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Quotient Permutation BF-Algebras and Quotient Maps

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Abstract

The permutation BF-algebras was given as a new class of BF-algebras and their basic features in this work were studied. We considered and discussed some new notions in permutation BF-algebras, like permutation BF_1/BF_2-algebras, permutation BF-ideals, permutation BF-subalgebras, normal permutation BF-subalgebras. We further looked at the homomorphism of BF-algebras, congruence relation, quotient permutation BF-algebras, and Quotient map.

Keywords: cycles, permutation, BF-algebras, filters, homomorphism, quotient.

1. Introduction

BF-algebras, which are a generalization of B-algebras, were introduced by A. Walendziakin [1]. In BF-algebras, Walendziak introduced the concepts of an ideal and a normal ideal, as well as their characteristics and characterizations. Different fields [2-8] examine symmetric and alternating groups, as well as their permutations. Some classes of algebra, group theory, and topology in non-classical sets, such as permutation sets [9-24], fuzzy sets [25-39], soft sets [40-53], nano sets [54], neutrosophic sets [55-62] and others [63-71] have been investigated in recent years.

In various domains, such as computer science, information science, cybernetics, and artificial intelligence, logic algebras serve as the algebraic underpinning of reasoning mechanisms. Imai and Iséki [72, 73] proposed the concepts of BCK-algebras and BCI-algebras in 1966. Agboola and Davvaz [74] used one of these sets on BCI/BCK-algebras; it is the neutrosophic set, which takes into account neutrosophic BCI/BCK-algebras. As a result, we will use the peremutation set on BF-algerbra.

In this paper, we introduced permutation BF-algebras as a new class of BF-algebras and gave their basic features. New notions in permutation BF-algebras, like permutation BF_1/BF_2-algebras, permutation BF-ideals, permutation BF-subalgebras, normal permutation BF-subalgebras were invvestigated. We further looked at homomorphism of BF-algebras, congruence relation, quotient permutation BF-algebras and Quotient map.

2. Preliminary Notes

We will go over the fundamental concepts and findings that are needed for this investigation in this section.

Definition 2.1: [1] Let $X \neq \emptyset$ and 0 be a constant with a binary operation $*$. We say that $(X,*,0)$ is a *B-algebra* if it satisfies the following conditions:

a) $x * x = 0$,
b) $x * 0 = x$,
c) $(x * y) * z = x * \left(z * (0 * y)\right), \forall x, y, z \in X$.

Definition 2.2:[1] Let $K(\tau)$ be the class of all algebras of type $\tau = (2,0)$. By a *BF-algebra* we mean a system $(X,*,0)$ in which the following axioms are satisfied:

(1) $x * x = 0, \forall x \in X$.
(2) $x * 0 = x, \forall x \in X$.
(3) $0 * (x * y) = y * x, \forall x, y \in X$.

We say that 0 is the unit in X.

Definition 2.3: [2]

For any permutation $\beta = \prod_{i=1}^{c(\beta)} \lambda_i$ in a symmetric group S_n, where $\{\lambda_i\}_{i=1}^{c(\beta)}$ is a composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$ where $\lambda_i = (t_1^i, t_2^i, \ldots, t_{\alpha_i}^i), 1 \leq i \leq c(\beta)$, for some $1 \leq \alpha_i, c(\beta) \leq n$. If $\lambda = (t_1, t_2, \ldots, t_k)$ is k −cycle in S_n, we define β −set as $\lambda^\beta = \{t_1, t_2, \ldots, t_k\}$ and is called β −set of cycle λ. So the β −sets of $\{\lambda_i\}_{i=1}^{c(\beta)}$ are defined by $\{\lambda_i^\beta = \{t_1^i, t_2^i, \ldots, t_{\alpha_i}^i\} | 1$

$\leq i \leq c(\beta)\}$.

Definition 2.4: [16] Let X be a collection of β-sets $\left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)}$, where β is a permutation in the symmetric group $G = S_n$ with $\{1\}$. Then $X = \left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)} \cup \{1\}$ with a binary operation $\#: X \times X \longrightarrow X$ is said to be a *permutation B-algebra* if $\#$ satisfies the condition:

(1) $\lambda_i^\beta \# \lambda_i^\beta = \{1\}$,
(2) $\lambda_i^\beta \# \{1\} = \lambda_i^\beta$,
(3) $\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_k^\beta = \lambda_i^\beta \# \left(\lambda_k^\beta \# \left(\{1\} \# \lambda_j^\beta\right)\right), \forall \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$.

Also, we say that $\{1\}$ is the fixed element in X. It is denoted by $(X, \#, \{1\})$.

3. Permutation BF-Algebras

In this section, we'll investigate new implications in permutation BF-algebras and look at some of their fundamental features.

