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Holomorphic–Geodesic Transformations

of Almost Hermitian Manifold

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Abstract

In this paper it has been proved that almost Hermitian manifold with a definite sign metric does not admit nontrivial geodesic transformation. It has also been proved that the almost Hermitian manifold belongs to one of the following Grey-Hervalla classes: $K, W_1, W_2, W_4, W_1 \oplus W_2, W_1 \oplus W_4, W_2 \oplus W_4,\ W_1 \oplus W_2 \oplus W_4$ do not admit nontrivial geodesic transformation, preserving the almost complex structure.

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1. Introduction

In the last years there are intensive researches for the basic problems of the geodesic transformation theory for Riemannian spaces, especially, these researches have studied geometric properties of Riemannian spaces that admit nontrivial geodesic transformation, also studied the problems of classification for these spaces, which are caused by the theory of geodesic transformation of Riemannian and affine spaces, and also are a generalization represent unconditional interest from the applied point of view. It is found, that the movement of many types of mechanical systems, and also bodies or particles in gravitational and electromagnetic fields, in continuous environment all occur in trajectories, which can be considered as geodesic lines of affine or Riemannian space.

The geometry of Riemannian spaces supplied with additional structure like complex structure, traditionally use the large popularity, especially these spaces which have the most interesting properties. So geodesic transformations of almost complex manifolds were studied by many researchers ([3], [5], [7], [8]), Yano [8] was proved, that Kahler space did not admit nontrivial geodesic transformations, preserving the complex structure. In [3] established, that Kahler and Nearly Kalhler spaces do not admit nontrivial geodesic transformations at preservation of structure. In work [5] had been proved, that conformal Kahler space of dimension more than 2 did not admit nontrivial geodesic transformations with a condition of preservation of structure.

There is an interesting question; whether the other classes of Gry-Harvella admit nontrivial geodesic transformation with a preservation of structure endomorphism. This paper deeply searches this question.

2. Preliminaries

Let M be an almost Hermitian manifold with an almost complex structure $(J, g =<. , .>)$, $J^2 = -id, < X, Y >=< JX, JY >$, $dimM = 2n > 2$ [2], and let $X(M)$ be the Lie algebra of smooth vector fields on M.

Diffeomorfism φ of Riemannian space V_n on Riemannian space $\bar{V}_n$ is called geodesic transformation, if geodesic line of space V_n transports to geodesic line of space $\bar{V}_n$ [6].

Suppose that M is an almost Hermitian manifold, consider on M geodesic transformation φ, preserving the complex structure J, i.e. $\varphi_*(J) = J$. We call this transformation a holomorphic-geodesic . Denote by $\tilde{g} = \varphi^*(g)$.

As far as $\tilde{g}(JX, JY) = (\varphi^*(g))(JX, JY) = g(J(\varphi_*X), J(\varphi_*Y)) = g(\varphi_*X, \varphi_*Y)$
$=\varphi^*(g)(X, Y) , X, Y \in X(M)$.

So$(J, \tilde{g})$ is an almost complex structure on M.

Suppose that ∇ is a Riemannian connection of the metric g, $\tilde{\nabla}$ is a Riemannian connection of the metric $\tilde{g}$, then the affine deformation tensor,

$T(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$ from the Riemannian connection∇ to the Riemannian connection $\tilde{\nabla}$ has the form $T^i_{jk} = \psi_j \delta^i_k + \psi_k \delta^i_j$ **,** where δ^i_j is the Kronecker delta and ψ any covector. We can rewrite this formula without indexes as the following:

$T(X,Y) = T^i_{jk}\, X^j Y^k e_i = \psi_j \delta^i_k X^j Y^k e_i + \psi_k \delta^i_j X^k Y^j e_i = \psi(X)Y + \psi(Y)X$.

