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Toeplitz Determinants for Inverse of Analytic Functions

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Abstract: Estimates bounds for Carathéodory functions in the complex domain are applied to demonstrate sharp limits for the inverse of analytic functions. Determining these values is considered a more difficult task compared to finding the values of analytic functions themselves. The challenge lies in finding the sharp estimate for the functionals. While some recent studies have made progress in calculating the sharp boundary values of Hankel determinants associated with inverse functions, the Toeplitz determinant is yet to be addressed. Our research aims to estimate the determinants of the Toeplitz matrix, which is also linked to inverse functions. We also focus on computing these determinants for familiar analytical functions (pre-starlike, starlike, convex, symmetric-starlike) while investigating coefficient values. The study also provides an improvement to the estimation of the determinants of the pre-starlike class presented by Li and Gou.

Keywords: analytic functions; bi-univalent functions; Toeplitz determinants

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1. Introduction

We symbolize the category of analytic functions f in $\mathbb{U}$ as $\mathcal{A}$, where

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \ (a_j = \frac{f^{(j)}}{j!}) \tag{1}$$

and

$$\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$$

is the open unit disc. The set $\mathcal{S} \subset \mathcal{A}$ comprises of all functions f that are both normalized $(f(0) = f'(0) - 1 = 0)$ and univalent. Bieberbach [1] initially proposed the well-known coefficient conjecture for the function $f \in \mathcal{S}$ of the form (1) in 1916, and de-Branges [2] proved it in 1985. Between 1916 and 1985, several studies attempted to validate or refute this conjecture. As a result, they identified many subfamilies $\mathcal{S}$ associated with various domains. The most basic subfamilies of $\mathcal{S}$ are the starlike function family $\mathcal{S}^*$, the convex function family $\mathcal{C}$ and the close-to-convex function family $\mathcal{K}$. These families have a geometric description. The following describes these families:

Definition 1 ([3])**.** *The families of pre-starlikeness $\mathcal{S}_{pre}$, starlikeness $\mathcal{S}^*$, convexity $\mathcal{C}$, and symmetric-starlike $\mathcal{S}_{sym}$, which are involved in univalent functions, can be stated as*

$$\mathcal{S}_{pre} := \{f \in \mathcal{S} : \Re f'(z) > 0\}, \tag{2}$$

$$\mathcal{S}^* := \left\{f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0\right\}, \tag{3}$$

$$\mathcal{C} := \left\{f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0\right\}, \tag{4}$$

$$\mathcal{S}_{sym} := \left\{f \in \mathcal{S} : \Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0\right\}, \tag{5}$$

respectively.

Further, the families $\mathcal{S}^*$ and $\mathcal{C}$ can be formulated by $zf'(z) = \gamma(z)f(z)$ and $f'(z) + zf''(z) = \gamma(z)f'(z)$, respectively, where $\gamma(z)$ is the familiar class of Carathéodory functions $\gamma(z) = 1 + \sum_{j=1}^{\infty} \gamma_j z^j$ with $\mathcal{R}(\gamma(z)) > 0$ pointed by $\mathcal{P}$.

We list some examples that satisfy the starlike and convex functions.

Example 1. *The Köebe function $k(\zeta) = \frac{\zeta}{(1-\zeta)^2}$ is starlike, since*

$$\Re\left(\frac{\zeta k'(\zeta)}{k(\zeta)}\right) = \Re\left(\frac{1-\zeta^2}{(1-\zeta)^2}\right) = \frac{(1 - rcos2\vartheta)}{(1 + r^2 - 2rcos\vartheta)} > 0, \quad (\zeta = re^{i\vartheta}; \zeta \in \mathbb{U}).$$

Example 2. i. *The function $h(\zeta) = \zeta + 1/4\zeta^2$ is convex, since*

$$\Re\left(1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right) = \Re\left(\frac{1+\zeta}{1+\frac{1}{2}\zeta}\right) = \Re\left(\frac{(1+\zeta)(1+\frac{1}{2}\bar{\zeta})}{(1+\frac{1}{2}\zeta)(1+\frac{1}{2}\bar{\zeta})}\right) > 0, \quad (\zeta \in \mathbb{U}).$$

ii. *Similarly, with $|\zeta| < 1$, the functions $\hbar_1(\zeta) = \frac{1}{2}log(\frac{1+\zeta}{1-\zeta})$ and $\hbar_2(\zeta) = -log(1-\zeta)$ are also convex functions.*