Definition 3.1: Let $\left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)}$ be a collection of β-sets, where β is a permutation in the symmetric group $G = S_n$. Then $X = \left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)} \cup \{1\}$ is said to be a *permutation BF-algebra* if there exists a mapping $\#: X \times X \longrightarrow X$ such that

(1) $\lambda_i^\beta \# \lambda_i^\beta = \{1\}, \quad \forall \lambda_i^\beta \in X$
(2) $\lambda_i^\beta \# \{1\} = \lambda_i^\beta, \quad \forall \lambda_i^\beta \in X$

(3) $\{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) = \lambda_j^\beta \# \lambda_i^\beta, \ \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

We say that $\{1\}$ is the fixed element in X.

Example 3.2:

Let (S_{10}, o) be a symmetric group and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 2 & 1 & 4 & 3 & 7 & 9 & 8 & 6 & 10 \end{pmatrix}$ be a permutation in S_{10}. So, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 2 & 1 & 4 & 3 & 7 & 9 & 8 & 6 & 10 \end{pmatrix} = (153)(2)(679)(8)(10)$. Therefore, we have $X = \left\{\lambda_i^\beta\right\}_{i=1}^{5} \cup \{1\} = \left\{\{1,5,3\}, \{2\}, \{6,7,9\}, \{8\}, \{10\}, \{1\}\right\}$. Define $\#: X \times X \longrightarrow X$ by $\#\left(\lambda_i^\beta, \lambda_j^\beta\right) = \lambda_i^\beta \# \lambda_j^\beta = \lambda_k^\beta$, where λ_k^β its cycle λ_k such that $\lambda_k = \lambda_i o \lambda_j^{-1}$, where λ_i and λ_j are cycles for λ_i^β and λ_i^β, respectively. Then X is a permutation BF-algebra.

Remark 3.3: In Example (3.2), we can consider that the three conditions in Definition (3.1) are hold for any $\lambda_i^\beta, \lambda_j^\beta \in X$ as following:

(1) $\lambda_i \# \lambda_i^{-1} = (1) \rightarrow \lambda_i^\beta \# \lambda_i^\beta = \{1\}$,

(2) $\lambda_i \# (1)^{-1} = \lambda_i \rightarrow \lambda_i^\beta \# \{1\} = \lambda_i^\beta$,

(3) $(1)o((\lambda_i o \lambda_j^{-1})))^{-1} = (1)o\left(\lambda_j o \lambda_i^{-1}\right) = (\lambda_j o \lambda_i^{-1})$. Hence $\{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) = \lambda_j^\beta \# \lambda_i^\beta$.

Proposition 3.4: Let $(X, \#, \{1\})$ be a permutation BF-algebra, then

(1) $\{1\} \# \left(\{1\} \# \lambda_i^\beta\right) = \lambda_i^\beta, \forall \lambda_i^\beta \in X$,

(2) If $\{1\} \# \lambda_i^\beta = \{1\} \# \lambda_j^\beta$, then $\lambda_i^\beta = \lambda_j^\beta, \quad \forall \lambda_i^\beta, \lambda_j^\beta \in X$,

(3) If $\lambda_i^\beta \# \lambda_j^\beta = \{1\}$, then $\lambda_j^\beta \# \lambda_i^\beta = \{1\}, \quad \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Proof: (1) Let $(X, \#, \{1\})$ be a permutation BF-algebra and $\lambda_i^\beta \in X$, then

$$\{1\} \# \left(\{1\} \# \lambda_i^\beta\right) = \lambda_i^\beta \# \{1\} = \lambda_i^\beta.$$

(2) If $\{1\} \# \lambda_j^\beta = \{1\} \# \lambda_j^\beta$, then $\{1\} \# \left(\{1\} \# \lambda_i^\beta\right) = \{1\} \# \left(\{1\} \# \lambda_j^\beta\right)$ and $\lambda_i^\beta = \lambda_j^\beta$ from (1).

(3) Let $\lambda_i^\beta, \lambda_j^\beta \in X$ and $\lambda_i^\beta \# \lambda_j^\beta = \{1\}$. Then

$$\{1\} = \{1\} \# \{1\} = \{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) = \lambda_j^\beta \# \lambda_i^\beta$$

Proposition 3.5: Any permutation BF-algebra $(X, \#, \{1\})$ that satisfies the identity

$\left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \left(\lambda_j^\beta \# \lambda_k^\beta\right) = \lambda_i^\beta \# \lambda_j^\beta$ is a permutation B-algebra.