Thus

$$T(X,Y) = \psi(X)Y + \psi(Y)X\ . \qquad (2.1)$$

Computing the covariant derivative of the structure operator J in the connection $\widetilde{\nabla}$:

$\tilde{\nabla}_X(J)Y = \tilde{\nabla}_X(JY) - J\tilde{\nabla}_X Y = \nabla_X(JY) + T(X,JY) - J\nabla_X Y - JT(X,Y)$
$\qquad = \nabla_X(J)Y + T(X,JY) - JT(X,Y)$

Hence

$$\tilde{\nabla}_X(J)Y = \nabla_X(J)Y + T(X,JY) - JT(X,Y) \qquad (2.2)$$

According to (2.1), we have:

$\tilde{\nabla}_X(J)Y = \nabla_X(J)Y + \psi(X)JY + \psi(JY)X - J(\psi(X)Y + \psi(Y)X)$
$\qquad = \nabla_X(J)Y + \psi(X)JY + \psi(JY)X - \psi(X)JY - \psi(Y)JX$.

Thus

$$\tilde{\nabla}_X(J)Y = \nabla_X(J)Y + \psi(JY)X - \psi(Y)JX \qquad (2.3)$$

3. Main results.

Consider the following tensors:

$B(X,Y) = \frac{1}{2}(\nabla_{JX}(J)Y - \nabla_X(J)(JY)$ (virtual tensor [4]).

$C(X,Y) = \frac{1}{2}(\nabla_{JX}(J)Y + \nabla_X(J)(JY)$ (structure tensor[4]).

Consider how these tensors change through geodesic transformation, preserving J.

For the structure tensor we have:

$2\tilde{C}(X,Y) = \tilde{\nabla}_{JX}(J)Y + \tilde{\nabla}_X(J)(JY) = \nabla_{JX}(J)Y + \psi(JY)JX$
$\qquad +\psi(Y)X + \nabla_X(J)(JY) - \psi(Y)X - \psi(JY)JX$
$\qquad = \nabla_{JX}(J)Y + \nabla_X(J)(JY) = 2C(X,Y)$.

Therefore we proved the following:

Proposition 3.1. The structure tensor does not change through geodesic transformation metric that preserving J. □

Now for the virtual tensor, according to (2.3), we have:

$2\tilde{B}(X,Y) = \tilde{\nabla}_{JX}(J)Y - \tilde{\nabla}_X(J)(JY) = \nabla_{JX}(J)Y + \psi(JY)JX + \psi(Y)X$
$\qquad -\nabla_X(J)(JY + \psi(Y)X + \psi(JY)JX$

Hence

$$\tilde{B}(X,Y) = B(X,Y) + \psi(Y)X + \psi(JY)JX \qquad (3.1)$$

Now we study the properties of the virtual tensor.

Proposition 3.2. The tensor B satisfies the following properties:

1) $JB(X,Y) = B(JX,Y) = -B(X,JY)$;
2) $< B(X,Y),Z >= -< B(X,Z),Y >$;
3) $\sum_{a=1}^{n} B(e_a, e_a) = -(\delta\Omega \circ J)^{\ddagger}$,

where $\{e_1, \dots, e_n\}$ is the basis of $T_p(M)$ as linear complex space, $(\delta\Omega \circ J)^{\ddagger}$ is vector field dual to the form $\delta\Omega \circ J$.

i.e. for each $X \in X(M)$, then $< X, (\delta\Omega \circ J)^{\ddagger} >= (\delta\Omega \circ J)(X)$.

Proof. 1) $2B(JX,Y) = -\nabla_X(J)Y - \nabla_{JX}(J)(JY)$
$2B(X,JY) = \nabla_{JX}(J)(JY) + \nabla_X(J)Y$
$2JB(X,Y) = J(\nabla_{JX}(J)Y - \nabla_X(J)(JY)) = -(\nabla_{JX}(J)(JY) + \nabla_X(J)Y)$
From these relations we directly get the property 1).
2) $2 < B(X,Y), Z >=< \nabla_{JX}(J)Y, Z > -< \nabla_X(J)(JY), Z >=$
$-< \nabla_{JX}(J)Z, Y > +< \nabla_X(J)(JZ), Y >= -2 < B(X,Z), Y >$
3) Note that $2JB(X,Y) = -\nabla_{JX}(J)(JY) - \nabla_X(J)Y$. Thus
$2\sum_{a=1}^{n} JB(e_a, e_a) = -\sum_{a=1}^{n}(\nabla_{Je_a}(J)(Je_a) + \nabla_{e_a}(J)e_a) = -\sum_{i=1}^{2n}\nabla_{e_i}(J)(e_i) =$
$-(\delta\Omega)^{\ddagger}$
Therefore
$\sum_{a=1}^{n} B(e_a, e_a) = J(\delta\Omega)^{\ddagger}$ □ (3.2)