The inverse function f^{-1} of the univalent function f has been satisfied utilizing the Köebe Theorem of $1/4$ (see [4]), which asserts that a disc with a radius of $f'(0)/4$ and a center at $f(0)$ is present in the image of f if a holomorphic function f is univalent in $\mathbb{U}$. The form of the inverse functions is stated by

$$f^{-1}(f(z)) = z,\ z \in \mathbb{U}, \quad \text{and} \quad f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right), \tag{6}$$

where

$$f^{-1}(w) = w + \sum_{j=2}^{\infty} u_j w^j.$$

Since $w = f\left(f^{-1}(w)\right)$, we attain

$$u_2 = -a_2, \tag{7}$$

$$u_3 = 2a_2^2 - a_3, \tag{8}$$

$$u_4 = -5a_2^3 + 5a_2a_3 - a_4, \tag{9}$$

and

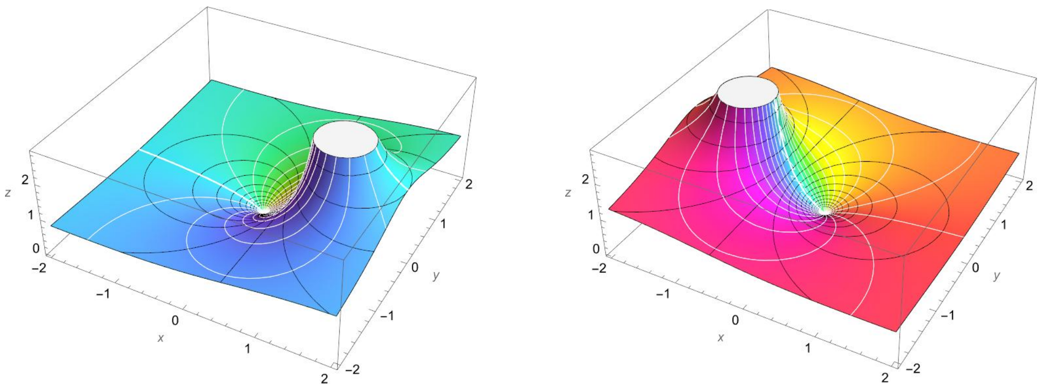
$$f^{-1}(w) = w + u_2w^2 + u_3w^3 + u_4w^4 + \dots. \tag{10}$$

The function f considers as a bi-univalent function if f and f^{-1} are both univalent. Σ represents the family of bi-univalent functions.

Example 3. *The familiar examples for this family are presented as follows:*

1. *The inverse of logarithm function* $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ *is* $\frac{e^{2w}-1}{e^{2w}+1}$.
2. *The inverse of function* $\frac{z}{1-z}$ *is* $\frac{w}{1+w}$.
3. *The inverse of function* $-\log(1-z)$ *is* $\frac{e^{w}-1}{e^{w}}$.

The subsequent illustrations (Figure 1) elucidate the concept of the analytic function 2 and its corresponding inverse. It also indicates that the structure (6) is satisfied.



(**a**) The function $f(z) = \frac{z}{1-z}$. (**b**) The inverse function $f^{-1}(w) = \frac{w}{1+w}$.

Figure 1. Graphs of the analytic function and its inverse.