Proof: If $(X, \#, \{1\})$ is a permutation BF-algebra, then Proposition 3.4 (1), shows that

$$\{1\} \# \left(\{1\} \# \lambda_i^\beta\right) = \lambda_i^\beta, \ \forall \lambda_i^\beta \in X.$$

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Now $\left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \left(\lambda_j^\beta \# \lambda_k^\beta\right) = \lambda_i^\beta \# \left[\left(\lambda_j^\beta \# \lambda_k^\beta\right) \# \left(\{1\} \# \lambda_k^\beta\right)\right]$

$$= \lambda_i^\beta \# \left[\lambda_j^\beta \# \left(\left(\{1\} \# \lambda_k^\beta\right) \# \left(\{1\} \# \lambda_k^\beta\right)\right)\right]$$

$$= \lambda_i^\beta \# \left[\lambda_j^\beta \# \{1\}\right] \quad \text{(From (1) of Definition 3.1)}$$

$$= \lambda_i^\beta \# \lambda_j^\beta \qquad \text{(From (2) of Definition 3.1)}$$

Now from Proposition 3.4 (1), $\lambda_i^\beta = \{1\} \# \left(\{1\} \# \lambda_i^\beta\right) = \left(\lambda_i^\beta \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_i^\beta\right) = \lambda_i^\beta \# \{1\}$.

Also from the condition, we have

$$\left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \left(\lambda_j^\beta \# \lambda_k^\beta\right) = \lambda_i^\beta \# \lambda_j^\beta, \text{take } \lambda_j^\beta = \{1\}$$

Then $\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right) = \lambda_i^\beta \# \{1\} = \lambda_i^\beta$ (from condition (2) of Definition 3.1)

Thus $\lambda_i^\beta \# \left(\lambda_k^\beta \# \left(\{1\} \# \lambda_j^\beta\right)\right) = \left[\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)\right] \# \left[\lambda_k^\beta \# \left(\{1\} \# \lambda_j^\beta\right)\right] = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_k^\beta$. Hence $(X, \#, 0)$ is a permutation B-algebra.

Definition 3.6:

A permutation BF-algebra is called a *permutation BF₁-algebra* if it such that $\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)$ and is said to be a *permutation BF₂-algebra* if it such that $\lambda_i^\beta \# \lambda_j^\beta = \{1\}$ and $\lambda_j^\beta \# \lambda_i^\beta = \{1\} \implies \lambda_i^\beta = \lambda_j^\beta$.

Note that every permutation B-algebra is a permutation BF_1/BF_2-algebra.

Proposition 3.7: A permutation algebra $(X, \#, \{1\})$ of type (2,0) is a permutation BF_1-algebra if and only if it such that:

(1) $\lambda_i^\beta \# \lambda_i^\beta = \{1\}$,

(2) $\{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) = \lambda_j^\beta \# \lambda_i^\beta$,

(3) $\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)$.

Proof: Suppose that $(X, \#, \{1\})$ satisfies condition (1), (2), and (3). Let $\lambda_i^\beta \in X$. Substituting $\lambda_j^\beta = \lambda_i^\beta$ in (3), we have

$$\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_i^\beta\right)$$

$$= \{1\} \# \left(\{1\} \# \lambda_i^\beta\right) \text{ (from (1))}$$

$$= \lambda_i^\beta \# \{1\} \qquad \text{(from (2))}$$

Then $(X, \#, \{1\})$ is a permutation BF_1-algebra.

Conversely, if $(X, \#, \{1\})$ is a permutation BF_1-algebra, then conditions (1), (2) and (3) holds from the definition.

Proposition 3.8: Let $(X, \#, \{1\})$ be a permutation algebra of type (2,0). Then $(X, \#, \{1\})$ is a permutation BF_2-algebra, if and only if it such that:

(1) $\lambda_i^\beta \# \{1\} = \lambda_i^\beta$,

(2) $\{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) = \lambda_j^\beta \# \lambda_i^\beta$,

(3) $\lambda_i^\beta \# \lambda_j^\beta = \{1\}$ if and only if $\lambda_i^\beta = \lambda_j^\beta$.

Proof: Let $(X, \#, \{1\})$ be a permutation BF_2-algebra. Then condition (1) and (2) satisfies $(X, \#, \{1\})$. Suppose that $\lambda_i^\beta \# \lambda_j^\beta = \{1\}$, $\forall \lambda_i^\beta, \lambda_j^\beta \in X$. We have from Proposition 3.4 (3) that $\lambda_j^\beta \# \lambda_i^\beta = \{1\}$. Now from Definition 3.2, $\lambda_i^\beta = \lambda_j^\beta$. If $\lambda_i^\beta = \lambda_j^\beta$, then $\lambda_i^\beta \# \lambda_j^\beta = \{1\}$ from Definition 3.2, then $\lambda_i^\beta \# \lambda_j^\beta = \{1\} \iff \lambda_i^\beta = \lambda_j^\beta$.

Now let $(X, \#, \{1\})$ satisfies $\lambda_i^\beta \# \{1\} = \lambda_i^\beta$, $\{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) = \lambda_j^\beta \# \lambda_i^\beta$ and $\lambda_i^\beta \# \lambda_j^\beta = \{1\} \Leftrightarrow \lambda_i^\beta = \lambda_j^\beta$. Thus if $\lambda_i^\beta = \lambda_j^\beta$, then $\lambda_j^\beta \# \lambda_i^\beta = \{1\} \implies \lambda_i^\beta = \lambda_j^\beta$.

