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Control Charts for Exponentially Distributed Characteristics: SD, ED, ESD with Taguchi's Loss Function

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ABSTRACT

This paper addresses the challenge of quality characteristics that follow an exponential distribution, which can significantly impact decision-making in various fields. Existing approaches rely on approximations to convert exponential distributions to normal distributions, upon which control charts are constructed. However, such conversions introduce errors that can lead to incorrect outcomes, particularly for highly sensitive characteristics. To address this limitation, we propose the development of control charts specifically designed for exponential characteristics, without relying on approximations. Our objective is to introduce four different schemes for constructing these control charts: a statistical scheme, an economic scheme, an economic-statistical scheme combined with Taguchi's loss function, and an economic-statistical scheme without the application of a loss function. To determine optimal design parameter values for each scheme, we employ the artificial bee colony algorithm. Additionally, we conduct a sensitivity analysis to investigate the impact of design parameters on each proposed design. To illustrate the practical implementation of these control charts, we provide a numerical example that demonstrates their effectiveness. By addressing the limitations of existing approaches and offering novel control chart designs, this paper contributes to enhancing decision-making accuracy and reliability in scenarios involving exponentially distributed quality characteristics.

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Introduction

Due to the competition of markets for customer satisfaction, more sales, and higher profits, quality has become one of the most important issues under consideration in industries. Among the various control tools, statistical process control (SPC) is of great importance to the improvement of quality. Among the diverse statistical methods, control charts are important tools which monitor and control processes. Today, these tools are widely used in

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industries. For this reason, researchers are constantly looking for optimal (accurate, low-cost, and fast) control charts.

In the construction of control charts, the distribution of quality characteristics is usually considered normal. However, in many cases, this assumption is not true. This happens for quality characteristics whose probability plots are highly skewed. For example, we can refer to qualitative characteristics with lifetime nature.

Among the well-known statistical distributions, the exponential distribution has a good fit for most of the skewed data (Santiago and Smith 2013) such as:

(1) Waiting times or inter-arrival times: When studying the time between consecutive events (e.g., customer arrivals, phone calls, or requests), the exponential distribution can be appropriate if the underlying process exhibits randomness and no memory, commonly known as the memoryless property.
(2) Survival analysis: In certain cases, the exponential distribution can be employed to model the survival times of a population when the hazard rate (probability of an event occurring at a given time) remains constant over time. This assumption is often referred to as an exponential survival function.
(3) Reliability analysis: When analyzing the lifetime or failure rate of systems or components, the exponential distribution can be utilized if the failure rate remains constant or exponentially decreasing over time. It assumes that the failure events are independent and do not depend on prior events.
(4) Queueing systems: In some queueing configurations, where the service times follow an exponential distribution and the arrival process satisfies certain assumptions (e.g., Poisson arrivals), the exponential distribution can be used to model the waiting times in the system accurately.

One of the reasons for using the exponential distribution for the distribution of a time characteristic is when the defect rate in a process is very low. In this case, instead of using C-charts and U-charts for the number of defects, they consider the times between the occurrence of two consecutive defects and plot a control chart for these times (Kumar 2022). If the defect rate in the process is low, for example, less than 1000 defects per million, the time between observing defective products will be long. Under such conditions, most samples will be defect-free, and a control chart that constantly plots a statistic at zero does not provide much useful information. Therefore, when the defect rates are expressed as parts per million (PPM), traditional C and U-charts are not effective. One way to address this issue is to use a new variable such as the time between consecutive defect observations (Kumar, Chakraborti, and

Castagliola 2022). The control chart for time between observations is one of the most successful methods for controlling processes with very low defect rates. Suppose defects or observations of interest follow a Poisson distribution. The choice of the Poisson distribution is commonly made in situations where we expect rare events to occur independently over a specific time or space interval. In our study, we have chosen to assume a Poisson distribution for defects or observations of interest based on the nature of the phenomena under investigation and previous research in the field. By assuming the Poisson distribution, we aim to simplify the analysis and make reasonable approximations that align with existing knowledge in the field. In this case, the time between observations will have a probability distribution. In such conditions, designing a control chart for time between observations involves a variable that follows an exponential distribution. However, the exponential distribution has high skewness, resulting in a highly asymmetric control chart (Montgomery 2020).