Lemma 3.3. $J(u)^{\ddagger} = -(u \circ J)^{\ddagger}$, where $u \in X^{*}(M)$

Proof. Let $v \in X^{*}(M)$ be any covector. We get:
$v(J(u^{\ddagger})) =< J(u)^{\ddagger}, v^{\ddagger} >= -< u^{\ddagger}, J(v^{\ddagger}) >= -u(J(v^{\ddagger})) = -(u \circ J)(v^{\ddagger})$
$= -< (u \circ J)^{\ddagger}, v^{\ddagger} >= -v((u \circ J)^{\ddagger})$
According to arbitrary choice of v, we get that $J(u^{\ddagger}) = -(u \circ J)^{\ddagger}$. □
Let $\varphi: M \to M$ be a diffeomorphism of pseudo-Riemannian space (M, g), $\tilde{g} = \varphi^{*}g$. So g and $\tilde{g}$ are pseudo-Riemannian structures on M, according to symmetry of the tensor $\tilde{g}$, there is an endomorphism $f \in T_1^1(M)$, such that
$\tilde{g}(X,Y) = g(X, fY), X, Y \in X(M)$ (3.3)
In particular, let g be a Riemannian structure, then $T_p(M)$ admit a basis consists of prober vectors of endomorphism f, i.e.of vectors Y, such that
$fY = \lambda Y,\ \lambda \in \mathbb{R}$. (3.4)
Thus according to (3.1) and the condition 2) of proposition 3.2 we get:
$0 = \tilde{g}(\tilde{B}(X,Y),Y) = \tilde{g}(B(X,Y),Y) + \psi(Y)\tilde{g}(X,Y) + \psi(JY)\tilde{g}(X,JY)$.
Note that, according to (3.3) and (3.4),
$\tilde{g}(B(X,Y),Y) = g(B(X,Y), fY) = \lambda\, g(B(X,Y),Y) = 0$.
So we get $\psi(Y)\ \tilde{g}(X,Y) + \psi(JY)\tilde{g}(X,JY) = 0$
According to nondegeneracy the metric $\tilde{g}$, $\psi(Y)Y + \psi(JY)JY = 0$. Since Y and JY are linearly independent, we obtained $\psi(Y) = 0$. As far as f admits the basis, consisting of prober vectors of this endomorfism, $\psi(Y) = 0$ for each $Y \in X(M)$. Therefore we proved the following theorem:

Theorem 3.4. The almost Hermitian manifold with the definite sign metric does not admit nontrivial holomorphic-geodesic transformation. □
The case of an indefinite metric needs more accurate dissection.

Consider 2- form $\ll X,Y \gg = < X,Y > + < X,JY > J$. Directly we can check this form C-linear by the first argument and C-antilinear by the second argument, also is Hermitian, i.e. $\overline{\ll X,Y \gg} = \ll Y,X \gg$.

Lemma 3.5. Let $p \in M, \{e_1, \dots, e_n\}$ be orthonormal basis of complex linear space $T_m(M)$, structure C-module in which a stander given through the relation $(\alpha + \sqrt{-1}\beta)\xi = \alpha\xi + \beta(J\xi)$; $\xi \in T_m(M)$, then $\forall \xi \in T_m(M)$,
$\sum_{a=1}^{n} \| e_a \|^2 \ll \xi, e_a \gg e_a = \xi$