Thus $(X, \#, \{1\})$ is a permutation BF_2-algebra.

Proposition 3.9: If $(X, \#, \{1\})$ is a permutation BF-algebra, then the following statements are equivalent:

(1) $(X, \#, \{1\})$ is a permutation BF_1-algebra.

(2) $\lambda_i^\beta = \left[\lambda_i^\beta \# \left(\{1\} \# \lambda_j^\beta\right)\right] \# \lambda_j^\beta, \quad \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

(3) $\lambda_i^\beta = \lambda_j^\beta \# \left[\left(\{1\} \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)\right], \ \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Proof: $(1) \Longrightarrow (2)$. Let $(X, \#, \{1\})$ be a permutation BF_1-algebra and $\lambda_i^\beta, \lambda_j^\beta \in X$. Substitute $\{1\} \# \lambda_j^\beta$ for λ_j^β in $\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)$, we have that

$$\lambda_i^\beta = \left(\lambda_i^\beta \# \left(\{1\} \# \lambda_j^\beta\right)\right) \# \left(\{1\} \# \left(\{1\} \# \lambda_j^\beta\right)\right) = \left[\lambda_i^\beta \# \left(\{1\} \# \lambda_j^\beta\right)\right] \# \lambda_j^\beta$$

$(2) \Longrightarrow (3)$ We have from (2), that

$$\{1\} \# \lambda_i^\beta = \left[\left(\{1\} \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)\right] \# \lambda_j^\beta$$

Hence $\{1\} \# \left(\{1\} \# \lambda_i^\beta\right) = \{1\} \# \left[\left(\left(\{1\} \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)\right) \# \lambda_j^\beta\right] \rightarrow \lambda_i^\beta = \lambda_j^\beta \# \left[\left(\{1\} \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)\right]$ From Proposition 3.4(1) and (3) of Definition (3.1).

$(3) \Longrightarrow (1)$. If (3) holds, then

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$$\{1\} \# \lambda_i^\beta = \left[\left(\{1\} \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)\right] \# \lambda_j^\beta$$

Substituting $\{1\} \# \lambda_i^\beta$ for λ_i^β and $\{1\} \# \lambda_j^\beta$ for λ_j^β, we have

$$\{1\} \# \left(\{1\} \# \lambda_i^\beta\right) = \left[\left(\{1\} \# \left(\{1\} \# \lambda_i^\beta\right)\right) \# \left(\{1\} \# \left(\{1\} \# \lambda_j^\beta\right)\right)\right] \# \left(\{1\} \# \lambda_j^\beta\right)$$

$$\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right) \quad \text{(From Proposition 3.4(1))}$$

Thus $(X, \#, \{1\})$ is a permutation BF_1-algebra.

Proposition 3.10: Let $(X, \#, \{1\})$ be a permutation BF- algebra. Then $\lambda_i^\beta \# \lambda_j^\beta = \{1\} \Longrightarrow \lambda_i^\beta = \lambda_j^\beta$.

Proof: If $\lambda_i^\beta \# \lambda_j^\beta = \{1\} \ \ \forall \lambda_i^\beta, \lambda_j^\beta \in X$. Then from

$$\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right) = \{1\} \# \left(\{1\} \# \lambda_j^\beta\right) = \lambda_j^\beta \quad \text{(From Proposition 3.4(1))}$$

Proposition 3.11: Every permutation BF_1-algebra is a permutation BF_2-algebra. Every permutation BF_2-algebra satisfying $\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)$ is a permutation BF_1-algebra.

Proof: From Proposition 3.10(1), it follows that every permutation BF_1-algebra is a permutation BF_2-algebra. The second part follows immediately from the definitions.

Note: From above we consider figure (1):

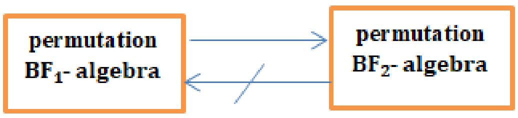

Figure (1)

Proposition 3.12: Let $(X, \#, \{1\})$ be a permutation BF_1- algebra. Then $(X, \#)$ is a quasi-group.

Proof: Let $(X, \#, \{1\})$ be a permutation BF_1-algebra and $\lambda_i^\beta, \lambda_j^\beta \in X$. Setting $\lambda_{k_1}^\beta = \lambda_i^\beta \# \left(\{1\} \# \lambda_j^\beta\right)$ and $\lambda_{k_2}^\beta = \left(\{1\} \# \lambda_i^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)$. Then from Proposition 3.9, we have $\lambda_i^\beta = \lambda_{k_1}^\beta \# \lambda_j^\beta$ and $\lambda_i^\beta = \lambda_j^\beta \# \lambda_{k_2}^\beta$. Thus Proposition 3.10 shows that $(X, \#)$ is a quasi-group.