The exponential distribution is used in many fields, including reliability engineering (time-to-failure of components), queueing theory (inter-arrival times of customers in a queue), telecommunications (time between successive arrivals of messages or packets), physics (decay of radioactive materials), finance (time to default of bonds and financial instruments), biology (waiting time between cell divisions, mutations, and evolutionary events), and epidemiology (time between infections during an epidemic).

Many researchers have focused on control charts for skewed type of quality characteristic. For example, Bai and Choi (1995) studied $\bar{X}$ and R control charts for skewed qualities, while Choobineh and Ballard (1987) developed a control chart for skewed data using weighted variance. Also Morales and Arturo Panza (2022) and Figueiredo and Ivette Gomes (2013) presented control charts for skewed data following a skew-normal distribution, while Al-Nuaami, Akbar Heydari, and Jabbari Khamnei (2023) proposed control charts specifically for counting data with overdispersion.

Santiago and Smith (2013) investigated the t-chart assuming exponentially distributed quality characteristics and time until the first deviation. They approximated the quality characteristic distribution to a normal distribution using Nelson's approximation (Nelson 1994) and constructed the appropriate control chart based on conventional 6σ (Shewhart) methods. Aslam et al. (2015) used a variable sampling model and the same approximation to normalize feature distribution. Tavakoli and Akbar Heydari (2021) built an exponential control chart using Nelson's approximation and integrated economic statistical design with a loss function.

However, it is important to acknowledge that such approximations can introduce errors in the obtained results. With our research, we aim to address this issue by directly constructing a control chart for exponentially distributed quality characteristics, thereby circumventing the need for such approximations. By eliminating the intermediate step of approximating to a normal distribution, we expect our proposed method to provide more accurate results. Thus, this paper proposes directly constructing a control chart for exponentially distributed features.

Control charts may have different purposes depending on the designer's strategy and the type of design. These designs may focus on statistical or economic aspects, or both. Thus, control charts can be designed in three ways: statistical design (SD), economic design (ED), and/or statistical-economic design (ESD). Statistical designs have desirable statistical properties, but ignore the cost of production, which can be high in some situations. On the other hand, some designers try to minimize production costs in production processes. Such a design is referred to as ED design. Statistical design, however, produces charts that have a high power and a low probability of type I error, but the cost is disregarded and is higher compared to economic design. On the other hand, EDs focus only on the cost and ignore statistical features. In economic design, the cost of all factors of production and control is applied to a cost function, and then the design parameters are obtained so that this function is minimized.

Paying attention to each of these two schemes without considering the other makes the controller away from an optimal model and a correct decision. To solve this problem, economic-statistical design (ESD) was proposed. In ESD, some statistical restrictions are applied to the economic model to minimize the average cost per unit of time under these conditions. These limitations depend on the opinion and needs of the process designer. Also, combining the optimal features of control charts from an economic dimension, with statistical advantages which are considered in the statistical-economic design, while saving the costs, can also take into account the statistical requirements to maintain product quality.

Duncan (1956) conducted the first study on the economic design of $\bar{X}$ control charts. Many researchers, such as (Duncan 1971; Lorenzen and Vance 1986), and (Banerjee and Rahim 1988), followed this approach in their research. But, none of the mere statistical or economic designs was optimal. The former approach had a high cost and the latter lacked statistically desirable features. To solve this problem, Saniga (1989) presented the economic-statistical design (ESD) for $\bar{X}$ and R charts. Later on, many researchers, such as (Faraz, Kazemzadeh, and Saniga 2010; Prabhu, Montgomery, and Runger 1997; Yang and Rahim 2005; Zhang and Berardi 1997), and (Tavakoli, Pourtaheri, and Moghadam 2017) applied this design to control charts. Recently (Ghanaatiyan, Amiri, and Sogandi 2017), proposed a modified multivariate weighted moving average control chart based on ESD. Khadem and Bameni Moghadam (2019) studied the economic-statistical design of the $\bar{X}$ control chart according to a modification of

Banerjee and Rahim's cost model. To learn about the latest articles on the economic-statistical design of control charts, please refer to (Taji, Farughi, and Rasay 2022; Wan 2020), and (Heydari, Tavakoli, and Rahim 2023).

