On Permutation G-part in Permutation Q-algebras

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ABSTRACT

Some of the novel algebraic topics explored in this paper, like Permutation Q - algebra, permutation G - part, permutation p-radical, permutation p - semisimple and permutation ideal are discussed and looked into. We show that if $(X, \#, \{1\})$ is a permutation Q - algebra, then $\left(\lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_j^\beta\right)\right) \# \lambda_j^\beta = \{1\}, \forall \lambda_i^\beta, \lambda_j^\beta \in X$. Also, in the permutation $G-\text{part}$ $G(X)$ of X, the left cancellation law is hold and for any β - set λ_i^β in Permutation Q - algebra $(X, \#, \{1\})$, we consider that λ_i^β belongs to $G(X)$ iff $\{1\} \# \lambda_i^\beta$ belongs to $G(X)$. Additionally permutation implicative, homomorphism, kernel and image of permutation Q - algebras were defined with specific results relating to our unique notions have been developed and examined.

Keywords: Permutation sets, Q-algebra, permutation Q-algebra, permutation $G-\text{part}$, permutation implicative.

1. INTRODUCTION

BCK/BCI-algebras were introduced by Iseki and Tanaka[1] and Iseki[2]. BCK-algebra is a proper subclass of BCI-algebra, as is well known. J. Neggers and H. S. Kim[3] established the concept of d-algebras, which is a useful generalization of BCK-algebras, and then examined numerous relationships between d-algebras and BCK-algebras, as well as several other intriguing relationships between d-algebras and oriented digraphs.

In 2021, the notion of Q-algebra is introduced and analyzed some of the features of a Q-algebra by Neggers et al[4]., which appear to be of some interest. Different fields[5-12] examine symmetric and alternating groups, as well as their permutations. Some kinds of pure mathematics using different types of the sets, such as permutation sets[13], fuzzy sets[14-20], and soft sets [21-26], have been researched and their properties debated and discussed their properties in recent years.

A number of the new algebraic problems investigated in this work, like Permutation Q-algebra, permutation $G-\text{part}$, permutation p-radical, permutation p-semisimple and permutation ideal are discussed and looked into. Additionally permutation implicative, homomorphism, kernel and image of permutation Q-algebras were defined with specific results relating to our unique notions have been developed and examined.

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2. PRELIMINARY

The definitions of Q-algebra and permutation sets are based on the[5, 13]. They are reviewed in this section.

Definition 2.1: Assume that $X \neq \emptyset$ and 0 is a fixed member with a map $*: X \times X \to X$. Then $(X,*,\emptyset)$ is called a $Q-$ algebra^4 if it satisfies the following conditions:

$(a) - x * x = 0,$

$(b) - x * 0 = x,$

$(c) - (x * y) * z = (x * z) * y, \ \ \forall x, y, z \in X.$

Definition 2.2:

For any permutation $\beta = \prod_{i=1}^{c(\beta)} \lambda_i$ in a symmetric group S_n, where $\lambda_i = (t_1^i, t_2^i, \dots, t_{\alpha_i}^i), 1 \leq i \leq c(\beta)$, for some $1 \leq \alpha_i, c(\beta) \leq n$. If $\lambda = (t_1, t_2, \dots, t_k)$ is $k-$cycle in S_n, $\beta-$set is defined as $\lambda^\beta = \{t_1, t_2, \dots, t_k\}$ and is called $\beta-$set of cycle λ. Then $\beta-$sets of $\{\lambda_i\}_{i=1}^{c(\beta)}$ are defined by $\{\lambda_i^\beta = \{t_1^i, t_2^i, \dots, t_{\alpha_i}^i\} | 1 \leq i \leq c(\beta)\}$.