Definition 3.13: Let $(X, \#, \{1\})$ be a permutation BF-algebra. A non-empty subset I of X is called apermutation BF-*ideal* of X if it such that

(1) $\{1\} \in I$,

(2) $\lambda_i^\beta \# \lambda_j^\beta \in I$ and $\lambda_j^\beta \in I \Longrightarrow \lambda_i^\beta \in I \ \ \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

A permutation BF-ideal I is said to be *normal* if for all $\lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X, \lambda_i^\beta \# \lambda_j^\beta \in I \Longrightarrow \left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \left(\lambda_k^\beta \# \lambda_j^\beta\right) \in I$.

A permutation BF-ideal I of $(X, \#, \{1\})$ is said to be *proper* if $I \neq X$. $\{\{1\}\}$ and X are obviously permutation BF-ideals of $(X, \#, \{1\})$. Also, X is normal but $\{\{1\}\}$ is not normal.

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Lemma 3.14: Let I be a normal permutation BF-ideal of a permutation BF-algebra $(X, \#, \{1\})$ and $\lambda_i^\beta, \lambda_j^\beta \in X$. Then

(1) $\lambda_i^\beta \in I \implies \{1\} \# \lambda_i^\beta \in I$,

(2) $\lambda_i^\beta \# \lambda_j^\beta \in I \implies \lambda_j^\beta \# \lambda_i^\beta \in I$.

Proof: (1) Let $\lambda_i^\beta \in I$. Then $\lambda_i^\beta = \lambda_i^\beta \# \{1\} \in I$. Since I is normal $\left(\{1\} \# \lambda_i^\beta\right) \# (\{1\} \# \{1\}) \in I$. Thus $\{1\} \# \lambda_i^\beta \in I$.

(2) Let $\lambda_i^\beta \# \lambda_j^\beta \in I$. By (1), $\{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) \in I$. Then from Definition 3.1, $\lambda_j^\beta \# \lambda_i^\beta \in I$.

Definition 3.15: Let $(X, \#, \{1\})$ be a permutation BF-algebra and $\emptyset \neq N \subseteq X$. We say N is a *permutation BF-subalgebra* of X if $\lambda_i^\beta \# \lambda_j^\beta \in N$ for all $\lambda_i^\beta, \lambda_j^\beta \in N$.

Lemma 3.16: Let N be a permutation BF-subalgebra of $(X, \#, \{1\})$ and let $\lambda_i^\beta, \lambda_j^\beta \in X$. If $\lambda_i^\beta \# \lambda_j^\beta \in N$, then $\lambda_j^\beta \# \lambda_i^\beta \in N$.

Proof: Let $\lambda_i^\beta \# \lambda_j^\beta \in N$. Then by Definition 3.1, $\lambda_j^\beta \# \lambda_i^\beta = \{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right)$. Since $\{1\} \in N$ and $\lambda_i^\beta \# \lambda_j^\beta \in N$, we have that $\{1\} \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) \in N$. Thus $\lambda_j^\beta \# \lambda_i^\beta \in N$.

Proposition 3.17: Let $(X, \#, \{1\})$ be a permutation BF-algebra. If I is a normal permutation BF-ideal of X, then I is a permutation BF-subalgebra of X such that:

If $\lambda_i^\beta \in X$ and $\lambda_j^\beta \in I$, then $\lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) \in I$.

Proof: Let $\lambda_i^\beta \in X$ and $\lambda_j^\beta \in I$. Lemma 3.14 (1), shows that $\{1\} \# \lambda_j^\beta \in I$. Since I is a normal permutation BF-ideal, we have that

$$\left(\lambda_i^\beta \# \{1\}\right) \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) \in I.$$

Thus $\lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) \in I$.

Now, let $\lambda_i^\beta, \lambda_j^\beta \in I$. Therefore $\lambda_i^\beta \# \left(\lambda_i^\beta \# \lambda_j^\beta\right) \in I$. Lemma 3.14 (2) shows that $\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \lambda_i^\beta \in I$. From the definition of an ideal, we have that $\lambda_i^\beta \# \lambda_j^\beta \in I$. Thus I is a permutation BF-subalgebra satisfying the condition.

Definition 3.18: Let $(X, \#, \{1\})$ be a permutation BF-algebra and N be a permutation BF-subalgebra of $(X, \#, \{1\})$. N is said to be a *normal permutation BF*-subalgebra if

$$\left(\lambda_i^\beta \# \lambda_m^\beta\right) \# \left(\lambda_j^\beta \# \lambda_n^\beta\right) \in N \ \forall \lambda_i^\beta \# \lambda_j^\beta, \lambda_m^\beta \# \lambda_n^\beta \in N.$$

Proposition 3.19: Let $(X, \#, \{1\})$ be a permutation B-algebra and let $N \subseteq (X, \#, \{1\})$. Then N is a normal permutation BF-subalgebra of $(X, \#, \{1\})$ if and only if N is a normal permutation BF-ideal.