In conventional economic models, only the fixed values of costs of the process are usually considered and monitored. However, the higher the deviation of the quality characteristic from the target value, the higher the cost of rework or defective product. Combining a loss function with the economic model leads to the conclusion that the greater the deviation from the ideal value, the higher the cost (loss), and the better the decision to continue or stop the process.

Safaei, Baradaran Kazemzadeh, and Niaki (2012) used Taguchi's loss function in their economic models. They showed the unrealistic results of the no-loss models in the control chart under consideration. In addition, Pasha et al. (2017) examined the effect of the distribution of quality characteristics on the economic model of Banerjee and Rahim, which was combined with Taguchi's loss function. Also, Pasha et al. (2018)examined the previous study under the economic model of Lorenzen and Vance. In the $\bar{X}$ control chart (Celano, Faraz, and Saniga 2014), proposed an on-line scheme to monitor the process losses under Taguchi's approach (Taguchi 1979, 1986; Taguchi, Elsayed, and Hsiang 1988).

Although control charts for exponential characteristics that do not rely on approximations have been studied by Chakraborti et al. (2014; Xie, Ngee Goh, and Ranjan 2002; Zhang, Megahed, and Woodall 2014), and others, but the economic and economic-statistical designs of an exponential-based control chart are not introduced yet. Therefore, in this paper, we aim to construct these types of control charts. This construction is carried out using four different schemes: statistical, economic, economic-statistical combined with Taguchi's loss function, and economic-statistical without the application of a loss function. We anticipate that our new methodology, which involves constructing control charts directly using the exponential distribution, will outperform traditional approximation-based methodologies for quality characteristics that follow the exponential distribution.

This paper introduces a novel approach to address the challenge of quality characteristics that follow an exponential distribution, which has significant implications for decision-making in various fields. Unlike existing approaches that rely on approximations, this manuscript proposes the development of control charts specifically designed for exponential characteristics without the need for conversions. This departure from traditional methods is a key innovative insight of this research.

The manuscript presents four different schemes for constructing these control charts: a statistical scheme, an economic scheme, an economic-statistical scheme combined with Taguchi's loss function, and an economic-statistical scheme without the application of a loss function. The utilization of the artificial bee colony algorithm to determine optimal design parameter values for each scheme further contributes to the novelty of this work.

Additionally, a sensitivity analysis is conducted to investigate the impact of design parameters on each proposed control chart design. The practical implementation of the control charts is demonstrated through a numerical example, showcasing their effectiveness in enhancing decision-making accuracy and reliability in scenarios involving exponentially distributed quality characteristics.

By addressing the limitations of existing approaches and offering new control chart designs tailored to exponential characteristics, this manuscript provides valuable insights and contributes to advancing the field of decision-making in quality control.

Since in each approach we aimed to find the optimal values of the design parameters to meet the desired conditions, the optimization problems were coded in MATLAB by using the artificial bee colony (ABC) algorithm.

In the next section, we obtain control limits for exponentially distributed quality characteristics. In Section 3, we introduce and present the economic design as well as Taguchi's loss function integrated with the economic model. In the fourth section, we determine the optimal design parameters based on the four considered schemes.

Statistical, Economic, and Economic-Statistical Designs of Control Charts for Exponentially Distributed, Individual Quality Characteristics

Most of the studies conducted on the development of economic models for control charts have assumed the normality of quality characteristics. However, many quality characteristics such as longevity, chemical characteristics, etc., are exponentially distributed. Incidentally, characteristics with these features are important.

The symbols and abbreviations used in this paper are summarized in Table 1.

