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ON VAISMAN-GRAY MANIFOLD WITH VANISHING CONHARMONIC CURVATURE TENSOR

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Abstract

In the present paper, we investigated the geometric meaning of vanishing conharmonic curvature tensor of Vaisman-Gray manifold. We proved that the conharmonic flat Vaisman-Gray manifold of dimension greater than four is a locally conformal Kähler manifold with vanishing scalar curvature tensor. Finally, an example of nearly Kähler manifold with vanishing conharmonic curvature tensor has been constructed.

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1. Introduction

The Vaisman-Gray manifold is a generalization of the classes nearly Kähler manifold and locally conformal Kähler manifold. Moreover, this class is not coincident with their union; therefore, it represents an interesting study. Most of the almost Hermitian manifold studies were done by using Kozal's operator method, but from our point of view, the study of almost Hermitian manifold by using adjoined *G*-structure space is to be more appropriate.

In [7], Kirichenko defined two tensors which were the structure and virtual tensors, then he used these tensors to find the structure equations of almost Hermitian manifold in adjoined *G*-structure space. Ignatochkina [4] investigated the conformal invariant of Vaisman-Gray manifold, especially she studied the geometric meaning of vanishing the conformal invariant of Vaisman-Gray manifold.

Ishi [5] studied the conharmonic transformation which was a conformal transformation that preserves the harmonicity of a certain function. He introduced a tensor that remains invariant under conharmonic transformation for an *n*-dimensional Riemannian manifold. This tensor is called a *conharmonic curvature tensor*. Kim et al. [6] found the necessary and sufficient condition for the invariance of the space of constant curvature by the conharmonic transformation. Siddiqui and Ahsan [9] investigated the 4-dimensional space-time with vanishing conharmonic curvature tensor. Finally, Abood and Abdulameer [1] studied the geometrical properties of conharmonic curvature tensor of Vaisman-Gray manifold. In particular, they have found the necessary and sufficient condition that flat conharmonic Vaisman-Gray manifold is an Einstein manifold.

2. Preliminaries

Let M be a smooth manifold of dimension $2n\ (n > 1)$, $X(M)$ be an algebra of smooth vector fields. Let $(M, J, g = \langle X, Y\rangle)$ be an almost Hermitian manifold with almost complex structure J and Riemannian metric

g, i.e., $J^2 = -id$, such that $g(JX, JY) = g(X, Y)$; $X, Y \in X(M)$. The pair (J, g) is called an *almost Hermitian structure* on the manifold *M*. All manifolds, tensor fields and other objects are presupposed to be smooth of class C^{∞}.

It is well known that the specifying of the almost Hermitian structure on *M* is equivalent to the specifying of the *G*-structure on *M* with structure group is the unitary group $U(n)$ whose space consists of frames constituted by pairwise conjugate eigenvectors of the structure endomorphism *J*. This frames are unitary in the natural Hermitian metric of the complexification of the corresponding tangent space, such frames are called *A-frames* [7]. This *G*-structure is called an *adjoined*. The integer indices $i, j, k, l, m, \ldots$ vary from 1 to 2*n*, while the integer indices $a, b, c, d, f, h, \ldots$ vary from 1 to *n*. We set $\hat{a} = a + n$. As usual, the putting of the indices on square brackets (respectively, on parentheses) means alternating (respectively, symmetrization) in these indices.

It is known that the giving of the almost Hermitian structure on *M* is equivalent to the giving of the *G*-structure on *M* with the structure group $U(n)$ in a fiber bundle of the frames $B(M)$.

The elements of *G*-structure are called *A-frames*. It can be easily shown [7] that the components of the tensors *g* and *J* on the adjoined *G*-structure space have the following matrices presentation, respectively:

$$(g_{ij}) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad \text{and} \quad (j^i_j) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix},$$

where I_n is the unit matrix of order *n*.