Proof: Let N be a normal permutation BF-subalgebra of $(X, \#, \{1\})$. Clearly, $\{1\} \in N$.

Suppose that $\lambda_i^\beta \# \lambda_j^\beta \in N$ and $\lambda_j^\beta \in N$. Then $\{1\} \# \lambda_j^\beta \in N$. Since N is a permutation BF-subalgebra, we have that $\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right) \in N$. But $\left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right) = \lambda_i^\beta$, because every permutation B-algebra satisfies

$\lambda_i^\beta = \left(\lambda_i^\beta \# \lambda_j^\beta\right) \# \left(\{1\} \# \lambda_j^\beta\right)$. Therefore $\lambda_i^\beta \in N$, and thus N is a permutation BF-ideal.

Let now $\lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$ and $\lambda_i^\beta \# \lambda_j^\beta \in N$. From Definition 3.18, $\left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \left(\lambda_k^\beta \# \lambda_j^\beta\right) \in N$.

Thus N is a permutation BF-normal.

Conversely, if N is a normal permutation BF-ideal, then from Proposition 3.17 and the fact that N is normal if and only if it satisfies the condition in Proposition 3.17, then N is a normal permutation BF-subalgebra.

Definition 3.20: Let $(X, \#, \{1\}_X)$ and $(Y, \#, \{1\}_Y)$ be two permutation BF-algebras. A mapping $\theta: X \longrightarrow Y$ is called a *homomorphism* from $(X, \#, \{1\}_X)$ to $(Y, \#, \{1\}_Y)$ if

$$\theta\left(\lambda_i^\beta \# \lambda_j^\beta\right) = \theta\left(\lambda_i^\beta\right) \# \theta\left(\lambda_j^\beta\right) \ \forall \lambda_i^\beta, \lambda_j^\beta \in X.$$

Note that $\theta(\{1\}_X) = \{1\}_Y$. The kernel of the homomorphism denoted by $Ker\theta$ is defined by

$$Ker\theta = \left\{\lambda_i^\beta \in X : \theta\left(\lambda_i^\beta\right) = \{1\}_Y\right\}.$$

Lemma 3.21: If $\theta: X \longrightarrow Y$ is a homomorphism from $(X, \#, \{1\}_X)$ to $(Y, \#, \{1\}_Y)$. Then $Ker\theta$ is an ideal of $(X, \#, \{1\}_X)$.

Proof: Clearly, $\{1\}_X \in Ker\theta$. Let $\lambda_i^\beta \# \lambda_j^\beta \in Ker\theta$ and $\lambda_j^\beta \in Ker\theta$. Then

$$\{1\}_Y = \theta\left(\lambda_i^\beta \# \lambda_j^\beta\right) = \theta\left(\lambda_i^\beta\right) \# \theta\left(\lambda_j^\beta\right) = \theta\left(\lambda_i^\beta\right) \# \{1\}_Y = \theta\left(\lambda_i^\beta\right).$$

Thus $\lambda_i^\beta \in Ker\theta$. Therefore, I is a permutation BF-ideal of $(X, \#, \{1\}_X)$.

Proposition 3.22: If $(X, \#, \{1\}_X)$ and $(Y, \#, \{1\}_Y)$ are two permutation BF_2-algebras and let $\theta: X \longrightarrow Y$ be a homomorphism from $(X, \#, \{1\}_X)$ to $(Y, \#, \{1\}_Y)$. Then

(1) $Ker\theta$ is a normal permutation BF-ideal,
(2) θ is one-to-one if and only if $Ker\theta = \{\{1\}_X\}$.

Proof: (1) From Lemma 3.21, $Ker\theta$ is a permutation BF-ideal of $(X, \#, \{1\}_X)$. Let $\lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta \in X$ and $\lambda_i^\beta \# \lambda_j^\beta \in Ker\theta$. Then $\{1\}_Y = \theta\left(\lambda_i^\beta \# \lambda_j^\beta\right) = \theta\left(\lambda_i^\beta\right) \# \theta\left(\lambda_j^\beta\right)$.

It follows from Proposition 3.8 (3), that $\theta\left(\lambda_i^\beta\right) = \theta\left(\lambda_j^\beta\right)$. Thus

$$\theta\left(\left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \left(\lambda_k^\beta \# \lambda_j^\beta\right)\right) = \theta\left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \theta\left(\lambda_k^\beta \# \lambda_j^\beta\right) = \left(\theta\left(\lambda_k^\beta\right) \# \theta\left(\lambda_i^\beta\right)\right) \# \left(\theta\left(\lambda_k^\beta\right) \# \theta\left(\lambda_i^\beta\right)\right) = \{1\}_Y,$$

And hence $\left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \left(\lambda_k^\beta \# \lambda_j^\beta\right) \in Ker\theta$.

