}}_{b,0}$ *and* $B_{abc} =$ $C^{ab} = \sqrt{-1}\Phi^a_{0,\hat{b}}$. *The tensors B, C and A are called the first, second and third structure tensors, respectively*.

Definition 2.5 [17]**.** A Riemann-Christoffel tensor of a smooth manifold M is a tensor of type $(4, 0)$ which is defined by

$$R(X, Y, Z, W) = g(R(Z, W)Y, X),$$

where $R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]Z})$, and satisfies the following properties:

(1) $R(X, Y, Z, W) = -R(Y, X, Z, W);$

(2) $R(X, Y, Z, W) = -R(X, Y, W, Z);$

(3) $R(X, Y, Z, W) = R(Z, W, X, Y);$

(4) $R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0.$

The components of Riemann-Christoffel tensor of *NC*-manifold are given in lemma below:

Lemma 2.3 [15]**.** *In the G-adjoined structure space, the components of Riemann-Christoffel tensor of NC-manifold have the following forms*:

(1) $R_{\hat{a}bcd} = 0;$

(2) $R_{abcd} = -2B_{ab[cd]};$

(3) $R_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd};$

(4) $R_{\hat{a}0b0} = C^{ac}C_{bc};$

(5) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}.$

The other components of Riemann-Christoffel tensor R can be obtained by the property of symmetry for R or equal to zero.

Definition 2.6 [19]**.** A tensor of type $(2, 0)$ which is defined as

$$r_{ij} = R^{k}_{ijk} = g^{kl}R_{kijl}$$

is called a *Ricci tensor*.

Lemma 2.4 [15]**.** *The components of Ricci tensor of NC-manifold in the G-adjoined structure space are given by the following forms*:

(1) $r_{ab} = 0;$

(2) $r_{a\hat{b}} = -A^{cb}_{ac} + 3B^{cbh}B_{hac} + \frac{2}{3}C^{bc}C_{ac};$

(3) $r_{a0} = 0;$

(4) $r_{00} = -2C^{cd}C_{cd};$

and the others are conjugate to the above components or equal to zero.

The previous definitions of Riemann-Christoffel and Ricci tensors completed the requirements of the projective tensor which is embodied in the next definition.

Definition 2.7 [11]**.** Let M^{2n+1} be an *AC*-manifold. Then a tensor of type $(4, 0)$ which is defined as

$$P_{ijkl} = R_{ijkl} - \frac{1}{2n}[r_{ik}g_{jl} - r_{jk}g_{il}]$$

is called a *projective curvature tensor*, where $P_{ijkl} = -P_{jikl} = -P_{ijlk} = P_{klij}$.

We will demonstrate the projective tensor on one of the *AC*-manifolds which is *NC*-manifold. The following theorem gives the components of this tensor on *NC*-manifold.

Theorem 2.1. *In the G-adjoined structure space, the components of projective curvature tensor of NC-manifold are given by the following forms*:

(1) $P_{abcd} = -B_{ab[cd]}$;

(2) $P_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd} - \frac{1}{2n}[r_c^a\delta_d^b - r_c^b\delta_d^a]$;

(3) $P_{\hat{a}bc\hat{d}} = A_{bc}^{ad} - B^{adh}B_{hac} - \frac{5}{3}C_{ac}^{ad} - \frac{1}{2n}r_c^a\delta_a^d$;

(4) $P_{\hat{a}0b0} = C^{ac}C_{bc} - \frac{1}{2n}r_b^a$;

and the others are conjugate to the above components or equal to zero.

Proof. By using Lemmas 2.3, 2.4 and Definition 2.7, directly we obtain the above components.

Definition 2.8 [12]**.** An *AC*-manifold *M* is called the *vanishing projective tensor*, if the projective tensor is equal to zero.

Definition 2.9 [13]**.** Let *M* be an *AC*-manifold. Then a Φ-holomorphic sectional curvature (Φ*HS*-curvature) of a manifold *M* in the direction $X \in X(M)$, $X \neq 0$ is a function $H(X)$ which is defined as:

$$H(X) = \langle R(X, \Phi X)X, \Phi X\rangle \| X \|^{-4}.$$

Definition 2.10 [13]**.** An *AC*-manifold is called a *manifold* of point constant Φ*HS*-curvature if

$$\langle R(X, \Phi X)X, \Phi X\rangle = c\| X \|^4,$$

where $c \in C^{\infty}(M)$, for all $X \in X(M)$.

Lemma 2.5 [13]**.** *An AC-manifold is a manifold of point constant* Φ*HS-curvature* C_0 *if and only if, on the G-adjoined structure, the following equation holds*:

$$R^{(ad)}_{bc} = \frac{C_0}{2}\tilde{\delta}^{ad}_{bc}, \tag{2.1}$$

where $C_0 \in C^{\infty}(M)$ *and* $\tilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^a_c\delta^d_b$.

Definition 2.11. An *NC*-manifold has Φ-invariant Ricci tensor, if $\Phi \circ r = r \circ \Phi$.

Lemma 2.6. *An NC-manifold has* Φ*-invariant Ricci tensor if and only if, in the G-adjoined structure space, the following condition*

$$r^{\hat{a}}_b = r_{ab} = 0$$

holds.

Concerning the projective tensor, we defined three special classes of *AC*-manifold which are given in the definition below.

