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On permutation topological groups with their basic properties

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Abstract

In this research, one of the non-classical sets, the permutation set, is utilized to analyze a new class of topological groups induced by β-sets, it is named permutation topological groups, and some of its basic features are investigated. In addition, several results are applied to our permutation topological group class.

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1. Introduction

In mathematics, the symmetric group S_n is extremely important. It occurs in a wide variety of contexts, hence its importance cannot be overstated. Thousands of pages of research papers in mathematics journals have a certain connection to this group. According to Cayley's theorem, every finite group can be regarded as a subgroup of S_n for some n. Many mathematicians debate the concept of topological groups [1-5].

In 2014, the technical [6] is given to find the link between each permutation in S_n and topological space. We say (Ω, τ_n^β) is permutation topological space (PTS). Assume (Ω, τ_n^β) is a (PTS). We say (Ω, τ_n^β) is a permutation indiscrete space (PIS) iff $\beta = (t_1, t_2, \ldots, t_n)$ and permutation discrete

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space (PIS) iff $\beta = (t_1)(t_2)\ldots(t_n)$ [7]. In this work, we will use a permutation set to consider a new class of ∘topological groups it is called the permutation topological group induced by β-sets and investigate some of their fundamental features. Next, several results are applied to this class of permutation topological groups.

2. Preliminaries

Here, we will recall basic ideas and results that are necessary in this research.

Definition 2.1: [6] Define $\wedge$ and $\vee$ on β – sets in Ω by:

$$\delta_i^\beta \wedge \delta_j^\beta = \begin{cases} \delta_i^\beta, if \sum_{k=1}^{\sigma} t_k^i < \sum_{k=1}^{v} t_k^j \\ \delta_j^\beta, if \sum_{k=1}^{\sigma} t_k^i > \sum_{k=1}^{v} t_k^j \\ \delta^\beta, if \delta_i^\beta = \delta_j^\beta = \delta^\beta \\ \varphi, if \delta_i^\beta \& \delta_j^\beta aredisjoint \end{cases} \text{ and } \delta_i^\beta \vee \delta_j^\beta = \begin{cases} \delta_i^\beta, if \sum_{k=1}^{\sigma} t_k^i > \sum_{k=1}^{v} t_k^j \\ \delta_j^\beta, if \sum_{k=1}^{\sigma} t_k^i < \sum_{k=1}^{v} t_k^j \\ \delta^\beta, if \delta_i^\beta = \delta_j^\beta = \delta^\beta \\ \Omega, if \delta_i^\beta \& \delta_j^\beta aredisjoint \end{cases}$$

Permutation subspaces 2.2: [6] Let(Ω, τ_n^β) be (PTS) , $\delta \subset \Omega$ and $T_i^\beta = \delta^\beta \wedge \delta_i^\beta$, for each proper $\delta_i^\beta \in \tau_n^\beta$, then

$$T_i^\beta = \begin{cases} \{h_1^i, h_2^i, \ldots, h_{i_k}^i\}, \text{if} \delta^\beta \& \delta_i^\beta \text{are not disjoint} \\ \varphi, if \delta^\beta \& \delta_i^\beta \text{are disjoint} \end{cases}.$$

Let $\Re = \{T_i^\beta | T_i^\beta$ nonempty open β-set$\}$. Let $h_k^i = Max\{h_1^i, h_2^i, \ldots, h_{i_k}^i\}$, and $m = Max\{b_k^i; T_i^\beta \in \Re\}, \forall T_i^\beta \in \Re$. Assume $\sum_{T_i^\beta \in \Re} |T_i| = s$, and $t = m - s$, so we consider $H = \{h_1, h_2, \ldots, h_t\}$, with $H = \bigcap_{T_i^\beta \in \Re} (\Omega' - T_i^\beta)$ and Ω'={1, 2,…, n}. Here we used normal intersection ($\cap$) . Hence, $T_i = (h_1^i, h_2^i, \ldots, h_{i_k}^i)$ is i_k-cycle in $S_m, \forall T_i^\beta \in \Re$ we get $T_i = (b_1^i, b_2^i, \ldots, b_{i_k}^i)$ is i_k-cycle in S_m. Hence $\{\{T_i\}_{T_i^\beta \in \Re}, \{(h_r)\}_{r=1}^t\}$ are disjoint cycles in S_m induced by δ^β say γ^{δ^β}. Then $(\Omega', t_m^{\gamma^{\delta^\beta}})$ is a permutation subspace of (Ω, t_n^β), where $t_m^{\gamma^{\delta^\beta}} = \{\Omega', \varphi, \{T_i^\beta\}_{T_i^\beta \in \Re}, \{h_r\}_{r=1}^t\}$ and Ω'={1,2,…,m}