As demonstrated by Löwner [5], the boundary values for the coefficients of inverse analytic function f^{-1} are estimated by

$$|u_j| \leq \frac{(2j)!}{j!(j+1)!}. \tag{11}$$

Afterward, many scholars provided that the sharpness is attained by the inverse of Köebe function. No known limits seem to exist for the difference in coefficients $|u_{j+1} - u_j|$, even when $j = 2$, despite the fact that sharp boundaries for $|u_j|$ for $j \geq 2$ are discovered. Sharp boundaries for $|u_3 - u_2|$ were given by Sim and Thomas [6] in 2020. Libera et al. [7] (also refer to [8,9]) determined the correlation between the f and f^{-1} coefficients. Conversely, Kapoor and Mishra [10] expanded on the findings of Krzyz et al. [11], who looked at bounds on the initial coefficients of the inverse of starlike functions. Furthermore, when a function comes into the family of strongly starlike functions, Ali [12] investigated the sharpness of the coefficients of inverse functions and the Fekete–Szegö problem. Determining how the inverse function f^{-1} provided in (10) behaves when the original function f is limited to certain valid geometric subfamilies of $\mathcal{S}$ has garnered a lot of attention. Yang [13] provided a more straightforward demonstration, while several writers have provided other proofs of the inequality (11). Very recently, Lecko and Śmiarowska [14] went towards studying the second-order Hankel determinant involving the logarithmic function $\frac{1}{2}\log\frac{f(z)}{z}$. For further investigation on the sharpness of inverse holomorphic functions, see [15,16].

The estimation of Hankel matrix limits has received a great deal of attention in the field of univalent function theory. Hankel matrices and determinants have a significant role in several mathematical applications (see [17,18]). There is a strong relationship between Toeplitz determinants and Hankel determinants. In contrast to Hankel matrices, which have constant entries along the inverse diagonal, Toeplitz matrices have constant entries along the diagonal.

The Toeplitz matrix defined by Ali et al. [19] for integers $j, \mathrm{m} \in \mathbb{N}$ is provided by

$$\mathcal{T}_{\mathrm{m}}^{j} = \begin{vmatrix} a_j & a_{j+1} & \cdots & a_{j+\mathrm{m}-1} \\ a_{j+1} & a_j & \cdots & a_{j+\mathrm{m}-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j+\mathrm{m}-1} & a_{j+\mathrm{m}-2} & \cdots & a_j \end{vmatrix}.$$

This leads to

$$\mathcal{T}_2^2 = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \mathcal{T}_2^3 = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \mathcal{T}_3^1 = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \text{ and } \mathcal{T}_3^2 = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

Rogozina [20] investigated the employee of Toeplitz matrices to find difference equations in numerical and time-dynamic models. For estimating the limits of Hankel and Toeplitz determinants for families when the coefficients are inverse functions, Maharana et al. [21] in 2020 investigated the Hankel determinants H_2^1 and H_3^1 for the starlike functions. Additionally, the analysis of Hankel inequalities for various families was looked at by Ali et al. [19] (also [22–29]). In 2023, the exact bounds of Hankel determinants for the strongly inverse analytic functions were derived by Allu and Shaji [30]. The fifth estimation $|u_5|$ for the families of strongly convex inverse functions was deduced by Daniswara et al. [31]. Numerous analyses concentrated on utilizing inverse analytic functions to estimate the limits of Hankel determinants (for further pertinent work; see [6,32–37]). Shi et al.'s [38] investigation of the Hankel determinant in relation to inverse functions is the most current. For the inverse holomorphic functions, Shi and Colleagues [39] revealed the limit of the third Hankel inequality H_3^1, adding a new dimension to this field. On the other hand, Kumar [40] computed the third Hankel determinant for the class of pre-starlike functions, providing accurate results that enhance mathematical understanding. Rath et al. [41] also made a significant contribution to clarifying the sharp limit of the third Hankel determinant for symmetric starlike functions, adding important insights to previous studies. The most recent study, by Abbas et al. [42], sheds light on the sharp limit of the Hankel functional for starlike functions relevant to the cosine function.

The coefficient boundaries of the Toeplitz determinant are not known at this time. Due to the difficulty of calculating the upper limit of the Toplitz inequalities, only one article was published in [43] for these inequalities related to the family of pre-starlike. Inspired by the research undertaken by the scholars Li and Gou [43] in their work, we attempt to give basic findings for the Toeplitz determinants whose coefficients depend on inverse analytic functions containing the families of pre-starlike, starlike, convex, and symmetric-starlike functions. The second and third-order $\mathcal{T}_2^2, \mathcal{T}_2^3, \mathcal{T}_3^1$, and $\mathcal{T}_3^2$ determinants are among them.