The region between the upper control limit (*UCL*), abbreviated as k_2, and the lower control limit (*LCL*), represented by k_1, is known as the control region. Out of this region is referred to as the out-of-control (action) region. At each period of sampling, a sample of size 1 is taken from the production process at h units of time, which is then compared with the control limits. If the value of the quality characteristic (X) (the taken sample) is located in the control range, the process is considered in the in-control state. Otherwise, the process is considered in the out-of-control state, and the search for the cause of deviation begins. These control limits are obtained by considering the exponential distribution for the quality characteristic X.

Thus, it is assumed that X has an exponential distribution with the probability density function (pdf) given by

Table 1. Symbols and abbreviations used in the content.

Symbol or abbreviation	Description
n	The sample size
h	The time interval between each sample
k_1	Lower control limit (LCL)
k_2	Upper control limit (UCL)
X	The distribution of quality characteristic
θ_0	The exponential distribution parameter desired for X, in the in-control state
θ_1	The exponential distribution parameter desired for X, in the out-of-control state
λ	The average of assignable cause occurrence rate based on a Poisson process
δ	The rate of change in parameter θ, when the process goes from in-control to out-of-control state.
ABC	Artificial Bee Colony
ARL_0	The average number of samples required to receive an alarm when the process is in-control.
ARL_1	The average number of samples required to receive an alarm when the process is out-of-control.
α	The type I error of control charts
β	The type II error of control charts
T	The time of the occurrence of an assignable cause
τ	The expected time length of being in control state, in the sampling interval that shift from in-control to out-of-control state occurs.
g	The time required to review and interpret an individual sample
d	The expected time to detect a deviation in the out-of-control state and correct the process
V_0	The average income per hour as long as the process is in-control
V_1	The average income per hour as long as the process is out-of-control
b	The fixed sampling cost
c	The cost per sampling unit
y	The cost of each false alarm
W	The expected cost of process repair and correction
$E(T)$	The expected time of production cycle
$E(C)$	The expected cost of production cycle
C_0	The average costs of the process when the process is in the in-control state
C_1	The average costs of the process when the process is in the out-of-control state
$E(CH)$	The average cost of the process per hour
k	The cost of reworking in the Taguchi loss function
L_0	The average costs of production of each defective product in the in-control state
L_1	The average costs of production of each defective product in the out-of-control state
n^*	The number of products produced per hour
ξ	The target value for a quality characteristic
Δ	The tolerance limit of the quality characteristic
A	The cost of reworking or scrapping a product unit
NIC	The expected net income
NIH	The expected net income per hour

$$f(x) = \theta e^{-\theta x}; \quad \theta > 0, \quad x > 0\,,$$

which can be represented as $X\tilde{E}(\theta)$. In the in-control state, $X\tilde{E}(\theta_0)$, and in the out-of-control state, $X\tilde{E}(\theta_1)$, where $\theta_1 = \delta\theta_0$. The parameters θ_0 and $\delta > 0$ are assumed to be known, while the control limits and sampling intervals should be obtained based on the considered design. In addition, it is assumed that the assignable cause occurs based on a Poisson process with an average of λ observations per hour. In other words, assuming that the process starts in the in-control mode, the length of time the process remains in this state will be an exponential random variable with an average of $1/\lambda$ hours.

Statistical Design Based on the Average Run Length (ARL) Approach

In order to statistically compare different control charts, it is common to use the ARL average length criterion. In fact, ARL is the average number of samples required to receive an alarm that the process is out of control. It is clear that when the process is in the in-control state, we expect a large ARL. But when the process is in the out-of-control state, we obtain a more powerful chart that has a smaller ARL compared to the other methods. Accordingly, one of the statistical control methods of the process is the control of ARL. In this method, the in-control ARL (ARL_0) is usually fixed (for example, 370) and the controller seeks to find the design parameters so that the out-of-control ARL (ARL_1) is minimized. By definition, when the process is in the in-control state, the ARL value can be obtained using the following relation:

$$ARL_0 = \frac{1}{\alpha}$$

where α is the type I error. In the exponential control chart with an individual sample, this is equal to

$$\alpha = P([X < LCL] \cup [X > UCL] | \theta = \theta_0) = 1 - e^{-\theta_0(LCL)} + e^{-\theta_0(UCL)}.$$