III. ON PERMUTATION Q-ALGEBRA

Definition 3.1: Assume that H is a family of $\beta-$ sets $\left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)}$, where β is a permutation in the symmetric group $G = S_n$. Let $X = H \cup \{1\}$, where $\{1\}$ is constant in X and let

$\#: X \times X \longrightarrow X$ be a map. Then $(X, \#, \{1\})$ is called a permutation $Q-$ algebra $(PQ-A)$, if $\#$ such that:

$(1) - \lambda_i^\beta \# \lambda_i^\beta = \{1\},$

$(2) - \lambda_i^\beta \# \{1\} = \lambda_i^\beta,$

$(3) - \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_k^\beta = \left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \lambda_j^\beta, \ \forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X.$

Example 3.2:

Assume that (S_{12},o) is symmetric group and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 3 & 9 & 2 & 8 & 7 & 10 & 6 & 1 & 11 & 12 & 5 \end{pmatrix}$ be a permutation in S_{11}. We have $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 3 & 9 & 1 & 8 & 6 & 5 & 7 & 2 & 11 & 12 & 10 \end{pmatrix} = (1\ 4\ 2\ 3\ 9)(5\ 8\ 6\ 7\ 10\ 11\ 12)$. Therefore, we have $X = \left\{\lambda_i^\beta\right\}_{i=1}^{5} \cup \{1\} = \{\{1,4,2,3,9\}, \{5,8,6,7,10,11,12\}, \{1\}\}$. Define $\#: X \times X \longrightarrow X$ by table (1)

Table (1): $(X, \#, T)$ is a $(PQ-A)$.

#	{1}	{1,4,2,3,9}	{5,8,6,7,10,11,12}
{1}	{1}	{1,4,2,3,9}	{5,8,6,7,10,11,12}
{1,4,2,3,9}	{1,4,2,3,9}	{1}	{5,8,6,7,10,11,12}
{5,8,6,7,10,11,12}	{5,8,6,7,10,11,12}	{1,4,2,3,9}	{1}

Then we see that $\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_k^\beta = \left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \lambda_j^\beta$. Then $(X, \#, \{1\})$ is a $(PQ-A)$.

Definition 3.3: Assume that $(X, \#, \{1\})$ is a $(PQ-A)$ X, then $\lambda_i^\beta \leq \lambda_j^\beta \leftrightarrow \lambda_i^\beta \# \lambda_j^\beta = \{1\}$.

Proposition 3.4: Suppose that $(X, \#, \{1\})$ is a $(PQ-A)$, then $\left(\lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_j^\beta\right)\right) \# \lambda_j^\beta = \{1\}, \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Proof: Now, $\left(\lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_j^\beta\right)\right) \# \lambda_j^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\lambda_i^\beta \# \lambda_j^\beta\right)$ (From (3) of Definition 3.1)

$= \{1\}$ (From (1) of Definition 3.1)

Proposition 3.5: Every $(PQ-A)(X,\#,\{1\})$ satisfying

$$\lambda_i^\beta \# (\lambda_i^\beta \# \lambda_j^\beta) = \lambda_i^\beta \# \lambda_j^\beta, \qquad \forall \lambda_i^\beta, \lambda_j^\beta \in X$$

Is a trivial permutation algebra.

Proof: Substituting $\lambda_i^\beta = \lambda_j^\beta$ into the equation

$$\lambda_i^\beta \# (\lambda_i^\beta \# \lambda_j^\beta) = \lambda_i^\beta \# \lambda_j^\beta,$$

We have that

$$\lambda_i^\beta \# (\lambda_i^\beta \# \lambda_i^\beta) = \lambda_i^\beta \# \lambda_i^\beta$$

$$\Longrightarrow \qquad \lambda_i^\beta \# \{1\} = \{1\} \qquad \text{(From (1) of Definition 3.1)}$$

$$\Longrightarrow \qquad \lambda_i^\beta = \{1\} \qquad \text{(From (2) of Definition 3.1)}$$

Thus $(X,\#,\{1\})$ is a trivial permutation group.

Proposition 3.6:

Every $(PQ-A)$ $(X,\#,\{1\})$ such that $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_k^\beta = \lambda_i^\beta \# (\lambda_j^\beta \# \lambda_k^\beta)$, $\forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$ is a group.