(2) Clearly, if θ is one-to-one, then $Ker\theta = \{\{1\}_X\}$.

On the other hand, suppose that $\lambda_i^\beta, \lambda_j^\beta \in X$ and $\theta\left(\lambda_i^\beta\right) = \theta\left(\lambda_j^\beta\right)$. Then

$$\theta\left(\lambda_i^\beta \# \lambda_j^\beta\right) = \theta\left(\lambda_i^\beta\right) \# \theta\left(\lambda_j^\beta\right) = \theta\left(\lambda_i^\beta\right) \# \theta\left(\lambda_i^\beta\right) = \{1\}_Y.$$

Hence $\lambda_i^\beta \# \lambda_j^\beta \in Ker\theta = \{\{1\}_X\}$, and so $\lambda_i^\beta \# \lambda_j^\beta = \{1\}_X$. From Proposition 3.8 (3), it follows that $\lambda_i^\beta = \lambda_j^\beta$. Therefore, θ is one-to-one.

Equivalence relation 3.23:

Let $(X, \#, \{1\})$ be a permutation BF-algebra and I be a normal permutation BF-ideal of $(X, \#, \{1\})$. For all $\lambda_i^\beta, \lambda_j^\beta \in X$, we define

$$\lambda_i^\beta \sim_I \lambda_j^\beta \iff \lambda_i^\beta \# \lambda_j^\beta \in I.$$

From Definition 3.13, $\lambda_i^\beta \# \lambda_i^\beta = \{1\} \in I$, that is, $\lambda_i^\beta \sim_I \lambda_i^\beta$ for all $\lambda_i^\beta \in X$. Thus $\sim_I$ is reflexive.

From Lemma 3.14 (2), $\lambda_i^\beta \# \lambda_j^\beta \in I \implies \lambda_j^\beta \# \lambda_i^\beta \in I$, that is $\lambda_i^\beta \sim_I \lambda_j^\beta \implies \lambda_j^\beta \sim_I \lambda_i^\beta$. Thus $\sim_I$ is transitive.

Let $\lambda_i^\beta \sim_I \lambda_j^\beta$ and $\lambda_j^\beta \sim_I \lambda_k^\beta$. Then $\lambda_i^\beta \# \lambda_j^\beta \in I$and $\lambda_j^\beta \# \lambda_k^\beta \in I$. Since I is normal permutation BF-ideal,

$$\left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \left(\lambda_k^\beta \# \lambda_j^\beta\right) \in I.$$

Thus $\lambda_k^\beta \# \lambda_j^\beta \in I$ because $\lambda_j^\beta \# \lambda_k^\beta \in I$. Hence we conclude that $\lambda_k^\beta \# \lambda_i^\beta \in I$ and thus $\lambda_i^\beta \# \lambda_k^\beta \in I$, and so $\lambda_i^\beta \sim_I \lambda_k^\beta$ and $\sim_I$ is transitive. Therefore $\sim_I$ is an equivalence relation on X.

Proposition 3.24: If I is a normal permutation BF-ideal of a permutation BF-algebra $(X, \#, \{1\})$. Then $\sim_I$ is a congruence relation of $(X, \#, \{1\})$.

Proof: Let $\lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta, \lambda_l^\beta \in X$. Suppose that $\lambda_i^\beta \sim_I \lambda_j^\beta$ and $\lambda_k^\beta \sim_I \lambda_l^\beta$. Then $\lambda_i^\beta \# \lambda_j^\beta \in I$ and $\lambda_k^\beta \# \lambda_l^\beta \in I$. Then $\lambda_i^\beta \# \lambda_j^\beta \in I$ and $\lambda_k^\beta \# \lambda_l^\beta \in I$. Since I is normal, $\left(\lambda_k^\beta \# \lambda_i^\beta\right) \# \left(\lambda_k^\beta \# \lambda_j^\beta\right) \in I$ and hence $\left[\{1\} \# \left(\lambda_k^\beta \# \lambda_i^\beta\right)\right] \# \left[\{1\} \# \left(\lambda_k^\beta \# \lambda_j^\beta\right)\right] \in I$.

From Definition 3.2, we have that

$$\left(\lambda_i^\beta \# \lambda_k^\beta\right) \# \left(\lambda_j^\beta \# \lambda_k^\beta\right) \in I.$$

Thus, $\lambda_i^\beta \# \lambda_k^\beta \sim_I \lambda_i^\beta \# \lambda_j^\beta$.

Since $\lambda_k^\beta \# \lambda_l^\beta \in I$ we have that $\left(\lambda_j^\beta \# \lambda_k^\beta\right) \# \left(\lambda_j^\beta \# \lambda_l^\beta\right) \in I$.