When the process is in the out-of-control state, ARL is equal to

$$ARL_1 = \frac{1}{1-\beta}. \quad (1)$$

Here, β is the type II error that can be obtained as follows:

$$\beta = P([LCL < X < UCL] | \theta = \theta_1) = e^{-\theta_1(LCL)} - e^{-\theta_1(UCL)}.$$

Based on this, the level of the type I error is usually considered to be 0.0027, which yields $ARL_0 = 370$. Also, in similar problems, we seek to maintain the type II error at the level of .75, and as a result, $ARL_1 = 4$. Therefore, the statistical design of the control chart will be such that $ARL_0 = 370$ is fixed, and the design parameters (n, h, k_1 and k_2) are obtained so that ARL_1 is less than a preset desired value. Therefore, the optimization process can be written as follows.

$$Min\, ARL_1$$

$$s.t.$$

$$ARL_0 = 370$$

$$\beta \leq 0.75. \quad (2)$$

This optimization problem was coded in MATLAB and the results were obtained for different values of the input parameters and process shift using the ABC algorithm. The results are presented in Table 3.

In order to compare the results of statistical design with other methods of designing control charts, such as economic design and economic-statistical, both with and without the presence of Taguchi loss function, it is necessary to explain that we have provided all the numerical results in Section 4.

A Modified Version of Duncan's Economic Model

In this section, we discuss one of the most widely studied economic models. In 1956, Duncan published a paper titled "The Economic Design of $\bar{X}$ Charts used to Maintain Current Control of a Process." This article was the main motivator for most of the subsequent research conducted in this field. Duncan found that the selection of parameters such as sample size, control limits, and sampling interval varied at different costs.

If we consider the control chart based on Duncan's model, then we need to assume that the process starts in the in-control state with the parameter θ_0, and that an assignable cause which occurs randomly, changes the distribution of the parameter of quality characteristic (X) from θ_0 to $\theta_1 = \delta\theta_0$.

In this model, the cost of eliminating an assignable cause and repairing the process is not deducted from the net income and the process is continuous. This means that the process continues as long as the search for the assignable cause is in progress. Figure 1 shows the quality cycle in Duncan's economic model.

This quality cycle consists of two periods, namely, the period when the process is in the in-control state, and the period when the process is in the out-of-control state. The process is assumed to be in the in-control state from the beginning of production and continues until an assignable cause occurs, and after this occurrence, the process enters the out-of-control state. This latter state in turn consists of three periods, namely, the time it takes for a reason deviation to occur until the alarm time, the time it takes to sample and check the chart, and the time it takes to detect a reason deviation and correct it. The last two times are shown together in Figure 1.

The Expected Time of the Quality Cycle in Duncan's Economic Model

The expected time of the quality cycle for this process is equal to the mathematical expectation of the sum of the four cycles expressed in the quality cycle. In this regard, the expected time for each period can be described as follows.

a- The expected time of the in-control period

Since the time of the in-control state follows the exponential distribution, the expected time of the in-control phase or the average duration of the