Definition 2.3: [8] A topological group $(G, *, \tau)$ is a group $(G, *)$ equipped with a topology τ on G such that the maps $\theta : G \times G \longrightarrow G$ are defined by $\theta(g, h) = g * h, \forall g, h \in G$ and the map $\vartheta : G \longrightarrow G$ defined by $\vartheta(g) = g^{-1}, \forall g \in G$ is continuous.

3. Permutation Topological Group Induced by β-Sets

In this section, we will give a new notion is called a permutation topological group induced by β-sets and study some of their basic properties.

Definition 3.1: Let $\{\delta_i^{\beta}\}_{i=1}^{c(\beta)}$ be a collection of β-sets, where β is a permutation in the symmetric group S_n Then Ω is said to be a permutation topological group (PTG) induced by β-sets if (Ω, #) is a group and there exists mappings θ:$\Omega \times \Omega \longrightarrow \Omega$, ϑ:$\Omega \longrightarrow \Omega$ such that $\theta(i,j) = i \# j$,$\forall i$, $j \in \Omega$ and $\vartheta(i) = i^{-1}$,$\forall$ $i \in \Omega$ are continuous. symbolized by $(\Omega, \#, \tau_n^{\beta})$.

Example 3.2: Assume (S_6,o) is a symmetric group and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 1 & 4 & 3 \end{pmatrix}$$

be a permutation in S_6. Since $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 1 & 4 & 3 \end{pmatrix} = (126354)$. Therefore, (Ω, τ_6^{β}) is a (PIS), where $\tau_6^{\beta} = \{\Omega, \varphi\}$, Define #:$\Omega \times \Omega \longrightarrow \Omega$ by the Table 1

Table 1
$(\Omega, \#)$ is a group

#	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Then (Ω,#) is a group, and hence $(\Omega, \#, \tau_6^{\beta})$is a (PTG) since (Ω, τ_6^{β}) is a (PIS).

Proposition 3.3:

(a) If $\acute{\Omega} = \{1,2, \dots, m\}$ is a subgroup of a (PTG) $(\Omega, \#, \tau_n^{\beta})$, then $(\acute{\Omega},, \#, t_m^{\gamma^{\delta\beta}})$ is a (PTG) in the induced topology,

(b) If $(\acute{\Omega}, \#)$ is a normal subgroup of a (PTG) $(\Omega, \#, \tau_n^{\beta})$, then the quotient group $(\Omega/\acute{\Omega}, \#, \tau)$ is a (PTG) concerning the quotient topology τ,

(c) A morphism of (PTG) $f: (\Omega_1, \#, \tau) \longrightarrow (\Omega_2, \#, \tau)$ is a continuous homomorphism between permutation topological groups (PTGs),

(d) A homomorphism of groups $f: (\Omega_1, \#, \tau) \longrightarrow (\Omega_2, \#, \tau)$ between (PTGs) is a morphism if and only if it is continuous at 1.

Proof: Straightforward.

Proposition 3.4: Assume that f is a bijective continuous map $f: (\Omega_1, \#, \tau) \longrightarrow (\Omega_2, \#, \tau)$ between two (PTGs), then f is a *homeomorphism* if its inverse is also continuous, that is, a bijection f is a homeomorphism if and only if U is open in Ω_1 implies that $f(U)$ is open in Ω_2 or $f^{-1}(V)$ is open in Ω_1 implies that V is open in Ω_2 holds for all $U \subseteq \Omega_1$ or $V \subseteq \Omega_2$.