2. Preliminary Lemmas

To be able to demonstrate our primary findings, we necessitate the following lemmas, which are considered part of the procedures used to acquire the asset results connected to the inverse of analytic functions.

Lemma 1 ([4])**.** *If the function $\gamma(z) \in \mathcal{P}$, then*

$$|\gamma_j| \le 2 \quad (j \ge 1).$$

Lemma 2 ([44])**.** *If the function $\gamma(z) \in \mathcal{P}$ and $\zeta \in \mathbb{C}$, then*

$$|\gamma_{n+k} - \zeta \gamma_n \gamma_k| \le 2 \max\{1; |2\zeta - 1|\}.$$

Lemma 3 ([8])**.** *If the function $\gamma(z) \in \mathcal{P}$, for some $\mu, z \in \overline{\mathbb{U}} = \mathbb{U} \cup \{1\}$ with $\gamma_1 \geq 0$, then*

$$2\gamma_2 = \gamma_1^2 + \mu\left(4 - \gamma_1^2\right)$$

and

$$4\gamma_3 = \gamma_1^3 + 2\left(4 - \gamma_1^2\right)\gamma_1\mu - \left(4 - \gamma_1^2\right)\gamma_1\mu^2 + 2\left(4 - \gamma_1^2\right)\left(1 - |\mu|^2\right)z.$$

3. Main Results

Inspired by initial coefficient bounds for pre-starlike holomorphic functions $\mathcal{S}_{pre}$, that were studied by Radhika et al. [29], Li and Gou [43] examined the upper bounds of Toeplitz determinants for inverse holomorphic functions. They claimed that the results were $|\mathcal{T}_2^2(f^{-1})| \leqq 0.73$, $|\mathcal{T}_3^1(f^{-1})| \leqq 1.4$, $|\mathcal{T}_2^3(f^{-1})| \leqq 3.023$, and $|\mathcal{T}_3^2(f^{-1})| \leqq 1.27$. With some simple calculations, noting that the results are not correct. In demonstrating these estimations, we notice that the values are inaccurate. Since the inequality $|\mathcal{T}_{\mathsf{m}}^j|$ is not rotational invariant because

$$\left|\mathcal{T}_{\mathsf{m}}^j(f)\right| \neq \left|\mathcal{T}_{\mathsf{m}}^j(f_\theta)\right|, \text{ for all } \theta \in \mathbb{R}, j, \mathsf{m} \in \mathbb{N}.$$

Similarly, the inequality $|\mathcal{T}_{\mathsf{m}}^j f^{-1}|$ is not rotational invariant because

$$\left|\mathcal{T}_{\mathsf{m}}^j\left(f^{-1}\right)\right| \neq \left|\mathcal{T}_{\mathsf{m}}^j\left(f_\theta^{-1}\right)\right|, \text{ for all } \theta \in \mathbb{R}, j, \mathsf{m} \in \mathbb{N}.$$

Particularly,

$$\left|\mathcal{T}_2^2\left(f^{-1}\right)\right| = \left|u_3^2 - u_2^2\right| \neq \left|\mathcal{T}_2^2\left(f_\theta^{-1}\right)\right| = \left|\left(e^{i2\theta}u_3\right)^2 - \left(e^{i\theta}u_2\right)^2\right|, \text{ for all } j, \mathsf{m} \in \mathbb{N}.$$

In Theorem 1 below, we demonstrate and discuss the limit of Toeplitz determinants.

Theorem 1. *If $f(z) \in \mathcal{S}_{pre}$, for $f(z)$ in (1), then the sharp bounds of the Toeplitz inequalities are*

1. $|\mathcal{T}_2^2(f^{-1})| \leqq 7.22$.
2. $|\mathcal{T}_2^3(f^{-1})| \leqq 168.694$.
3. $|\mathcal{T}_3^1(f^{-1})| \leqq 3.88$.
4. $|\mathcal{T}_3^2(f^{-1})| \leqq 64.79$.