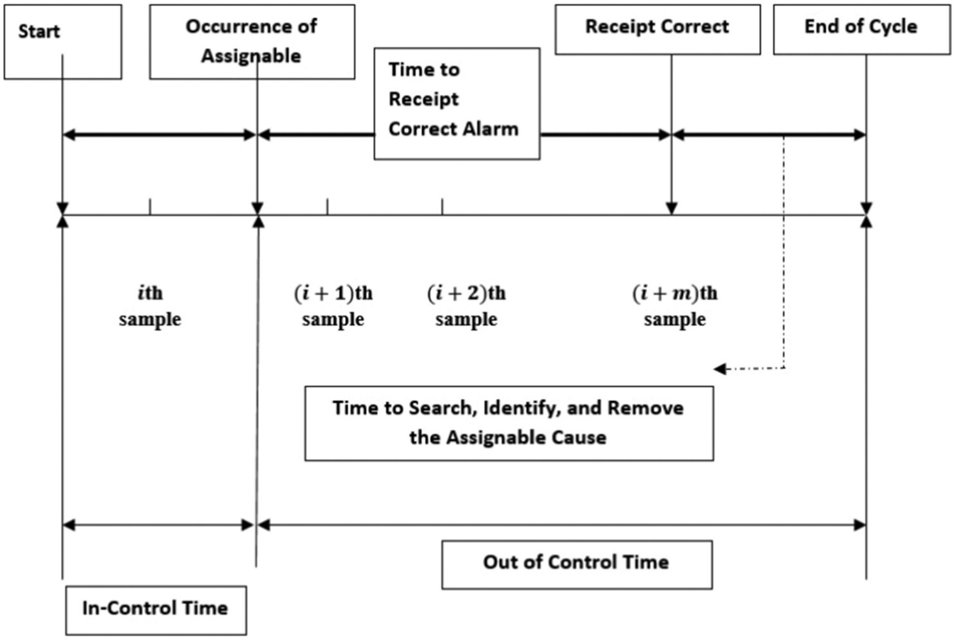

Figure 1. Quality cycle in Duncan's economic model.

process in the in-control state is equal to the average exponential distribution, which is equal to $1/\lambda$.

b- The time expected to receive an alarm

Assume that the random variable T represents the time of the occurrence of an assignable cause. In this case, if the samples are taken from the process once every h hour and an assignable cause occurs between the ith and $(i+1)$th samples, the average time to observe this reason deviation at this distance is equal to

$$\tau = E(T - ih | ih < T < (i+1)h)$$

$$= \int_{ih}^{(i+1)h} (t - ih) f(t | ih < T < (i+1)h) dt.$$

For more information, see Figure 1. According to the definition of the conditional probability density function:

$$f(t | ih < T < (i+1)h) = \frac{f(t)}{\int_{ih}^{(i+1)h} f(t) dt}.$$

According to the hypotheses of Duncan's model, T has an exponential distribution with an average of $1/\lambda$. Therefore, the exact denominator of the above fraction is $e^{-\lambda(ih)} - e^{-\lambda(i+1)h)}$, which does not depend on t and can be taken out of the integral in the calculation of τ. Therefore,

$$\tau = \frac{\int_{ih}^{(i+1)h} e^{-\lambda(ih)}\lambda(t-ih)dt}{\int_{ih}^{(i+1)h} e^{-\lambda(ih)}\lambda dt} = \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})} \approx \frac{h}{2} - \frac{\lambda h^2}{12}.$$

When an assignable cause occurs, the probability of finding it in the next sample is

$$p = P(X < LCL \cup > UCL | \theta = \theta_1).$$

If the parameter of quality characteristic changes from θ_0 to θ_1, then it is $X \sim E(\theta_1)$. So, in this case,

$$\begin{aligned} p &= P(X < LCL|\theta_1) + P(X > UCL|\theta_1) \\ &= P(X < LCL|\theta_1) + 1 - P(X < UCL|\theta_1) \\ &= 1 - e^{-\theta_1(LCL)} + 1 - \left(1 - e^{-\theta_1(UCL)}\right) \\ &= 1 - e^{-\theta_1(LCL)} + e^{-\theta_1(UCL)}. \end{aligned}$$

Therefore, the number of samples expected to observe an assignable cause is a geometric random variable with an average of $1/p$. So, the expected time to receive an alarm (the average time when the process is in the out-of-control state) is equal to

$$\frac{h}{p} - \left(\frac{h}{2} - \frac{\lambda h^2}{12}\right) = h\left(\frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12}\right).$$

c- The expected time for sample selection and interpretation of the results

The time required to review and interpret an individual sample (single observation) is considered to be g. Therefore, for any process with a sample size of n, the expected time to select a sample and interpret is equal to gn.

d- The expected time for repair

The expected time to detect a deviation in the out-of-control state and correct the process is considered to be d.