Proof: Straightforward.

Proposition 3.5: Assume that $(\Omega, \#, \tau)$ is a (PTG), the for all $a \in \Omega$, both the left and right translations $f^a(x) = x \# a$, and $f_a(x) = a \# x$, $x \in \Omega$ of a are homeomorphism.

Proof: Straightforward.

Proposition 3.6: (Translation Invariance)
For any (PTG) $(\Omega, \#, \tau)$, $U \subseteq \Omega$ and $g \in \Omega$, then the following are equivalent: (a) U is open,(b) gU is open, (c) Ug is open,(d) U^{-1} is open.

Proof: We will prove the first implication only since the remaining ones are similar. Let $g \in \Omega$ be a fixed element and U be an open subset of Ω. We will show that gU is open since, for each element $x \in gU$, we can find an open set U_x containing x such that $U_x \subseteq gU$. Then the equality $gU = \bigcup_{x \in gU} U_x$ will prove that gU is open. Now, let $x \in gU$. One can therefore find $u \in U$ such that $x = gu$. Then the continuous function $\#$ defined satisfies $g \# u = x$ and since gU is an open set that contains x, then the definition of product topology implies that one can find open sets U_g, U_u contains g and u such that $U_g U_u \subseteq gU$. We can then define $U_x = U_g U_u$ and observe that $x = gu \in U_g U_h = U_x$ and $U_x = \bigcup_{h \in U_u} U_g h$ is open. This completes the proof.

Proposition 3.7: Assume that $(\Omega, \#, \tau)$ is a (PTG). Then $(\Omega, \#, \tau)$ has an open basis $\mathfrak{W}$ at e that consists of symmetric neighborhood U of e. (A Symmetric neighborhood of e is a set U that contains an open set containing e such that $U = U^{-1}$).

Proof: Straightforward.

Proposition 3.8: The closure $\bar{A}$ of a subset A of Ω, where $(\Omega, \#, \tau)$ is a (PTG) is given by $\bar{A} = \bigcap A \# U$, where U is open and $e \in U$.

Proof: Straightforward.

Proposition 3.9: Assume that H is a subgroup of Ω, where $(\Omega, \#, \tau)$ be a (PTG). Then $\overline{H}$ is also a subgroup of Ω.

Proof: Suppose that H is a subgroup of $(\Omega, \#, \tau)$. We are required to prove that $\overline{H}$ is also a subgroup. Now $e \in H$, and $H \subset \overline{H}$. So, $\overline{H}$ is non-empty since $e \in \overline{H}$. Again $H \subset h^{-1}\overline{H}$ for all $h \in H$ because $hH = H$, so $\overline{H} \subset \overline{h^{-1}\overline{H}} = h^{-1}\overline{H}$ for all $h \in H$, thus $h\overline{H} \subset \overline{H}$ for all $h \in H$. Next, if $x \in \overline{H}$ then $Hx \subset \overline{H}$, so $H \subset \overline{H}x^{-1}$ which gives $\overline{H} \subset \overline{\overline{H}x^{-1}} = \overline{H}x^{-1}$ (since $\overline{H}x^{-1}$ is closed). Thus $\overline{H}\,\overline{H} \subset \overline{H}$. So, if

u, v are in $\overline{H}$ then $uv \in \overline{H}$. To complete the proof, we need to show that if $u \in \overline{H}$, then $u^{-1} \in \overline{H}$. Now since $g: w \longmapsto w^{-1}$ is a homeomorphism, we have that g is a closed map. So, $g(\overline{H})$ is a closed set. Now, $g(\overline{H}) \subset \overline{g(H)} \subset \overline{g(\overline{H})} \subset g(\overline{H})$. And so $\overline{g(H)} = g(\overline{H})$ which gives $\overline{(H^{-1})} = (\overline{H})^{-1}$. Since H is a subgroup, we have that $H = H^{-1}$. Thus, $\overline{H} = (\overline{H})^{-1}$. So, if $u \in \overline{H}$ then $u^{-1} \in (\overline{H})^{-1} = \overline{H}$. That completes the proof.