These estimates are sharp (except inequality $\mathcal{T}_3^2$) for when

$$f(z) = \frac{1 + iz}{1 - iz} = 1 + 2iz - 2z^2 - 2iz^3 + 2z^4 + \cdots.$$

Proof. The initial coefficients $|a_j|$ analyzed by Radhika et al. [29] are given by

$$a_2 = \frac{1}{2}\gamma_1, \tag{12}$$

$$a_3 = \frac{1}{3}\gamma_2, \tag{13}$$

$$a_4 = \frac{1}{4}\gamma_3. \tag{14}$$

Correspondence of the values $|a_j|$, $(a_j = 2,3,4)$ above to the coefficients (7)–(9), it follows that

$$u_2 = -\frac{1}{2}\gamma_1, \tag{15}$$

$$u_3 = -\frac{1}{6}\left(2\gamma_2 - 3\gamma_1^2\right), \tag{16}$$

$$u_4 = -\frac{1}{12}\left(15\gamma_1^3 - 10\gamma_1\gamma_2 + 3\gamma_3\right). \tag{17}$$

1. In investigating this estimate, the researchers claimed that the maximum value is achieved by $\gamma = \sqrt{\frac{17}{14}}$. In fact, the authors made incorrect simplifications to find out the value of $\Phi(\gamma, 1)$. We show, in short, that the maximum value happens when $\gamma = 2$.

$$\mathcal{T}_2^2(f^{-1})| = |u_3^2 - u_2^2| \leqq \left|u_3^2\right| + \left|u_2^2\right| = |\frac{1}{36}(4\gamma_2^2 - 12\gamma_1^2\gamma_2 + 9\gamma_1^4)| + |\frac{1}{4}\gamma_1^2|$$
$$|\frac{4}{36}(|\gamma_2||\gamma_2^2 - 3\gamma_1^2| + \frac{9}{4}|\gamma_1^4|)| + \frac{1}{4}|\gamma_1|^2 \leqq \frac{56}{9} + 1 = 7.22.$$

2. We conclude by using u_3 and u_4 in (16) and (17), respectively, that

$$|\mathcal{T}_2^3(f^{-1})| = |u_4^2 - u_3^2| \leqq \left|u_4^2\right| + \left|u_3^2\right| \leqq (\frac{25}{16}|\gamma_1^6| + \frac{25}{36}|\gamma_1^2||\gamma_2||\gamma_2 - 3\gamma_1^2|$$
$$+ \frac{5}{2}|\gamma_1||\gamma_3||\gamma_2 - \frac{12}{8}\gamma_1^2| + \frac{1}{16}|\gamma_3^2|) + \frac{56}{9} = 168.694.$$

The expected values are derived by considering Lemmas 1 and 2. It has been shown that the maximum values occur at $\gamma = 2$. Subsequently, the estimate is sharp.

3. Lemmas 1 and 2, u_2 and u_3 in Theorem 1 allow us to deduce that

$$|\mathcal{T}_3^1(f^{-1})| = \left|1 + 2u_2^2(u_3 - 1) - u_3^2\right| \leqq 1 + 2\left|u_2^2\right| + |u_3|\left|u_3 - 2u_2^2\right|$$
$$\leqq 1 + 2 + \frac{4}{9}|\gamma_2| \leqq 1 + 2 + \frac{8}{9} = 3.88$$

and the sharp bound of the Toeplitz inequality $\mathcal{T}_3^1$ is achieved.

4. The precise formula for the functional $|\mathcal{T}_3^2(f^{-1})|$ is

$$|\mathcal{T}_3^2(f^{-1})| = \left|(u_2 - u_4)(u_2^2 - 2u_3^2 + u_2u_4)\right|. \tag{18}$$

By computing the values of $|u_2 - u_4|$ and $\left|u_2^2 - 2u_3^2 + u_2u_4\right|$, we may be estimated the boundary value of this determination.
Firstly, it is clear that

$$|u_2 - u_4| \leqq |u_2| + |u_4| = |\frac{1}{2}\gamma_1| + \frac{1}{12}|15\gamma_1^3 - 10\gamma_1\gamma_2 + 3\gamma_3|$$
$$\leqq 1 + \frac{5}{6}|\gamma_1||\gamma_2 - \frac{6}{4}\gamma_1^2| + \frac{1}{4}|\gamma_3| \leqq 1 + \frac{20}{3} + \frac{1}{2} = 8.16.$$