Recall that [3] an almost Hermitian structure $(J, g = \langle X, Y \rangle)$ is called a *structure of class* $W_1 \oplus W_4$ or *Vaisman-Gray structure* if

$$\nabla_X(F)(X, Y) = \frac{-1}{2(n-1)}\{\langle X, X\rangle \delta F(Y) - \langle X, Y\rangle \delta F(X) - \langle JX, Y\rangle \delta F(JX)\},$$

where ∇ is the Riemannian connection of g, $F(X, Y) = \langle JX, Y\rangle$ is the Kähler form, δ is a coderivative and $X, Y \in X(M)$. An almost Hermitian structure $(J, g = \langle X, Y\rangle)$ is called a *structure of class* W_1 or a *nearly Kählerian* if its Kähler form is the Killing form, or, equivalently,

$$\nabla_X(J) = 0, \quad X \in X(M).$$

An almost Hermitian structure $(J, g = \langle X, Y\rangle)$ is called a *structure of* class W_4 if

$$\nabla_X(F)(Y, Z) = \frac{-1}{2(n-1)} \times \{\langle X, Y\rangle\delta F(Z) - \langle X, Z\rangle\delta F(Y)$$
$$- \langle X, JY\rangle\delta F(JZ) + \langle X, JZ\rangle\delta F(JY)\}.$$

For each almost Hermitian manifold, in particular for Vaisman-Gray manifold, defined a Lie form by the formula

$$\alpha = \frac{1}{n-1}\delta F \circ J.$$

Let $\{\omega^i\}$ be the components of the solder form, $\{\omega^i_j\}$ be the components of the connection form for Riemannian metric g. Recall that a component of a tensor field t on the adjoined G-structure space is given by the functions

$$t^{i_1\cdots i_r}_{j_1\cdots j_s} = t_m(e_{j_1}, \ldots, e_{j_s}, e^{i_1}, \ldots, e^{i_r}),$$

where $p = (m, e_{j1}, \ldots, e_{js})$ is any frame, $m \in M$ and $(e^{i1}, \ldots, e^{i_r})$ is the dual base.

By $\{j^i_{j,k}\}$, we denote the components of covariant differential for almost complex structure. Then we have four groups of functions on adjoined G-structure space

$$B^{abc} = \frac{\sqrt{-1}}{2} J^a_{[\hat{b}, \hat{c}]}, \quad B_{abc} = \frac{-\sqrt{-1}}{2} J^{\hat{a}}_{[b, c]},$$

$$B^{ab}{}_{c} = \frac{-\sqrt{-1}}{2} J^{a}_{\hat{b},c}, \quad B_{ab}{}^{c} = \frac{\sqrt{-1}}{2} J^{\hat{a}}_{b,\hat{c}}.$$

It is proved [7] that these functions define four complex tensors, so there are the following differential equations:

$$dB^{abc} - B^{dbc}\omega^{a}_{d} - B^{adc}\omega^{b}_{d} - B^{abd}\omega^{c}_{d} = B^{abcd}\omega_{d} + B^{abc}{}_{d}\omega^{d};$$

$$dB_{abc} + B_{dbc}\omega^{d}_{a} + B_{adc}\omega^{d}_{c} + B_{abd}\omega^{c}_{d} = B_{abcd}\omega^{d} + B_{abc}{}^{d}\omega_{d};$$

$$dB^{ab}{}_{c} - B^{db}{}_{c}\omega^{a}_{d} - B^{ad}{}_{c}\omega^{b}_{d} + B^{ab}{}_{d}\omega^{d}_{c} = B^{ab}{}_{cd}\omega^{d} + B^{ab}{}_{c}{}^{d}\omega_{d};$$

$$dB_{ab}{}^{c} + B_{db}{}^{c}\omega^{d}_{a} + B_{ad}{}^{c}\omega^{d}_{b} - B_{ab}{}^{d}\omega^{c}_{d} = B^{cd}{}_{ab}\omega_{d} + B_{ab}{}^{c}{}_{d}\omega^{d},$$

where $\{B^{abcd}, B_{abcd}, B^{abc}, B_{abc}, B^{ab}{}_{c}{}^{d}, B_{ab}{}^{c}{}_{d}, B^{ab}{}_{cd}, B^{cd}{}_{ab}\}$ are some functions on adjoined G-structure space. The tensors $\{B^{abc}\}$ and $\{B_{abc}\}$ are called the *structure tensors* and the tensors $B^{ab}{}_{c}$ and $B_{ab}{}^{c}$ are called the *virtual tensors*. It is obvious that

$$\overline{B}^{abc} = B_{abc}; \quad \overline{B}^{ab}{}_{c} = B_{ab}{}^{c}.$$

It is known [4] that the structure equations of Riemannian connection of the Vaisman-Gray structure on adjoined G-structure space which are called the *structure equations* of Vaisman-Gray structure have the forms