Proposition 3.10: If $(\Omega, \#, \tau)$ is a (PTG), and if (Ω, τ) is a T_0-space, then (Ω, τ) is Hausdorff.

Proof: Straightforward.

Proposition 3.11: If $(\Omega, \#, \tau)$ is a (PTG), and if (Ω, τ) is a T_0-space, then (Ω, τ) is Hausdorff.

Proof: Since $(\Omega, \#, \tau)$ is a T_0-space, we have that for $x \in \Omega$ with $x \neq e$, either there is $U(x)$ with $e \notin U(x)$, or there is $U(e)$, where $U(g)$ means that $U(g)$ is open and contains $g \in \Omega$. If the former holds then, $x \notin \{e\}'$ ($\{e\}'$ is the derived set of $\{e\}$), which means that $\{e\}' =$ the empty set $\emptyset_\Omega$ of Ω. If the latter holds then $e \in \overline{\{e\}}$. In either case, $\overline{\{e\}} = \{e\}$, since $\bar{A} = A \cup A'$ for every subset A of Ω in (Ω, τ). Next, if $x \neq y$, (x, y are in Ω) then $x \# y^{-1} \in \Omega \backslash \{e\} = \{e\}^c$, since $x \# y^{-1} = e$ would imply $x = y$. Since $\{e\}$ is closed from $\overline{\{e\}} = \{e\}$, we have that $\Omega \backslash \{e\} = \{e\}^c$ is open. By the continuity of $f: (u, v) \longmapsto u \# v$, we have that there exist U, W in τ with $x \in U$ and $y^{-1} \in W$ such that $f(U \times W) \subset \Omega \backslash \{e\}$. So, $U \times W \subset f^{-1}(\Omega \backslash \{e\})$ and since $y^{-1} \in W$ and $W \in \tau$, we have by the continuity of $h: u \longmapsto u^{-1}$ that there is $V \in \tau$ with $y \in V$ such that $h(V) \subset W$. Lastly, U and V are disjoint because if there is a point z of Ω with $z \in U$ and $z \in V$, then we would have $z^{-1} = h(z) \in W$ which implies that $f\big((z, z^{-1})\big) \in G \backslash \{e\}$implying that $e = z \# z^{-1} \in \Omega \backslash \{e\}$; a contradiction. So U, V are open and $U \cap V = \emptyset$, with $x \in U$ and $y \in V$. Therefore, $(\Omega, \#, \tau)$ is a T_2-space.

Corollary 3.12: *If* $\{e\}$ *is a closed-in (PTG)* $(\Omega, \#, \tau)$*, then* $(\Omega, \#, \tau)$ *is a* T_2*-space (Hausdorff space) and infact a* T_3*-space (Regular space).*

Proof: Suppose that $\{e\}$ is closed. Then $\overline{\{e\}} = \{e\}$ holds. So, $(G, \#, \tau)$ is a T_2-space from the proof of Proposition 3.11since $\{e\}^c = \Omega \backslash \{e\}$ is open. Now for each open set U containing e, there is an open set V containing e that satisfies $VV \subset U$ by the continuity of $f: (u, v) \longmapsto u \# v$ implying that $\bar{V} \subset VV \subset U$. We get that $(\Omega, \#, \tau)$ is a T_3-space.

Definition 3.13: Suppose that f is a function from a permutation measurable function into itself. Then,

a) $f: \Omega \longrightarrow \Omega$ is said to be *right uniformly continuous* if, for each neighborhood V of e, there exists a neighborhood U of e such that $f(x) \# \big(f(y)\big)^{-1} \in V$ for all $x, y \in \Omega$ such that $x \# y^{-1} \in U$.

b) $f: G \longrightarrow G$ is said to be *left uniformly continuous* if, for each neighborhood V of e, there exists a neighborhood U of e such that $\left(f(y)\right)^{-1} \# f(x) \in V$ for all $x, y \in \Omega$ such that $y^{-1} \# x \in U$.

c) $f: \Omega \longrightarrow \Omega$ is said to be *uniformly continuous* if it is right uniformly continuous and left uniformly continuous.