Proof. $\sum_{a=1}^{n} \| e_a \|^2 \ll \xi, e_a \gg e_a = \sum_{a=1}^{n} \| e_a \|^2 < \xi, e_a > e_a +$
$\| e_a \|^2 \sum_{a=1}^{n} < \xi, Je_a > Je_a = \xi$. This because $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ is an orthonormal basis of $\mathbb{R}$-linear space $T_m(M)$. □

Lemma 3.6. $\ll B(X,Y), Y \gg = 0$

Proof. $\ll B(X,Y), Y \gg = < B(X,Y), Y > + < B(X,Y), JY > J = < B(X,Y), Y >$
$- < JB(X,Y), Y > J = < B(X,Y), Y > - < B(JX,Y), Y > J = 0.$
According to conditions 1) and 2) of proposition 3.2, $< B(X,Y), Y > = 0$. □

Proposition3.7. Let (M, g, J) be *AH*-manifold , then $f \circ J = J \circ f$.

Proof. By (3.3), $\tilde{g}(JX,Y) = g(JX, fY) = -g(X, J \circ fY)$.
On the other hand, $\tilde{g}(JX,Y) = -\tilde{g}(X,JY) = -g(X, f \circ JY)$
According to nondegeneracy the metric $\tilde{g}$, we get $J \circ fY = f \circ JY (Y \in X(M)$, and this means
$f \circ J = J \circ f$. □

Corollary 3.8.Let $h = \ll \cdot,\cdot \gg, \tilde{h} = \varphi^* h$, then $\tilde{h}(X,Y) = \ll X, fY \gg = \ll fX, Y \gg$

Proof. $\tilde{h}(X,Y) = \tilde{g}(X,Y) + \tilde{g}(X,JY)J = < X, fY > + < X, f \circ JY > J$
$= < X, fY > + < X, J \circ fY > J = \ll X, fY \gg$.
On the other hand,
$\ll fX, Y \gg = < fX, Y > + < fX, JY > J = < X, fY > + < X, f \circ JY > J$
$= < X, fY > + < X, J \circ fY > J = \ll X, fY \gg$. □
Note that equation (3.1) can be written as the form:
$\tilde{B}(X,Y) = B(X,Y) + < \psi^\#, Y > X + < \psi^\#, JY > JX = B(X,Y) +$
$\ll \psi^\#, Y \gg X$. Thus we get:
$$\tilde{B}(X,Y) = B(X,Y) + \ll \psi^\#, Y \gg X \quad (3.5)$$
Consider a tensor B^* which is defined as the form:
$\ll B(X,Y), Z \gg = \ll X, B^*(Y,Z) \gg$. Clearly, $B^*(X,Z) = -B^*(Z,X)$, since

$\ll X, B^*(Y,Z) \gg = \ll B(X,Y), Z \gg = -\ll B(X,Z), Y \gg = \ll X, B^*(Z,Y) \gg$, and the metric is nondegenerate.
The tensor $\tilde{B}^*$ has the same properties of the tensor B^*which is defined as the form: $\tilde{h}(\tilde{B}(X,Y),Z) = \tilde{h}\left(X, \tilde{B}^*\,(Y,Z)\right)$. According to these notes we get:

$$0 = \tilde{h}\big(\tilde{B}\,(X,Y),Y\big) = \tilde{h}(B(X,Y),Y) + \ll \psi^\#, Y \gg \tilde{h}(X,Y)$$
$$= \ll B(X,Y), fY \gg + \ll \psi^\#, Y \gg \ll X, fY \gg$$

From this we obtained:
$\ll X, B^*(Y,fY) \gg = -\ll \psi^\#, Y \gg \ll X, fY \gg$, and according to nondegeneracy the metric we get:

$$B^*(Y,fY) = -\ll Y, \psi^\# \gg fY\ . \qquad (3.6)$$

Proposition 3.9. The virtual tensor can be written as the form:

$$2B(X,Y) = \ll \xi, Y \gg X - \ll X, Y \gg \xi + 2B_\circ(X,Y)\ , \qquad (3.7)$$

where ξ is a vector that must be coincide with a vector Lie, $B_\circ$ is a tensor have the properties of virtual tensor such that$\sum_a \parallel e_a \parallel^2 B_\circ(e_a, e_a) = 0$.