Now, estimating the second part of the functional with the assistance of u_2, u_3, u_4, Lemmas 1 and 2 as follows:

$$\left|u_2^2 - 2u_3^2 + u_2u_4\right| = \frac{1}{72}\left|9\gamma_1^4 + 18\gamma_1^2 + 18\gamma_1^2\gamma_2 - 16\gamma_2^2 + 9\gamma_1\gamma_3\right|$$
$$\leqq \frac{1}{72}\left[9|\gamma_1|^4 + 18|\gamma_1|^2 + 16|\gamma_2||\gamma_2 - \frac{9}{8}\gamma_1^2| + 9|\gamma_1||\gamma_3|\right] \leqq 7.94.$$

□

Theorem 2. *If $f(z) \in \mathcal{S}^*$, for $f(z)$ appearing in (1), the sharp bounds of the Toeplitz inequalities are*

1. $|\mathcal{T}_2^2(f^{-1})| \leqq 51.$
2. $|\mathcal{T}_2^3(f^{-1})| \leqq 116.33.$
3. $|\mathcal{T}_3^1(f^{-1})| \leqq 24.$
4. $|\mathcal{T}_3^2(f^{-1})| \leqq 650.56.$

All sharp estimates are derived from the inverse of

$$f(z) = \frac{z}{(1-iz)^2} = z + 2iz^2 - 3z^3 - 4iz^4 + \cdots .$$

Proof. Building on simplification of the starlike form $zf'(z) = \gamma(z)f(z)$, we deduce

$$a_2 = \gamma_1, \tag{19}$$

$$a_3 = \frac{1}{2}\left(\gamma_2 + \gamma_1^2\right) \tag{20}$$

and

$$a_4 = \frac{1}{6}\gamma_1^3 + \frac{1}{2}\gamma_1\gamma_2 + \frac{1}{3}\gamma_3. \tag{21}$$

Comparing (7)–(9) with (19)–(21), we have

$$u_2 = -\gamma_1, \tag{22}$$

$$u_3 = -\frac{1}{2}\left(\gamma_2 - 3\gamma_1^2\right), \tag{23}$$

$$u_4 = -\frac{1}{3}\left(\gamma_3 - 6\gamma_1\gamma_2 + 8\gamma_1^3\right). \tag{24}$$

1. Since $f \in \mathcal{S}^*$, then via Lemma 2, we have

$$|\mathcal{T}_2^2(f^{-1})| = |u_3^2 - u_2^2| \leqq \left|u_3^2\right| + \left|u_2^2\right| = |(\frac{1}{2}(\gamma_2 - 3\gamma_1^2))^2| + |\gamma_1|^2 \tag{25}$$

$$\leqq \frac{9}{4}|\gamma_1^4| + \frac{1}{4}|\gamma_2||\gamma_2 - 6\gamma_1^2| + |\gamma_1^2| \leqq 47 + 4 = 51. \tag{26}$$

2. By considering $|\gamma_j| \leqq 2$, the upper bound of the inequality's permissible range, we may now easily proceed to the largest possible value of $\mathcal{T}_2^3(f^{-1})$.

$$|\mathcal{T}_2^3(f^{-1})| = |u_4^2 - u_3^2| \leqq \left|u_4^2\right| + \left|u_3^2\right| \leqq \frac{64}{9}|\gamma_1|^6 + \frac{4}{3}|\gamma_1||\gamma_3||\gamma_2 - \frac{4}{3}\gamma_1^2|$$
$$+ 4|\gamma_1^2||\gamma_2||\gamma_2 - \frac{8}{3}\gamma_1^2| \leqq \frac{208}{3} + 47 = 116.33.$$