Proposition 3.14: Let $(\Omega, \#, \tau)$ be a (PTG).

a) $\overline{\{e\}}$ is a closed, normal subgroup of Ω, and the smallest closed subgroup of Ω,

b) If $a \in \Omega$, we have that $\overline{\{a\}}$ is the closet $a \# \overline{\{e\}}$ of $H = \overline{\{e\}}$.

Proof:

a) If x, z belong to H, where $H = \overline{\{e\}}$, then $x \# z$ belongs to H because $x \# z$ belongs to $H \# H \subset \overline{\{e \# e\}} = \overline{\{e\}} = H$. If $y \in H$ then $y^{-1} \in H^{-1} = \left(\overline{\{e\}}\right)^{-1} = \overline{(\{e\})^{-1}} = \overline{\{e\}} = H$. Since $e \in H$, H is non-empty. So, H is a subgroup of Ω. Now $\bar{A} = \bigcap A \# U$, where U is open and $e \in U$. So, $e \in \bar{A}$ if A is a subgroup of Ω. Therefore $\{e\} \subset \bar{A}$ for every subgroup of Ω. Thus, $\overline{\{e\}} \subset \overline{(\bar{A})} = \bar{A}$, if A is a subgroup of Ω. Next, H is a normal subgroup of Ω because if $a \in \Omega$ then $a \# \overline{\{e\}} \# a^{-1} = \overline{\{a \# e \# a^{-1}\}}$ because $x \# \bar{A} \# y = \overline{x \# A \# y}$ for any points $x, y \in \Omega$ and any subset A of Ω since the map $h: z \longmapsto x \# z \# y$ is a homeomorphism of Ω onto Ω, so $x \# \bar{A} \# y$ is closed, (as h is a closed map), hence $x \# \bar{A} \# y \supset \overline{x \# A \# y}$, the reverse inclusion holds by the continuity of h, (since $h(\bar{A}) \subset \overline{h(A)}$). So, $a \# \overline{\{e\}} \# a^{-1} = \overline{\{e\}} = H$. Therefore, H is a normal subgroup of Ω.

b) Since a Right translation $\rho_a: w \longmapsto a \# w$ of $a \in \Omega$, is a homeomorphism, we have that $\overline{\{a\}} = a \# \overline{\{e\}}$, now as ρ_a is a closed map and hence $a \# \overline{\{e\}}$ is a closed set which implies that $a \# \overline{\{e\}} \supset \overline{a \# e} = \bar{a}$, the reverse inclusion holds because $\rho_a(\bar{B}) \subset \overline{\rho_a(B)}$ for each subset B of Ω since ρ_a is continuous. Similarly, because a left translation $f^a: w \longmapsto w \# a$ of $a \in \Omega$, is also a homeomorphism on e, we have that $\overline{\{a\}} = \overline{\{e\}} \# a$ for all $a \in \Omega$. This completes the proof.

Proposition 3.15: A subgroup H of a (PTG) $(\Omega, \#, \tau)$ is discrete if and only if H has an isolated point.

Proof: Suppose H has an isolated point x, say in the relative topology $\tau_H = \tau \cap H = \{U \cap H : U \in \tau\}$ of the subgroup H. Then there is an open neighborhood V of e in $(\Omega, \#, \tau)$ such that $(xV) \cap H = \{x\}$. Then, for all $y \in H$, we have $(yV) \cap H = (yV) \cap (yx^{-1}H) = yx^{-1}\left((xV) \cap H\right) = \{y\}$. So, every point of H is isolated and therefore H is a discrete space. Conversely, if H is discrete, then each point of H is an isolated point. So, e being in H, is an isolated point.

4. Conclusion

In this work, we provided the first definition of a PTG. We explored some of the basic features of a new class of topological groups this class it is induced by β-sets. Here we used the permutation sets and in future work we will try to extend this class using other non-classical classes to consider new structure and discuss their results.

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