$$d\omega^{a} = \omega^{a}_{b} \wedge \omega^{b} + B^{ab}{}_{c}\omega^{c} \wedge \omega_{b} + B^{abc}\omega_{b} \wedge \omega_{c};$$

$$d\omega_{a} = -\omega^{b}_{a} \wedge \omega_{b} + B_{ab}{}^{c}\omega_{c} \wedge \omega^{b} + B_{abc}\omega^{b} \wedge \omega^{c};$$

$$\begin{aligned} d\omega^{a}_{b} = {} & \omega^{a}_{c} \wedge \omega^{c}_{b} + (2B^{adh}B_{hbc} + A^{ad}_{bc})\omega^{c}_{d} \\ & + (B^{ah}_{c}B_{dbh} + A^{a}_{bcd})\omega^{c} \wedge \omega^{d} \\ & + (B^{c}_{bh}B^{dah} + A^{acd}_{b})\omega_{c} \wedge \omega_{d}, \end{aligned}$$

where $\omega_a = \omega^{\hat{a}}$, $\{A^{ad}_{bc}, A^{a}_{bcd}, A^{acd}_{b}\}$ are some functions on adjoined G-structure space. The functions $\{A^{ad}_{bc}\}$ define tensor field on the manifold M, this tensor is called a *tensor of holomorphic sectional curvature*. It is known that $\overline{A}^{ad}_{bc} = A^{bc}_{ad}$.

Lemma 2.1 [2]. *An almost Hermitian manifold is Vaisman-Gray manifold if and only if*

$$B^{[abc]} = B^{abc}; \quad B_{[abc]} = B_{abc}; \quad B^{ab}{}_{c} = \alpha^{[a}\delta^{b]}_{c}; \quad B_{ab}{}^{c} = \alpha_{[a}\delta^{c}_{b]},$$

where $\{\alpha_a, \alpha^a \equiv \alpha_{\hat{a}}\}$ *are the components of the Lie form*.

Using Lemma 2.1 and the classification of almost Hermitian manifold [2], we find the following proposition directly:

Proposition 2.1. (i) *A Vaisman-Gray manifold M of dimension* $2n > 2$ *is nearly Kählerian manifold if and only if* $\alpha_a = \alpha^a = 0$.

(ii) *A Vaisman-Gray manifold M of dimension* $2n > 2$ *is manifold of the class* W_4 *if and only if* $B_{abc} = B^{abc} = 0$.

(iii) *A Vaisman-Gray manifold M of dimension* $2n > 2$ *is Kählerian manifold if and only if* $\alpha_a = \alpha^a = 0$ *and* $B_{abc} = B^{abc} = 0$.

Lemma 2.2 [4]. *In the adjoined G-structure space, the components of Riemannian curvature tensor R of Vaisman-Gray manifold are given as the following forms*:

(i) $R_{abcd} = 2(B_{ab[cd]} + \alpha_{[a}B_{b]cd});$

(ii) $R_{\hat{a}bcd} = 2A^{a}_{bcd};$

(iii) $R_{\hat{a}\hat{b}cd} = 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]});$

(iv) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d},$

where $\{\alpha^{a}{}_{b},\ \alpha_{a}{}^{b},\ \alpha_{ab},\ \alpha^{ab}\}$ *are some functions on adjoined G-structure space such that*

$$d\alpha_a + \alpha_b \omega_a^b = \alpha_a^b + \alpha_{ab}\omega^b; \quad d\alpha^a \alpha^{\ b}\omega_b^a = \alpha^{a}{}_{b} + \alpha^{ab}\omega_b.$$

The others components of Riemannian curvature tensor R can be obtained by the properties of the symmetry for R. It is proved [4] that for all Vaisman-Gray manifolds of dimension $2n > 4$, the following equality is fair:

$$\alpha^{a}{}_{b} = \alpha_{b}{}^{a}.$$

We shall denote $\alpha^{a}{}_{b} = \alpha_{b}{}^{a} = \alpha_b^a$ for Vaisman-Gray manifold of dimension $2n > 4$.