3. With analogous technique to Theorem 1, we readily procure that

$$|\mathcal{T}_3^1(f^{-1})| \leqq 1 + 2\left|u_2^2\right| + |u_3|\left|u_3 - 2u_2^2\right|$$
$$\leqq 1 + 2\left|\gamma_1^2\right| + \frac{5}{2}\left|\gamma_2 + \gamma_1^2\right| \leqq 1 + 8 + 15 = 24.$$

4. The technique used for the functional $|\mathcal{T}_3^2(f^{-1})|$ in (18) yields

$$|u_2 - u_4| \leqq |u_2| + |u_4| \leqq |\gamma_1| + \frac{1}{3}|\gamma_3| + 2|\gamma_1||\gamma_2 - \frac{4}{3}\gamma_1^2| \leqq 2 + \frac{2}{3} + \frac{40}{3} = 16.$$

Afterwards

$$\left|u_2^2 - 2u_3^2 + u_2u_4\right| = \frac{1}{6}\left|6\gamma_1^2 - 11\gamma_1^4 + 6\gamma_1^2\gamma_2 - 3\gamma_2^2 + 2\gamma_1\gamma_3\right|$$
$$\leqq \frac{1}{6}\left[11|\gamma_1|^4 + 6|\gamma_1|^2 + 3|\gamma_2||\gamma_2 - 2\gamma_1^2| + 2|\gamma_1||\gamma_3|\right] \leqq 40.66,$$

which leads to the result.

□

The following investigations give the sharp boundary values of the convex functions associated with the inverse functions:

Theorem 3. *If $f(z) \in \mathcal{C}$, for $f(z)$ appearing in (1), then the bounds of the Toeplitz inequalities are*

1. $|\mathcal{T}_2^2(f^{-1})| \leqq 2.7.$
2. $|\mathcal{T}_3^1(f^{-1})| \leqq 4.$
3. $|\mathcal{T}_2^3(f^{-1})| \leqq 10.27.$
4. $|\mathcal{T}_3^2(f^{-1})| \leqq 7.24.$

All sharp estimates are derived from the inverse of $f(z) := z/(1 - iz)$.

Proof. By the simplification of the convex form $f'(z) + zf''(z) = \gamma(z)f'(z)$, we obtain

$$a_2 = \frac{1}{2}\gamma_1, \tag{27}$$
$$a_3 = \frac{1}{6}\left(\gamma_2 + \gamma_1^2\right), \tag{28}$$
$$a_4 = \frac{1}{24}\left(\gamma_1^3 + 3\gamma_1\gamma_2 + 2\gamma_3\right). \tag{29}$$

Comparing (7)–(9) with (27)–(29), we have

$$u_2 = -\frac{1}{2}\gamma_1, \tag{30}$$
$$u_3 = -\frac{1}{6}\left(\gamma_2 - 2\gamma_1^2\right), \tag{31}$$
$$u_4 = -\frac{1}{24}\left(2\gamma_3 - 7\gamma_1\gamma_2 + 6\gamma_1^3\right). \tag{32}$$

1. Substituting in the form of $\mathcal{T}_2^2$ as in (25) and employing Lemmas 1 and 2, we consider

$$\mathcal{T}_2^2(f^{-1})| = \left|u_3^2 - u_2^2\right| \leqq \left|u_3^2\right| + \left|u_2^2\right| = |(-\frac{1}{6}(\gamma_2 - 2\gamma_1^2))^2| + |(-\frac{1}{2}\gamma_1)^2|$$
$$\leqq \frac{1}{9}|\gamma_1|^4 + \frac{1}{36}|\gamma_2||\gamma_2 - 4\gamma_1^2| + \frac{1}{4}|\gamma_1|^2 \leqq \frac{16}{9} + \frac{7}{9} + 1 = 2.7.$$

2. With follow-up from values of u_2 and u_3 along with Lemmas 1 and 2, we attain that

$$|\mathcal{T}_3^1(f^{-1})| \leqq 1 + 2\left|u_2^2\right| + |u_3|\left|u_3 - 2u_2^2\right|$$
$$\leqq 1 + \frac{1}{2}|\gamma_1|^2 + \frac{1}{6}|\gamma_2 + \gamma_1^2| \leqq 1 + 2 + 1 = 4.$$