Proof: Substituting $\lambda_i^\beta = \lambda_j^\beta = \lambda_k^\beta$ into the associative lawa

$$(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_k^\beta = \lambda_i^\beta \# (\lambda_j^\beta \# \lambda_k^\beta)$$

We have that

$$(\lambda_i^\beta \# \lambda_i^\beta) \# \lambda_i^\beta = \lambda_i^\beta \# (\lambda_i^\beta \# \lambda_i^\beta)$$

$$\Longrightarrow \qquad \{1\} \# \lambda_i^\beta = \lambda_i^\beta \# \{1\} \quad \text{(From (1) of Definition 3.1)}$$

$$\Longrightarrow \qquad = \lambda_i^\beta \qquad \text{(From (2) of Definition 3.1)}$$

Therefore $\{1\}$ is the identity of $(X,\#,\{1\})$. From (1) of Definition 3.1, we have $\forall \lambda_i^\beta \in X$, $\exists {\lambda_i^\beta}^{-1} = \lambda_i^\beta \in X$. Therefore, $(X,\#)$ is a group.

Lemma 3.7: If $(X,\#,\{1\})$ is a $(PQ-A)$and $\lambda_i^\beta \# \lambda_j^\beta = \lambda_i^\beta \# \lambda_k^\beta$, $\forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$, then $\{1\} \# \lambda_j^\beta = \{1\} \# \lambda_k^\beta$.

Proof: Now, $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_i^\beta = (\lambda_i^\beta \# \lambda_i^\beta) \# \lambda_j^\beta$ (From (3) of Definition 3.1)

$$= \{1\} \# \lambda_j^\beta \qquad \text{(From (2) of Definition 3.1)}$$

And

$$(\lambda_i^\beta \# \lambda_k^\beta) \# \lambda_i^\beta = (\lambda_i^\beta \# \lambda_i^\beta) \# \lambda_k^\beta \qquad \text{(From (3) of Definition 3.1)}$$

$$= \{1\} \# \lambda_k^\beta$$

Since $\lambda_i^\beta \# \lambda_j^\beta = \lambda_i^\beta \# \lambda_k^\beta$, then

$$\{1\} \# \lambda_j^\beta = \{1\} \# \lambda_k^\beta.$$

Definition 3.8: Assume that $(X,\#,\{1\})$ is a $(PQ-A)$ & $\emptyset \neq S \subseteq X$. Let $G(S) = \left\{\lambda_i^\beta \in S \mid \{1\} \# \lambda_i^\beta = \lambda_i^\beta\right\}$. We say $G(X)$ is *permutation* $G-part$ of X, whenever $S = X$.

Corollary 3.9: In $G(X)$, the left cancellation law is held.

Proof: Assume that $\lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in G(X)$ with

$$\lambda_i^\beta \# \lambda_j^\beta = \lambda_i^\beta \# \lambda_k^\beta.$$

From Lemma 3.7, we have that

$$\{1\} \# \lambda_j^\beta = \{1\} \# \lambda_k^\beta.$$

Since $\lambda_j^\beta, \lambda_k^\beta \in G(X)$, we have that

$$\lambda_j^\beta = \lambda_k^\beta.$$

Proposition 3.10: $\lambda_i^\beta \in G(X) \leftrightarrow \{1\} \# \lambda_i^\beta \in G(X)$, where $(X,\#,\{1\})$ is a $(PQ-A)$.

Proof: Suppose that $G(X)$ has λ_i^β, then $\{1\} \# \lambda_i^\beta = \lambda_i^\beta$ & $\{1\} \# (\{1\} \# \lambda_i^\beta) = \{1\} \# \lambda_i^\beta$. Hence

$$\{1\} \# \lambda_i^\beta \in G(X).$$

Conversely, if $\{1\} \# \lambda_i^\beta \in G(X)$, then $\{1\} \# (\{1\} \# \lambda_i^\beta) = \{1\} \# \lambda_i^\beta$. From Corollary 3.9, we have that $\{1\} \# \lambda_i^\beta = \lambda_i^\beta$. Therefore, $\lambda_i^\beta \in G(X)$.

Definition 3.11: Assume that $(X, \#, \{1\})$ is a $(PQ-A)$, then $B(X)=\left\{\lambda_i^\beta \in X \mid \{1\} \# \lambda_i^\beta = \{1\}\right\}$ is said to be permutation $\mathrm{p}-\mathrm{radical}$ of X. If $B(X)=\left\{\{1\}\right\}$, then $(X, \#, \{1\})$ is called permutation semi $\mathrm{p}-\mathrm{semisimple}$ $\mathrm{Q}-$ algebra $(PPSQ-A)$.