Proof. $\sum_a \parallel e_a \parallel^2 B\ \ (e_a, e_a) = \sum_a \parallel e_a \parallel^2 \{\ll \xi, e_a \gg e_a - \ll e_a, e_a \gg\}$
$= \xi - n\xi$.
So we have

$$\xi = \frac{2}{1-n} \sum_a \parallel e_a \parallel^2 B_\circ(e_a, e_a) = \frac{1}{n-1} (\delta\Omega \circ J)^\#.$$

If $\alpha(X) = < \xi, X >$, then $\alpha(X) = \frac{1}{n-1} < (\delta\Omega \circ J)^\#, X > = \frac{1}{n-1}\ \delta\Omega \circ J(X),\ X \in X(M)$.
So we have $\alpha = \frac{1}{1-n}\ \delta\Omega \circ J$ and this means α is Lie form and its dual vector ξ is Lie vector.
Therefore, if this presentation exists, then it is unique and ξ is Lie vector,
Conversely, put $B_\circ(X,Y) = B(X,Y) - \ll \xi, Y \gg X + \ll X, Y \gg \xi +$, where ξ is Lie vector, we obtained:

$$\sum_a \parallel e_a \parallel^2 B_\circ(e_a, e_a) = \sum_a \parallel e_a \parallel^2 B(e_a, e_a) - \sum_a\{\parallel e_a \parallel^2 \ll \xi, e_a \gg$$
$$e_a - n\xi\} = \sum_a B(e_a, e_a) + (n-1)\xi = -(\delta\Omega \circ J)^\#$$
$$+(n-1)\xi = 0.$$

Therefore the proof is complete. □
Note that equation(3.7) can be written as the following form:

$$\ll X, B^*(Y,Z) \gg = \ll B(X,Y), Z \gg = \ll \xi, Y \gg \ll X, Z \gg$$
$$- \ll X, Y \gg \ll \xi, Z \gg + \ll B_\circ(X,Y), Z \gg$$
$$= \ll \xi, Y \gg \ll X, Z \gg - \ll X, Y \gg \ll \xi, Z \gg$$
$$+ \ll X, B_\circ^*(Y,Z) \gg$$

From this and according to nondegeneracy metric,
$B^*(Y,Z) = \ll Y, \xi \gg Z - \ll Z, \xi \gg Y + B_\circ^*(Y,Z)$. Put $Z = fY$, we get:
$B^*(Y,fY) = \ll Y, \xi \gg f(Y) - \ll f(Y), \xi \gg Y + B_\circ^*(Y,fY)$.
In particular put $B_\circ = 0$, so $B^*(Y, f(Y)) = \{\ll Y, \xi \gg Z - \ll Z, \xi \gg Y\}$.
According to (3.6), $\ll Y, \xi + \psi^\# \gg f(Y) - \ll f(Y), \xi \gg Y = 0$.

If f is not a scalar endomorphism, then there is $Y \in X(M)$ such that $\{f(Y), Y\}$ is a linear independent. Thus

$\ll f(Y), \xi \gg = 0$, and since Y is arbitrary element of the basis and f is a nondegenerate , then $\xi = 0$.

If f is a scalar endomorphism, then we get $\ll Y, \psi^{\#} \gg = 0$, so we obtained:

$\psi^{\#} = 0.$

Therefore, in all cases we have $\psi = 0$, and that means, the geodesic transformation is trivial. Note that the condition $B_{\circ} = 0$ equivalent to that *AH*-structure belongs to class U, such that $U \cap W_3 = \{0\}$, this means one of the following classes:

$K, W_1, W_2, W_4, W_1 \oplus W_2, W_1 \oplus W_4, W_2 \oplus W_4, W_1 \oplus W_2 \oplus W_4$ [1].

Therefore we proved the following:

Theorem 3.10. The almost Hermitian manifolds of the above computed of Gry-Harvella classes do not admit nontrivial holomorphic-geodesic transformation.

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