3. Similarly for $\mathcal{T}_2^3(f^{-1})$, we have

$$|\mathcal{T}_2^3(f^{-1})| = |u_4^2 - u_3^2| \leqq \left|u_4^2\right| + \left|u_3^2\right| \leqq \frac{139}{18} + \frac{23}{9} = 10.27.$$

4. By leveraging the aforementioned technique, the subsequent inequalities hold:

$$\begin{aligned}|u_2 - u_4| &\leqq |u_2| + |u_4| \\ &\leqq \frac{1}{2}|\gamma_1| + \frac{1}{24}(2|\gamma_3| + 7|\gamma_1||\gamma_2 - \frac{6}{7}\gamma_1^2|) \leqq 1 + \frac{32}{24} = 2.33,\end{aligned}$$

$$\begin{aligned}\left|u_2^2 - 2u_3^2 + u_2u_4\right| &= \frac{1}{144}|36\gamma_1^2 - 14\gamma_1^4 + 11\gamma_1^2\gamma_2 - 8\gamma_2^2 + 6\gamma_1\gamma_3| \\ &\leqq \frac{1}{144}[36|\gamma_1|^2 + 14|\gamma_1|^4 + 11|\gamma_1||\gamma_1\gamma_2 + \frac{6}{11}\gamma_3| + 8|\gamma_2|^2] \leqq 3.11\end{aligned}$$

and

$$|T_3^2(f^{-1})| \leqq 7.24.$$

This deduces the demonstration. □

Finally, the method of computing the previous theories will help us to calculate the upper limits of the family $\mathcal{S}_{sym}$ as the following.

Theorem 4. *If $f(z) \in \mathcal{S}_{sym}$, for $f(z)$ appearing in (1), the bounds of the Toeplitz inequalities are*

1. $|\mathcal{T}_2^2(f^{-1})| \leqq 8.$
2. $|T_3^1(f^{-1})| \leqq 4.$
3. $|\mathcal{T}_2^3(f^{-1})| \leqq 2.$
4. $|T_3^2(f^{-1})| \leqq 39.$

Proof. By the simplification of the symmetric-starlike form $\frac{2zf'(z)}{(f(z)-f(-z))} = \gamma(z)$, we have

$$a_2 = \frac{1}{2}\gamma_1, \tag{33}$$

$$a_3 = \frac{1}{2}\gamma_2, \tag{34}$$

$$a_4 = \frac{1}{8}(\gamma_1\gamma_2 + 2\gamma_3). \tag{35}$$

Comparing (7)–(9) with (33)–(35), we observe

$$u_2 = -\frac{1}{2}\gamma_1, \tag{36}$$

$$u_3 = -\frac{1}{2}\left(\gamma_2 - \gamma_1^2\right), \tag{37}$$

$$u_4 = -\frac{1}{8}\left(2\gamma_3 - 9\gamma_1\gamma_2 + 5\gamma_1^3\right). \tag{38}$$

With analogous methods to Theorem 1, we gain that

$$|\mathcal{T}_2^2(f^{-1})| \leqq 8, |\mathcal{T}_3^1(f^{-1})| \leqq 4, |\mathcal{T}_2^3(f^{-1})| \leqq 2, \text{ and } |T_3^2(f^{-1})| \leqq 39.$$

The demonstration of Theorem is carried out. □

4. Conclusions

The study of the bounds of Hankel determinants has expanded as a result of scholars' interest in determining the initial coefficients of inverse analytical functions. In our investigation, we were able to estimate the upper and lower bounds of the Toeplitz determinant with different degrees for the family of holomorphic functions related to its inverse. The primary techniques employed in our findings, together with the outcome relating to the parameters of Carathéodory's functions, seem to be really helpful for the estimates.

Additionally, the inverse coefficient's sharp estimate has been determined. This paper's approach may be used with a variety of holomorphic function families to determine the first inverse coefficients' upper estimates. Regarding the sharp values of the starlike function, it is verified by the inverse of the function $f(z) = z/(1-iz)^2$. As for the convex function, it is proven using the function $f(z) = z/(1-iz)$.

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