Remark 3.12: $G(X) \cap B(X)=\left\{\{1\}\right\}$.

Proposition 3.13: Assume that $(X, \#, \{1\})$ is a $(PQ-A)$ & $\lambda_i^\beta, \lambda_j^\beta \in X$, then

$$\lambda_j^\beta \in B(X) \iff \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_i^\beta = \{1\}.$$

Proof: Now

$$\begin{aligned}\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_i^\beta &= \left(\lambda_i^\beta \# \lambda_i^\beta\right) \# \lambda_j^\beta \quad \text{(From (3) of Definition 3.1)}\\ &= \{1\} \# \lambda_j^\beta \quad \text{(From (3) of Definition 3.1)}\\ &= \{1\} \leftrightarrow \lambda_j^\beta \in B(X).\end{aligned}$$

Definition 3.14: Suppose that $(X, \#, \{1\})$ is a $(PQ-A)$ & $I(\neq \emptyset) \subseteq X$. We say I is a permutation $\mathrm{Q}-$ ideal $(PQ-I)$of X if $\forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$,

(1) $\{1\} \in I$,

(2) $\lambda_i^\beta \# \lambda_j^\beta \in I$ and $\lambda_j^\beta \in I \Longrightarrow \lambda_i^\beta \in I$.

Moreover, each one of X and $\left\{\{1\}\right\}$ is $(PQ-I)$ of X. Then X and $\left\{\{1\}\right\}$ are called *trivial and fixed* of X, respectively. A $(PQ-I)$ is said to be *proper* if I does not equal X. Also, for any $I(\neq \emptyset) \subseteq X$. We say I is a permutation subalgebra of X if $\lambda_i^\beta \# \lambda_j^\beta \in I, \forall \lambda_i^\beta, \lambda_j^\beta \in I$.

Proposition 3.15: Assume $(X, \#, \{1\})$ is a $(PQ-A)$. Then $B(X)$ is a $(PQ-I)$of X.

Proof: Since $(\{1\} \# \{1\}) \# \{1\} = \{1\}$, from Proposition 3.13, $\{1\} \in B(X)$.

Let $\lambda_i^\beta \# \lambda_j^\beta \in B(X)$ and $\lambda_j^\beta \in B(X)$. Then by Proposition 3.13,

$$\left(\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_i^\beta\right) \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) = \{1\}.$$

By (3) of Definition 3.1, we have that

$$\begin{aligned}\left(\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\lambda_i^\beta \# \lambda_j^\beta\right)\right) \# \lambda_i^\beta &= \{1\} \# \lambda_i^\beta \quad \text{(From (2) of Definition 3.1)}\\ &= \{1\}\end{aligned}$$

Hence $\lambda_i^\beta \in B(X)$. Therefore $B(X)$ is a $(PQ-I)$ of X.

Proposition 3.16: Suppose that H is permutation subalgebra of a $(PQ-A)$ $(X, \#, \{1\})$, then $G(X) \cap H = G(H)$.

Proof: Since $G(X) \cap H \subseteq G(H)$, so $\{1\} \# \lambda_i^\beta = \lambda_i^\beta$ & $\lambda_i^\beta \in H \subseteq X$ whenever $\lambda_i^\beta \in G(S)$. Hence $\lambda_i^\beta \in G(X)$ and this implies $\lambda_i^\beta \in G(X) \cap H$.

Proposition 3.17: Assume that $(Y, \#, \{1\})$ is a $(PQ-A)$. If $Y = G(Y)$, then Y is a $(PPSQ-A)$.

Proof: Assume that $G(Y) = Y$. By Remark 3.12

$$\{1\} = G(Y) \cap B(Y) = Y \cap B(Y) = B(Y).$$

Hence Y is a $(PPSQ-A)$.

Proposition 3.18: Suppose that $(Y, \#, \{1\})$ is a $(PQ-A)$ of order 3, then $|G(Y)| \neq 3$, that is, $G(Y) \neq Y$.