Therefore, using the above discussion, the expected time of a cycle can be obtained as follows.

$$E(T) = \frac{1}{\lambda} + h\left(\frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12}\right) + gn + d. \tag{3}$$

The Expected Cost of the Quality Cycle in Duncan's Economic Model

Duncan expressed a simple economic principle for his model: the average net income equals total income minus total cost, in which total income is divided into two parts, namely, income as long as the process is under control (V_0), and income as long as the process is out of control (V_1). It also divides the total cost into three parts: the cost of sampling, the false alarm and, process correction and repair. Sampling cost is determined according to the type of product and the number of samples taken at each sampling interval. The cost of a false alarm is the cost of searching for a deviation when no one exists. The cost of correction is the cost of finding a reason for a deviation when there is a deviation. Also, the cost of repair is a cost that, after discovering the cause of the deviation, tries to correct the process and return it to the in-control state. Therefore, net income can be considered as the algebraic sum of income and deduction of costs:

a- The expected income as long as the process is in the in-control state

Since the expected time that the process is under control is equal to $1/\lambda$, the expected income as long as the process is in the in-control state is equal to V_0/λ.

b- The expected income as long as the process is out of control

Since the expected time during the out-of-control state for each cycle is

$$h\left(\frac{1}{p}-\frac{1}{2}+\frac{\lambda h}{12}\right)+gn+d.$$

the expected income in the out-of-control period is as follows

$$V_1\left[h\left(\frac{1}{p}-\frac{1}{2}+\frac{\lambda h}{12}\right)+gn+d\right].$$

c- The expected cost of sampling

Since the expected number of sampling times per cycle is equal to $(E(T))/h$, and the expected cost per sampling time is equal to $(b+cn)$, where b is the fixed sampling cost, c is the cost per sampling unit, and n is the number of samples taken each time, the expected cost of sampling in each period can be obtained as follows.

$$\frac{E(T)}{h}(b+cn)$$

d- The expected cost of false alarms

This value is equal to the expected number of false alarms per period multiplied by the cost of the false alarm. The expected number of false alarms in each period is calculated by multiplying the probability of false alarms by the expected number of sampling times before the process gets out of control. The expected number of sampling times before the process gets out of control is

$$\sum_{i=0}^{\infty} iP(ih < T < (i+1)h).$$

By considering the exponential distribution for the out-of-control times, this can be calculated as follows.

$$\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} i\lambda e^{-\lambda t} dt = \sum_{i=0}^{\infty} i\left(e^{-ih\lambda} - e^{-(i+1)h\lambda}\right) = \left(1 - e^{-\lambda h}\right) \sum_{i=0}^{\infty} ie^{-ih\lambda}$$
$$= \frac{e^{-\lambda h}}{1 - e^{-\lambda h}}$$

As shown by the numerical results of Duncan (1956), if one can ignore expressions with a degree of $\lambda^2 h^2$ or higher, the above expression is approximately equal to $\frac{1}{\lambda h}$. The probability of a false alarm is the probability of the type I error (α), which can be calculated as follows:

$$\alpha = P([X < LCL] \cup [X > UCL] | \theta = \theta_0) = 1 - e^{-\theta_0(LCL)} + e^{-\theta_0(UCL)}.$$

Therefore, the expected number of false alarms in each period is equal to $\alpha/\lambda h$. Since the cost of each false alarm is y, the expected cost of false alarms is equal to $\alpha y/\lambda h$.

e- The expected cost of process repair and correction

This value is considered equal to W.

As a result, the expected net income is equal to

$$E(NIC) = \frac{V_0}{\lambda} + V_1\left[h\left(\frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12}\right) + gn + d\right] - \frac{E(T)}{h}(b + nc) - \frac{\alpha y}{\lambda h} - W. \quad (4)$$

In addition, the net income per hour can be obtained as follows:

$$E(NIH) = \frac{E(NIC)}{E(T)} = V_0 - l. \quad (5)$$

Here, l is a cost loss function that can be obtained using equations (3–5) as follows:

$$l = \frac{b + cn}{h} + \frac{\lambda MB + \frac{\alpha y}{h}\lambda W}{1 + \lambda B}.$$

Herein,

$$B = h\left(\frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12}\right) + gn + d$$

and

$$M = V_0 - V_1.$$

Since in the next section we seek to combine the economic model with the loss function, and for this purpose the economic model should be expressed based on the cost and not income, we modified Equation (4) as follows

$$E(C) = \frac{C_0}{\lambda} + C_1\left[h\left(\frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12}\right) + gn + d\right] + \frac{E(T)}{h}(b + nc) + \frac{\alpha y}{\lambda h} + W. \tag{6}$$