Lemma 2.3 [4]. *In the adjoined G-structure space, the components of Ricci tensor* $r_{ij} = R_{ijk}^{k}$ *of Vaisman-Gray manifold are given as the following forms*:

(i)

$$r_{ab} = \frac{1-n}{2}(\alpha_{ab} + \alpha_{ba} + \alpha_a\alpha_b);$$

(ii)

$$r_{\hat{a}b} = r_b^a = 3B^{cah}B_{cbh} - A_{bc}^{ca} + \frac{n-1}{2}(\alpha^a\alpha_b - \alpha^h\alpha_h) - \frac{1}{2}\alpha^{h}{}_{h}\delta_b^a + (n-2)\alpha^{a}{}_{b}.$$

Lemma 2.4 [4]. *The scalar curvature tensor* $\chi = g^{ij}r_{ij}$ *is given as the following form*:

$$\chi = 6B^{abc}B_{abc} - 2A_{ba}^{ab} - \frac{(n-1)^2}{2}\alpha^h\alpha_h - 2(n-1)\alpha^{h}{}_{h}.$$

Let f be a smooth function on a manifold M. The almost Hermitian manifold $\{M,\ J,\ \tilde{g} = e^{2f}g\}$ is called a *conformal transformation* of almost Hermitian manifold. The basic invariant of conformal transformation is a conformal curvature tensor W. The conformal curvature tensor is given by the following formula [8]:

$$W_{ijkl} = R_{ijkl} + \frac{1}{2(n-1)}(r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il})$$

$$+ \frac{\chi(g_{jk}g_{il} - g_{jl}g_{ik})}{(2n-1)(2n-2)}.$$

It is well known [8] that the Riemannian manifold is conformal flat if and only if $W = 0$.

Lemma 2.5 [4]**.** *In the adjoined G-structure space, the components of conformal curvature tensor of Vaisman-Gray manifold are given as the following forms*:

(i) $W_{abcd} = R_{abcd}$;

(ii) $W_{\hat{a}bcd} = R_{abcd} - \frac{1}{2(n-1)}(r_{bc}\delta_d^a - r_{bd}\delta_c^a)$;

(iii) $W_{\hat{a}\hat{b}cd} = R_{\hat{a}\hat{b}cd} + \frac{2}{(n-1)}r^{[a}_{[c}\delta^{b]}_{d]} - \frac{\chi^{ab}_{cd}}{2(n-1)(2n-1)}$;

(iv) $W_{\hat{a}bc\hat{d}} = R_{\hat{a}bcd} + \frac{1}{2(n-1)}(r_c^a\delta_b^d + r_b^d\delta_c^a) - \frac{\chi\delta_c^a\delta_{bc}^{ad}}{2(n-1)(2n-1)}$,

where $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$.

It is well known that the conformal curvature tensor has properties which are similar to the properties of Riemannian curvature tensor. Therefore, W has four basic components; W_{abcd}, $W_{\hat{a}bcd}$, $W_{\hat{a}\hat{b}cd}$, $W_{\hat{a}bc\hat{d}}$.

Lemma 2.6 [4]**.** *The vanishing of the basic components of conformal curvature tensor defines four conformal invariant classes* C_0, C_1, C_2, C_3 *of Vaisman-Gray manifold,*

(i) $M \in C_0$ *if and only if* $W_{abcd} = 0$;

(ii) $M \in C_1$ *if and only if* $W_{\hat{a}bcd} = 0$;

(iii) $M \in C_2$ *if and only if* $W_{\hat{a}\hat{b}cd} = 0$;

(iv) $M \in C_3$ *if and only if* $W_{\hat{a}bc\hat{d}} = 0$.

Lemma 2.7 [4]**.** *For Vaisman-Gray manifold, we have* $C_1 \subseteq C_0$ *and* $C_3 \subseteq C_2$.

Lemma 2.8 [4]**.** (i) *For Vaisman-Gray manifold of dimension* $2n > 4$, *the class* C_1 *coincides with class of locally conformal nearly Kählerian manifolds such that each point of* $M \in C_1$ *has a neighborhood U and a smooth function f for which* $\{U, J, e^{2f}\}$ *is a nearly Kählerian manifold.*

(ii) *Any manifold of class* W_4 *of dimension* $2n > 4$ *is locally conformal Kählerian manifold.*

3. Main Results

Definition 3.1 [5]**.** The conformal transformation of Riemannian manifold is called a *conharmonic transformation* if any harmonic function changes to a harmonic function under this transformation.