Proof: Assume that $X = \left\{\{1\}, \lambda_i^\beta, \lambda_j^\beta\right\}$ is a $(PQ-A)$ with $|G(Y)| = 3$, so $G(Y) = Y$. Therefore

$$\{1\} \# \{1\} = \{1\}, \qquad \{1\} \# \lambda_i^\beta = \lambda_i^\beta \qquad \text{and} \qquad \{1\} \# \lambda_j^\beta = \lambda_j^\beta.$$

From (1) and (2) of Definition 3.1, we have that

$$\lambda_i^\beta \# \lambda_i^\beta = \{1\}, \lambda_j^\beta \# \lambda_j^\beta = \{1\}, \lambda_i^\beta \# \{1\} = \lambda_i^\beta, \text{and } \lambda_j^\beta \# \{1\} = \lambda_j^\beta.$$

Now let $\lambda_i^\beta \# \lambda_j^\beta = \{1\}$. Then $\{1\}, \lambda_i^\beta$ and λ_j^β are elements of the computation.

If $\lambda_j^\beta \# \lambda_i^\beta = \{1\}$, then $\lambda_i^\beta \# \lambda_j^\beta = \{1\} = \lambda_j^\beta \# \lambda_i^\beta$ and so

$$\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_i^\beta = \left(\lambda_j^\beta \# \lambda_i^\beta\right) \# \lambda_i^\beta.$$

By (3) of Definition 3.1,

$$\left(\lambda_i^\beta \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \left(\lambda_j^\beta \Vdash \lambda_i^\beta\right) \Vdash \lambda_i^\beta.$$

Hence

$$\{1\} \Vdash \lambda_j^\beta = \{1\} \Vdash \lambda_i^\beta.$$

and hence $\lambda_i^\beta = \lambda_j^\beta$[By Corollary 3.9], but it is a contradiction. Also, if $\lambda_j^\beta \Vdash \lambda_i^\beta = \lambda_i^\beta$, then

$$\lambda_i^\beta = \lambda_j^\beta \Vdash \lambda_i^\beta = \left(\{1\} \Vdash \lambda_j^\beta\right) \Vdash \lambda_i^\beta = \left(\{1\} \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \lambda_i^\beta \Vdash \lambda_j^\beta = \{1\},$$

but this a contradiction.

For the case $\lambda_j^\beta \Vdash \lambda_i^\beta = \lambda_j^\beta$, we have that

$$\lambda_j^\beta = \lambda_j^\beta \Vdash \lambda_i^\beta = \left(\{1\} \Vdash \lambda_j^\beta\right) \Vdash \lambda_i^\beta = \left(\{1\} \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \lambda_i^\beta \Vdash \lambda_j^\beta = \{1\},$$

Which is also a contradiction.

Next, if $\lambda_i^\beta \Vdash \lambda_j^\beta = \lambda_i^\beta$, then

$$\left(\lambda_i^\beta \Vdash \left(\lambda_i^\beta \Vdash \lambda_j^\beta\right)\right) \Vdash \lambda_j^\beta = \left(\lambda_i^\beta \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \{1\} \Vdash \lambda_j^\beta = \lambda_j^\beta \neq \{1\}.$$

This gives the conclusion that Proposition 3.4 is not satisfy, and this a contradiction. Now, suppose that $\lambda_i^\beta \Vdash \lambda_j^\beta = \lambda_j^\beta$. If $\lambda_j^\beta \Vdash \lambda_i^\beta = \{1\}$, then

$$\lambda_j^\beta = \lambda_i^\beta \Vdash \lambda_j^\beta = \left(\{1\} \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \left(\{1\} \Vdash \lambda_j^\beta\right) \Vdash \lambda_i^\beta = \lambda_j^\beta \Vdash \lambda_i^\beta = \{1\},$$

a contradiction.

If $\lambda_j^\beta \Vdash \lambda_i^\beta = \{1\}$, then

$$\lambda_j^\beta = \lambda_i^\beta \Vdash \lambda_j^\beta = \left(\{1\} \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \left(\{1\} \Vdash \lambda_j^\beta\right) \Vdash \lambda_i^\beta = \lambda_j^\beta \Vdash \lambda_i^\beta = \{1\},$$

a contradiction.