Definition 2.12. Let M be an *AC*-manifold. Then M is a manifold of class:

(1) P_1 if, $P(X, Y, Z, W) = P(X, Y, \Phi Z, \Phi W)$,

(2) P_2 if, $P(X, Y, Z, W) = P(\Phi X, \Phi Y, Z, W) + P(\Phi X, Y, \Phi Z, W) + P(\Phi X, Y, Z, \Phi W)$,

(3) P_3 if, $P(X, Y, Z, W) = P(\Phi X, \Phi Y, \Phi Z, \Phi W)$,

where $X, Y, Z, W \in X(M)$.

3. The Main Results

This section highlights on the study of *NC*-manifold of point constant ΦHS-curvature. In particular, we find the necessary conditions in which *M* is an Einstein manifold.

Theorem 3.1. *Suppose that M is an NC-manifold. Then the necessary and sufficient condition in which M is a manifold of point constant* ΦHS*-curvature* C_0 *is*

$$A_{bc}^{ad} = B^{adh}B_{hbc} + \frac{5}{3}C^{ad}C_{bc} + \frac{C_0}{2}\tilde{\delta}_{bc}^{ad}. \tag{3.1}$$

Proof. The relation (3.1) can be found directly from Lemma 2.3 and equation (2.1).

Theorem 3.2. *Suppose that M is an NC-manifold of* ΦHS*-curvature tensor and is projectively vanishing with* Φ*-invariant Ricci tensor. Then M is an Einstein manifold.*

Proof. Let *M* be an *NC*-manifold of ΦHS-curvature tensor and be projectively vanishing. Then by using Theorem 2.1 and equation (3.1), we get

$$r_c^a\delta_b^d = nC_0[\delta_b^a\delta_c^d + \delta_c^a\delta_b^d]. \tag{3.2}$$

Contracting (3.2) by induces (b, a), we have

$$r_c^d = e\delta_c^d,$$

where $e = n(n+1)C_0$. According to the Φ-invariant Ricci tensor, we get that *M* is an Einstein manifold.

Theorem 3.3. *Suppose that M is an NC-manifold of* ΦHS*-curvature tensor with* Φ*-invariant Ricci tensor. Then the necessary condition in which M is an Einstein manifold is* $A_{ac}^{ac} = \frac{-5}{6}e + C_1$.

Proof. Let M be an NC-manifold of ΦHS-curvature tensor. From Theorem 3.1, we have

$$A_{bc}^{ad} = B^{adh}B_{hbc} + \frac{5}{3}C^{ad}C_{bc} + \frac{C_0}{2}[\delta_b^a\delta_c^d + \delta_c^a\delta_b^d]. \tag{3.3}$$

Symmetrizing and antisymmetrizing (3.3) by the indices (d, h), we have

$$A_{bc}^{ad} = \frac{5}{3}C^{ad}C_{bc} + \frac{C_0}{2}[\delta_b^a\delta_c^d + \delta_c^a\delta_b^d].$$

Contracting the indices (c, d), we obtain

$$A_{bc}^{ac} = \frac{5}{3}C^{ac}C_{bc} + \frac{C_0(n+1)}{2}\delta_b^a.$$

Contracting the indices (a, b), we get

$$A_{ac}^{ac} = \frac{5}{3}C^{ac}C_{ac} + \frac{n(n+1)C_0}{2}. \tag{3.4}$$

According to Lemma 2.2, equation (3.4) becomes

$$A_{ac}^{ac} = \frac{-5}{6}r_{00} + \frac{n(n+1)C_0}{2}.$$

Since M is an Einstein manifold, we conclude

$$A_{ac}^{ac} = \frac{-5}{6}e + C_1.$$

Theorem 3.4. *Suppose that M is an NC-manifold of class P_2 and ΦHS-curvature tensor with Φ-invariant Ricci tensor. Then M is an Einstein manifold.*

Proof. Let M be an NC-manifold of class P_2. From Definition 2.12 and Theorem 2.1, we have

$$-2B_{ab[cd]} = -2B^{abh}B_{hcd} - \frac{1}{2n}[r_c^a\delta_d^b - r_c^b\delta_d^a] - A_{bd}^{ac} + B_{hbd}^{ach}$$
$$+\frac{5}{3}C_{bd}^{ac} + \frac{1}{2n}r_d^a\delta_b^c + A_{bc}^{ad} - B_{hbc}^{adh} - \frac{5}{3}C_{bc}^{ad} - \frac{1}{2n}r_c^a\delta_b^d.$$

Since M is a manifold of ΦHS-curvature, according to Lemma 2.5, we obtain

$$-2B_{ab[cd]} = -2B^{abh}B_{hcd} - \frac{1}{2n}[r_c^a\delta_d^b - r_c^b\delta_d^a] - \frac{C_0}{2}\tilde{\delta}_{bd}^{ac}$$

$$+\frac{1}{2n}r_d^a\delta_d^c + \frac{C_0}{2}\tilde{\delta}_{bc}^{ad} - \frac{1}{2n}r_c^a\delta_c^d.$$

By symmetrization and anti-symmetrization of the indices (a, b) and (a, c), we get

$$\frac{C_0}{2}[\delta_b^a\delta_c^d + \delta_c^a\delta_b^d] = \frac{1}{2n}r_c^a\delta_b^d. \tag{3.5}$$

Contracting equation (3.5) by the indices (b, d), it follows that

$$r_c^a = C_0(n+1)\delta_c^a.$$

Since M is Φ-invariant Ricci tensor, M is an Einstein manifold.

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