Definition 3.2 [5]**.** A tensor *T* of type (4, 0) which is invariant under conharmonic transformation and defined by the formula

$$T_{ijkl} = R_{ijkl} - \frac{1}{2(n-1)}\left(r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il}\right)$$

is called a *conharmonic curvature tensor*, where *r*, *R* and *g* are, respectively, Ricci tensor, Riemannian curvature tensor and Riemannian metric.

It is known [5] that the Riemannian manifold is conharmonic flat if and only if $T \equiv 0$.

Lemma 3.1. *Let any Riemannian manifold* $M(\dim M > 2)$ *be a conharmonic flat. Then M is conformal flat and the scalar curvature tensor of M is vanished.*

Let M be any conharmonic flat Riemannian manifold M with $\dim M > 2.$ By using the definition of conharmonic transformation, we get that M is conformal flat.

Definition 3.3. Let M be Vaisman-Gray manifold of dimension $2n > 2.$ we have four basic components in adjoined G-structure space, namely T_{abcd}, $T_{\hat{a}bcd}$, $T_{\hat{a}\hat{b}cd}$ and $T_{\hat{a}bc\hat{d}}$. So we can define the following conharmonically invariants of Vaisman-Gray manifold:

(i) $M \in T_0$ if and only if $T_{abcd} = 0;$

(ii) $M \in T_1$ if and only if $T_{\hat{a}bcd} = 0;$

(iii) $M \in T_2$ if and only if $T_{\hat{a}\hat{b}cd} = 0;$

(iv) $M \in T_3$ if and only if $T_{\hat{a}bc\hat{d}} = 0.$

Obviously, $T_0 \cap T_1 \cap T_2 \cap T_3$ is the class of conharmonic flat Vaisman-Gray manifolds.

Lemma 3.2. *By the direct calculation, we get*

(i) $T_{abcd} = R_{abcd};$

(ii) $T_{\hat{a}bcd} = R_{\hat{a}bcd} - \frac{1}{2(n-1)}(r_{bc}\delta_d^a - r_{bd}\delta_c^a);$

(iii) $T_{\hat{a}\hat{b}cd} = R_{\hat{a}\hat{b}cd} + \frac{2}{(n-1)} r_{[c}^{[a}\delta_{d]}^{b]};$

(iv) $T_{\hat{a}bc\hat{d}} = R_{\hat{a}bc\hat{d}} + \frac{1}{2(n-1)}(r_c^a\delta_b^d + r_b^d\delta_c^a).$

According to Definition 3.3, Lemmas 2.6-2.8, we get the following result:

Theorem 3.1. (i) *In the class of Vaisman-Gray manifolds of dimension* $2n > 2,$ $T_0 = C_0,$ $T_1 \subset C_1,$ $T_1 \subset T_0$ *and* $T_1 \subset T_2.$

(ii) *In the class of Vaisman-Gray manifolds of dimension* $2n > 4$, T_1 *coincides with the class of the locally conformal nearly Kählerian manifolds.*

Remark 3.1. According to Definition 3.3 and Lemma 2.6, we get $T_2 = C_2$ and $T_3 = C_3$ if and only if the scalar curvature tensor of M is vanished.

Theorem 3.2. *Let M be a Vaisman-Gray manifold of dimension* $2n > 2$. *If M is a manifold of class* T_3, *then its scalar curvature tensor not positive.*

Proof. Let M be Vaisman-Gray manifold of class T_3 of dimension $2n > 2$. Therefore, $M \in T_2$. Using Definition 3.3 and Lemma 2.3, we get:

$$-2B^{abh}B_{hcd} + \frac{n}{n-1}\alpha^{[a}_{[c}\delta^{b]}_{d]} + \frac{6}{n-1}B^{[a}_{[c}\delta^{b]}_{d]} - \frac{2}{n-1}A^{[a}_{[c}\delta^{b]}_{d]}$$

$$-\frac{1}{4}\alpha^h\alpha_h\delta^{ab}_{cd} + \frac{1}{2}\alpha^{[a}\alpha_{[c}\delta^{b]}_{d]} - \frac{1}{2}\alpha^h\alpha_h\delta^{ab}_{cd} = 0, \tag{1}$$

$$A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \frac{1}{2(n-1)}(3B^d_b\delta^d_c - 3B^a_c\delta^d_b - A^d_b\delta^a_c - A^a_c\delta^d_b$$