For the case $\lambda_j^\beta \Vdash \lambda_i^\beta = \lambda_j^\beta$, we have that

$$\lambda_i^\beta = \{1\} \Vdash \lambda_i^\beta = \left(\lambda_j^\beta \Vdash \lambda_i^\beta\right) \Vdash \lambda_i^\beta = \left(\lambda_j^\beta \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \lambda_j^\beta \Vdash \lambda_j^\beta = \{1\},$$

This is yet another contradiction. This brings the proof to a close.

Proposition 3.19: Assume $(X, \Vdash, \{1\})$ is a $(PQ-A)$ with $|X| = 2$, then for any state, the $G-$part $G(X)$ is a $(PQ-I)$ of X.

Proof: Suppose that $|X| = 2$. Then either $G(X) = \{\{1\}\}$ or $G(X) = X$. In either case, $G(X)$ is a $(PQ-I)$ of X.

Proposition 3.20: Assume $(X, \Vdash, \{1\})$ is a $(PQ-A)$ with $|X| = 3$. Then $G(X)$ is a $(PQ-I)$ iff $|G(X)| = 1$.

Proof: Suppose that $X = \left\{\{1\}, \lambda_i^\beta, \lambda_j^\beta\right\}$ is a $(PQ-A)$ with $|G(X)| = 1$, hence $G(X) = \{\{1\}\}$ is the trivial $(PQ-I)$. Now, let $G(X)$ be a $(PQ-I)$. From Proposition (3.18), we consider that $|G(X)|$ is either equal 1 or 2. Therefore $G(X)$ is equal either $\left\{\{1\}, \lambda_i^\beta\right\}$ or $\left\{\{1\}, \lambda_j^\beta\right\}$, whenever $|G(X)| = 2$.. Also, $\lambda_j^\beta \Vdash \lambda_i^\beta \notin G(X)$, whenever $G(X) = \left\{\{1\}, \lambda_i^\beta\right\}$ [Since $G(X)$ is a $(PQ-I)$]. Thus $\lambda_j^\beta \Vdash \lambda_i^\beta = \lambda_j^\beta$ and this implies that$\lambda_i^\beta = \{1\} \Vdash \lambda_i^\beta = \left(\lambda_j^\beta \Vdash \lambda_j^\beta\right) \Vdash \lambda_i^\beta = \left(\lambda_j^\beta \Vdash \lambda_i^\beta\right) \Vdash \lambda_j^\beta = \lambda_j^\beta \Vdash \lambda_j^\beta = \{1\}$, but this contradiction. By the same way, $G(X) = \left\{\{1\}, \lambda_j^\beta\right\}$ gives a contradiction. Hence $G(X)$ dose not equal 2. Therefore it is equal1.

Definition 3.21: A $(PQ-I)$ of a $(PQ-A)$ $(X, \Vdash, \{1\})$ is said to be *permutation implicative Q-ideal* $(PIQ-I)$ if

$$\left(\lambda_i^\beta \Vdash \lambda_j^\beta\right) \Vdash \lambda_k^\beta \in I \text{ and } \lambda_j^\beta \Vdash \lambda_k^\beta \in I,$$

Then

$$\lambda_i^\beta \Vdash \lambda_k^\beta \in I, \qquad \forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X.$$

Proposition 3.22: Assume $(X, \Vdash, \{1\})$ is a $(PQ-A)$ with I is a $(PIQ-I)$of X. Then $G(X) \subseteq I$.

Proof: Let $\lambda_i^\beta \in G(X)$, thus $\left(\{1\} \Vdash \lambda_i^\beta\right) \Vdash \lambda_i^\beta = \lambda_i^\beta \Vdash \lambda_i^\beta = \{1\} \in I$ and $\lambda_i^\beta \Vdash \lambda_i^\beta = \{1\} \in I$. Since I is a $(PIQ-I)$, it follows that $\lambda_i^\beta = \{1\} \Vdash \lambda_i^\beta \in I$. Therefore $G(X) \subseteq I$.

