AN IMPROVEMENT OF FINITE ELEMENT METHODS FOR SOLVING THE CONVECTION-DIFFUSION-REACTION EQUATION

A Thesis

Submitted to College of Science, University of Basrah

in Partial Fulfillment of the Requirements for

The Degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Hameeda Oda Mezban AL - Humedi

(2006)

Abstract

For the convective-diffusive-reactive problems, the behaviour of the solution depends on the magnitude of the convectivity coefficient (velocity field) and the diffusivity coefficient.

In this thesis the aim has been to move towards a more advanced simulation for Convection Diffusion Reaction (CDR) problems when the convection term is dominated which require special stabilization techniques to obtain meaningful numerical results.

A number of new finite element techniques which are constructed are effective for governing equation in case of small diffusion term and show that these techniques are stable provided that the space-time grid is appropriately constructed .

An improvement of finite element approximations
Galerkin-Modified Parameters (Modified Parameters-Galerkin),
Galerkin-New Schemes (New Schemes-Galerkin) and
Galerkin-Modified Parameters-New Schemes (Modified Parameters-New Schemes-Galerkin)
which are discussing to solve two dimensional time-dependent CDR equation with dominating convection term in equal space to reduce or completely eliminate numerical oscillations .

These modified schemes are more accurate and uniformly convergent (exhibit a higher order of convergence) especially Galerkin-Modified Parameters-New Schemes(Modified Parameters-New Schemes-Galerkin) finite element method with respect to diffusion parameter both the time step and the small perturbation parameter with a wider range of time compare with a standard Galerkin finite element method which exhibit oscillation at any time.

The main idea of such stabilization techniques prevent numerical oscillations and other instabilities in solving problem with high Peclet num-

ber (convection term is dominant) by introducing sophisticated way additional viscosity into the discrete equations.

Stabilized finite element methods were constructed by modifying the variational form of a particular problem, such enhanced numerical stability is achieved without compromising consistency, so, introduced a new analysis of error estimate of the stabilized Galerkin finite element method and all improvement methods which were related to standard Galerkin method which was applied to the CDR problem shows that the discretization error in all these methods are $O(\Delta t + h^{k+1} + \parallel u^0 - U^0 \parallel)$. where the new analysis of error estimate in this thesis consist of bounds on bilinear form as follows:

1) Galerkin finite element method

$$\sum_{n=1}^{m} \triangle t[(a\nabla(u^n - U^n), \nabla(u^n - U^n)) + (c(u^n - U^n), (u^n - U^n))]$$
$$\leq C\{E_n + \sum_{n=1}^{m} \sum_{K\in T_h} \triangle t \ z^{2r} \parallel u \parallel^2_{L^\infty(L^2(\Omega))}\}.$$

2) Galerkin-Modified Parameters (Modified Parameters-Galerkin) finite element method

$$\sum_{n=1}^{m} \triangle t[(a\nabla(u^n - U^n), \nabla(u^n - U^n)) + (c(u^n - U^n), (u^n - U^n))$$
$$+(\tau \mathbf{b}^2 \cdot \nabla(u^n - U^n), \nabla(u^n - U^n)) - (\tau\xi c^2(u^n - U^n), (u^n - U^n))]$$
$$\leq C\{E_n + \sum_{n=1}^{m} \sum_{K\in T_h} \triangle t \ z^{2r} \parallel u \parallel^2_{L^\infty(L^2(\Omega))}\}.$$

3) Galerkin-New Schemes (New Schemes-Galerkin) finite element method

$$\sum_{n=1}^{m} \triangle t[(a_p\nabla(u^n - U^n), \nabla(u^n - U^n)) + (c(u^n - U^n), (u^n - U^n))]$$

$$\leq C\{E_n + \sum_{n=1}^{m} \sum_{K \in T_h} \triangle t \ z^{2r} \parallel u \parallel^2_{L^\infty(L^2(\Omega))}\}.$$

4) Galerkin-Modified Parameters-New Schemes (Modified Parameters-New Schemes-Galerkin) finite element method

$$\sum_{n=1}^{m} \triangle t[(a_p \nabla(u^n - U^n), \nabla(u^n - U^n)) + (c(u^n - U^n), (u^n - U^n))$$

$$+(a_1 \tau \mathbf{b}^2 \cdot \nabla(u^n - U^n), \nabla(u^n - U^n)) - (\tau \xi c^2 (u^n - U^n), (u^n - U^n))]$$

$$\leq C\{E_n + \sum_{n=1}^{m} \sum_{K \in T_h} \triangle t \ z^{2r} \parallel u \parallel^2_{L^\infty(L^2(\Omega))}\}.$$

Where

$$E_n = \parallel u^0 - U^0 \parallel^2 + \{\sum_{n=1}^{m} (\sum_{K \in T_h} [\int_{t_{n-1}}^{t_n} z^{r+1} \parallel u_t \parallel_{L^\infty(L^2(\Omega))} \ dt]^2)^{1/2}\}^2 +$$

$$[\sum_{n=1}^{m} \parallel (R_n - R_{n-1}) u^{n-1} \parallel]^2 + \sum_{n=1}^{m} \triangle t [\sum_{K \in T_h} z^{2(r+1)} \parallel u \parallel^2_{L^\infty(L^2(\Omega))}$$

$$+\triangle t (\int_{t_{n-1}}^{t_n} \parallel u_{tt} \parallel_{L^2(L^2(\Omega))} \ dt)^2],$$

and $R_n u$ is the elliptic projection.
In many applications, it is important to design approximation methods guaranteeing that the discrete solution satisfies some maximum principle,

1) Galerkin finite element method satisfies the discrete maximum principle when

$$a + \frac{h}{6} + \frac{ch^2}{12} \leq 0$$

where a and c are diffusion and reaction coefficients

2) Galerkin-Modified Parameters (Modified Parameters-Galerkin) finite element method satisfy the discrete maximum principle when

$$\tau \geq (\ a + \frac{h}{6} + \frac{ch^2}{12})/(\ -\frac{1}{2} + \frac{(\xi+1)ch}{6} + \frac{\xi c^2 h^2}{12}),$$

where $\xi \in [-1, 1]$ and the stabilization parameter τ are weights for stability respectively.

3) Galerkin-New Schemes (New Schemes-Galerkin) finite element method satisfies the discrete maximum principle when

$$a_p + \frac{h}{6} + \frac{ch^2}{12} \leq 0,$$

where $a_p = a\ a_1$ and

$$a_1 = \begin{cases} Pe\ \coth(Pe)\ , & New\ Scheme1 \equiv NS1 \\ 1/(1+Pe)\ , & New\ Scheme2 \equiv NS2 \\ 1/(1-Pe^2)\ , & New\ Scheme3 \equiv NS3 \end{cases}$$

where $Pe = (\mathbf{b}\ h)/(2\ a)$ is mesh Peclet number and $\mathbf{b}$ is convection term.

4) Galerkin-Modified Parameters-New Schemes (Modified Parameters-New Schemes-Galerkin) finite element method satisfies the discrete maximum principle when

$$\tau \geq (\ a_p + \frac{h}{6} + \frac{ch^2}{12}\ )/(\ -\frac{a_1}{2} + \frac{(\xi+1)ch}{6} + \frac{\xi c^2 h^2}{12}).$$