$$-\frac{1}{8}(2\alpha^h\alpha_h\delta^a_c\delta^d_b - \alpha^d\alpha_b\delta^a_c - \alpha^a\alpha_c\delta^d_b) - \frac{1}{4(n-1)}(\alpha^h{}_h\delta^a_c\delta^d_b + (n-2)\alpha^d{}_b\delta^a_c$$

$$+(n-2)\alpha^a{}_c\delta^d_b) = 0, \tag{2}$$

where $A^a_c = A^{ah}_{hc}$, $B^a_c = B^{ahf}B_{fch}$, $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^b_c\delta^a_d$.

Contracting (1) by indexes a, c and b, d, we get:

$$B - A - \frac{(n-1)^2}{4}\alpha^h\alpha_h = 0, \tag{3}$$

where $B = B^h_h$, $A - A^{jh}_{hf}$.

Contracting (2) by indexes a, c and b, d, it follows that

$$-A + (2n+1)B - \frac{(n-1)^2}{4}\alpha^h\alpha_h - n(n-1)\alpha^h_h = 0. \tag{4}$$

Combining (3) and (4), we obtain

$$B = \frac{n-1}{2}\alpha_h^h A = B - \frac{(n-1)}{4}\alpha^h \alpha_h.$$

Making use of Lemma 2.4, we get

$$\chi = -2nB.$$

We see that $B = \sum |B_{abc}|^2 \geq 0,$ therefore, $\chi \leq 0.$

And this completes the proof. □

Theorem 3.3. *Any conharmonic flat Vaisman-Gray manifold of dimension $2n > 4$ is a locally conformal Kählerian manifold with the vanishing scalar curvature tensor.*

Proof. Let M be a conharmonic flat Vaisman-Gray manifold of dimension $2n > 4,$ therefore, $M \in T_1$ and $M \in T_3.$ By using Theorem 3.1, we get M is locally conformal nearly Kählerian manifold. According to Theorem 3.2 and Remark 3.1, we obtain $B = 0,$ i.e., $B_{abc} = B^{abc} = 0,$ therefore M is locally conformal nearly Kählerian manifold and $\chi = 0.$ □

Lemma 3.3 [6]**.** *The function f defines a conharmonic transformation if and only if* $\tilde{\chi} = e^{-2f}\chi.$

Theorem 3.4. *Let M be a conharmonic flat locally conformal nearly Kählerian manifold. Then the function f that defines the local conformal equivalence between M and Kählerian manifold is the function of conharmonic transformation.*

Proof. If M is conharmonic flat, then it is conformal flat manifold. According to [4], we get the function *f* locally transforms M to the complex Euclidean space $C^n.$ We know that C^n is the flat manifold. In particular, it has a vanishing scalar curvature tensor. Then we have $\tilde{\chi} = e^{-2f}\chi,$ where $\tilde{\chi}$ is the scalar curvature tensor of $C^n.$ By using Lemma 3.3, we get that *f* is the function of conharmonic transformation. □

Theorem 3.5. *Any conharmonic flat nearly Kählerian manifold of dimension* $2n > 4$ *is locally holomorphically isometric to the complex Euclidean space* C^n *with canonical Kählerian structure.*

Proof. Let M be a conharmonic flat nearly Kählerian manifold.

Using Theorem 3.3, we get $B_{abc} = B^{abc} = 0.$

According to Proposition 2.1, we conclude that M is Kählerian manifold.

Applying (1) and (2) for the nearly Kählerian manifold, we obtain

$$-A_d^a \delta_c^b + A_d^b \delta_c^a - A_c^b \delta_d^a + A_c^a \delta_d^b = 0, \tag{5}$$

$$A_{bc}^{ad} + \frac{1}{2(n-1)}(-A_b^d \delta_c^a - A_c^a \delta_b^d) = 0. \tag{6}$$

Contracting (5) by indexes b and c, it follows that $(n-2)A_d^a = 0.$

If the dimension M is $2n > 4,$ then $A_d^a = 0.$ Substituting this formula in (6), we get $A_{bc}^{ad} = 0.$

Using Lemma 2.2, we get $R_{ijkl} = 0.$

Therefore, M is a flat Kählerian manifold, i.e., it is locally holomorphically isometric to C^n with canonical Kählerian structure. □

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