Definition 3.23: Assume $(X, \Vdash, \{1\}_X)$ & $(Y, \Vdash, \{1\}_Y)$ are two $(PQ-As)$. We say $f: X \longrightarrow Y$ is a *homomorphism* if

$$f\left(\lambda_i^\beta \Vdash \lambda_j^\beta\right) = f\left(\lambda_i^\beta\right) \Vdash f\left(\lambda_j^\beta\right), \qquad \forall \lambda_i^\beta, \lambda_j^\beta \in X.$$

Also, we say for any homomorphism is *epimorphism* (resp. *monomorphism*) if it is surjective (resp. injective). Moreover, for any two $(PQ-As)$ $(X, \Vdash, \{1\}_X)$ & $(Y, \Vdash, \{1\}_Y)$ are called *isomorphic*, symbolized as $X \cong Y$, if there exists surjective and injective map $f: X \longrightarrow Y$. Furthermore, $Ker(f) = \left\{\lambda_i^\beta \in X \mid f\left(\lambda_i^\beta\right) = \{1\}\right\}$ and $Im(f) = \left\{f\left(\lambda_i^\beta\right) | \lambda_i^\beta \in X\right\}$ are called *kernel and image* of f, respectively. We define $Hom(X, Y)$ by $Hom(X, Y) = \{\, f\ is$ homomorphism of $(PQ - As) | \, f: X \longrightarrow Y\}$.

Proposition 3.24: Let $f: X \longrightarrow Y$ be a homomorphism of $(PQ - As)$. Then

(1) $f(\{1\}_X) = \{1\}_Y$.

(2) f is isotone, i.e. if $\lambda_i^\beta \ \#\ \lambda_j^\beta = \{1\}_X$ then $f\left(\lambda_i^\beta\right) \# f\left(\lambda_j^\beta\right) = \{1\}_Y, \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Proof: (1) Now, $f(\{1\}_X) = f(\{1\}_X \# \{1\}_X) = f(\{1\}_X) \# f(\{1\}_X) = \{1\}_Y \# \{1\}_Y = \{1\}_Y$.
(2) If $\lambda_i^\beta, \lambda_j^\beta \in X$ and $\lambda_i^\beta \leq \lambda_j^\beta$, that is, $\lambda_i^\beta \ \#\ \lambda_j^\beta = \{1\}_X$, then by (1),

$$f\left(\lambda_i^\beta\right) \# f\left(\lambda_j^\beta\right) = f\left(\lambda_i^\beta \ \#\ \lambda_j^\beta\right) = f(\{1\}_X) = \{1\}_Y.$$

Hence

$$f\left(\lambda_i^\beta\right) \leq f\left(\lambda_j^\beta\right).$$

Proposition 3.25: Assume $(X, \#, \{1\}_X)$ & $(Y, \#, \{1\}_Y)$ are two $(PQ - As)$ with B is a $(PQ - I)$ of Y. Then for any $f \in Hom(X, Y)$, $f^{-1}(B)$ is a $(PQ - I)$ of X.
Proof: We have $\{1\}_Y \in f^{-1}(\{1\}_X)$ [From Proposition 3.24 (1)]. Let $\lambda_i^\beta \ \#\ \lambda_j^\beta \in f^{-1}(B)$ & $\lambda_j^\beta \in f^{-1}(B)$. Thus

$$f\left(\lambda_i^\beta\right) \# f\left(\lambda_j^\beta\right) = f\left(\lambda_i^\beta \ \#\ \lambda_j^\beta\right) \in B.$$

Then $f\left(\lambda_i^\beta\right) \in B$, and $\lambda_i^\beta \in f^{-1}(B)$ [Since B is a $(PQ - I)$ of Y]. Therefore $f^{-1}(B)$ is a $(PQ - I)$ of X.

Corollary 3.26: $Ker(f)$ is a $(PQ - I)$ of X.
Proof: Since $\{\{1\}_Y\}$ is a $(PQ - I)$ of Y,

$$Ker(f) = f^{-1}(\{\{1\}_Y\}) \quad \text{for any } f \in Hom(X, Y).$$

4. CONCLUSION

Some new extensions of $\mathrm{Q}-$ algebras are introduced in this paper, and their properties are investigated using β −sets. which are different sets. Also, some different sets like nano sets[27] and neutrosophic sets[28-33] have been used to study numerous mathematical problems in recent work. As a result, instead of employing permutation sets in a future study, we will extend our notions and conclusions in this paper using nano and neutrosophic sets.

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