Therefore, $\lambda_j^\beta \# \lambda_k^\beta \sim_I \lambda_j^\beta \# \lambda_l^\beta$. Thus $\sim_I$ is a congruence relation of $(X, \#, \{1\})$.

Definition 3.25: Let I be a normal permutation BF-ideal of $(X, \#, \{1\})$. For $\lambda_i^\beta \in X$, we write λ_i^β / I for the congruence class containing λ_i^β, i.e.

$$\lambda_i^\beta / I = \left\{\lambda_j^\beta \in X : \lambda_i^\beta \sim_I \lambda_j^\beta\right\}.$$

We note that $\lambda_i^\beta \sim_I \lambda_j^\beta \iff \lambda_i^\beta / I = \lambda_j^\beta / I$.

Denote $X/I = \left\{\lambda_i^\beta / I : \lambda_i^\beta \in X\right\}$ and set $\lambda_i^\beta / I \#' \lambda_j^\beta / I = \left(\lambda_i^\beta \# \lambda_j^\beta\right)/I$.

The operation $\#'$ is well defined, since $\sim_I$ is congruence relation of $(X, \#, \{1\})$. It is easy to see that $(X, \#, \{1\})/I = (X/I, \#', \{1\}/I)$ is a permutation BF-algebra. The permutation BF- algebra $(X, \#, \{1\})/I$ is called the *quotient permutation* BF-algebra of $(X, \#, \{1\})$ modulo I. there is a normal map θ_I, called the *quotient map*, from $(X, \#, \{1\})$ and $(X, \#, \{1\})/I$ defined by

$$\theta_I\left(\lambda_i^\beta\right) = \lambda_i^\beta / I \quad \forall \lambda_i^\beta \in X.$$

θ_I is clearly a homomorphism from $(X, \#, \{1\})$ to $(X, \#, \{1\})/I$.

Note that $Ker(\theta_I) = I$. Indeed $\lambda_i^\beta / I = \{1\}/I \iff \lambda_i^\beta \sim_I \{1\} \iff \lambda_i^\beta \# \{1\} \in I \iff \lambda_i^\beta \in I$.

Proposition 3.26: If $(X, \#, \{1\}_X)$ and $(X, \#, \{1\}_Y)$ are two permutation BF_2-algebras and let $\theta: X \longrightarrow Y$ be a homomorphism from $(X, \#, \{1\}_X)$ onto $(Y, \#, \{1\}_Y)$. Then $(X, \#, \{1\}_X)/Ker(\theta)$ is isomorphic to $(Y, \#, \{1\}_Y)$.

Proof: From Proposition 3.22 (1), $I = Ker(\theta)$ is a normal permutation BF-ideal of $(X, \#, \{1\}_X)$. Define a mapping

$$\varphi\left(\lambda_i^\beta / I\right) = \theta\left(\lambda_i^\beta\right) \quad \forall \lambda_i^\beta \in I.$$

Let $\lambda_i^\beta / I = \lambda_j^\beta / I$. Hence $\lambda_i^\beta \sim_I \lambda_j^\beta$, i.e. $\lambda_i^\beta \# \lambda_j^\beta \in I$. Thus $\theta\left(\lambda_i^\beta\right) \# \theta\left(\lambda_j^\beta\right) = \{1\}_Y$. From Proposition 3.8 (3), we have $\theta\left(\lambda_i^\beta\right) = \theta\left(\lambda_j^\beta\right)$. Thus, $\varphi\left(\lambda_i^\beta / I\right) = \varphi\left(\lambda_j^\beta / I\right)$.

This means that φ is well defined. It is easy to see that φ is a homomorphism from $(X, \#, \{1\}_X)/I$ to $(Y, \#, \{1\}_Y)$. Observe that $Ker\varphi = \{\{1\}_X \in I\}$. Indeed, $\lambda_i^\beta / I \in Ker\varphi \iff \varphi\left(\lambda_i^\beta / I\right) = \{1\}_Y \iff \theta\left(\lambda_i^\beta\right) = \{1\}_Y \iff \lambda_i^\beta \in I \iff \lambda_i^\beta / I \in \{1\}_X / I$. From Proposition 3.22 (2), it follows that φ is one-to-one. Thus φ is an isomorphism from $(X, \#, \{1\}_X)/I$ to $(Y, \#, \{1\}_Y)$.

4. Conclusion

Some novel conceptions that are extensions of BF-algebras are explored in this research, like permutation BF-algebras, permutation BF_1/BF_2-algebras, permutation BF-ideals, permutation BF-subalgebras, normal permutation BF-subalgebras, homomorphism of BF-algebras, congruence relation, quotient permutation BF-algebras and Quotient maps. On the other hand, their characteristics are specified using permutation sets, which have also been used to study various mathematical concepts in recent work. As a result, in future study, instead of using permutation sets, we will use nano and neutrosophic sets to extend our concepts and outcomes in this research using other non-classical sets.

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