Here, C_0 and C_1 are the average costs of the process when the process is in the in-control and out-of-control phases, respectively. Therefore, the average cost of the process per hour is equal to

$$E(CH) = \frac{E(C)}{E(T)}. \tag{7}$$

Accordingly, to perform the economic design of the considered control charts based on Duncan's economic model, we determine the design parameters in such a way that the cost function $E(CH)$ is minimized. According to the above discussion, the input parameters of the economic design based on Duncan's economic model are λ, C_0, C_1, g, d, c, b, y and W.

This optimization problem was coded in MATLAB and the results for different values of the input parameters and process shift were obtained using the ABC algorithm. The results are shown in Table 4.

The Economic-Statistical Design

As expressed in the first section, in the definition of economic-statistical design (ESD), if some statistical restrictions are applied to the economic design, the process design is considered as economic-statistical. In this sub-section, we present an economic-statistical design by using the last two sub-sections: the average run length approach (which is a statistical feature) and a modification of Duncan's economic model. To do so, we apply the constraints considered in the statistical design to the economic model. This optimization problem can be written as follows:

$$Min\, E(CH)$$

$$s.t.$$

$$ARL_0 = 370$$

$$\beta \leq 0.75. \tag{8}$$

The optimization problem was coded in MATLAB and the results were obtained for different values of the input parameters and process shift using the ABC algorithm. The results are shown in Table 5.

The Loss Function and Its Combination with the Cost Model

As seen in the previous section, the cost function was considered based on some fixed values. However, many hidden factors and unforeseen costs may arise during the process. On the other hand, the cost of a defective product depends on how defective it is, which may be remedied with a brief rework or it may turn into waste, which is also variable. To address this weakness, a loss function is selected according to the process and combined with the economic model.

One of the most common loss functions is Taguchi's loss function, which is based on Taguchi's definition. In (Taguchi 1986), Taguchi put a cost factor (k) in the quadratic loss function called the cost of reworking and scrap cost.

The method of combining the loss function with the cost model is that if we consider $L(X)$ as the desired loss function and define L_0 and L_1 as the average costs of production of each defective product in the in-control and out-of-control conditions, then C_0 and C_1 in relation (4) become $C_0 = n^*.L_0$ and $C_1 = n^*.L_1$, respectively.Here, n^* equals the number of products produced per hour, $L_0 = E_{\theta_0}(L(X))$ and $L_1 = E_{\theta_1}(L(X))$.

According to Taguchi's philosophy of loss function, the greater the degree of deviation of the quality characteristic from the ideal value, the higher the social cost of quality. In other words, if we consider the target value for a quality characteristic as ξ, the quality loss becomes zero when the value of the quality characteristic is equal to ξ, and it is clear that the distance from ξ increases the cost.

With this description, Taguchi's loss function was defined as $L(X) = k(x-\xi)^2$. Herein, k is the loss coefficient which is equal to A/Δ^2, wherein Δ is the tolerance limit of x, and A is the cost of reworking or scrapping a product unit. Given that the quality characteristics in the in-control and out-of -control states are exponentially distributed with the probability density function $f(x) = \theta e^{-\theta x}$ and the parameters θ_0 and $\theta_1 = \delta\theta_0$, respectively, we obtain

$$\begin{aligned} L_0 &= {}_0^\infty k(x-\xi)^2 f(x)dx \\ &= k\,{}_0^\infty \left[\left(x-\frac{1}{\theta_0}\right)+\left(\frac{1}{\theta_0}-\xi\right)\right]^2 f(x)dx \\ &= k\left[Var(X|\theta_0)+\left(\frac{1}{\theta_0}-\xi\right)^2+2\left(\frac{1}{\theta_0}-\xi\right)E\left(X-\frac{1}{\theta_0}\right)\right] \end{aligned}$$

$$= k\left[Var(X|\theta_0) + \left(\frac{1}{\theta_0} - \xi\right)^2\right] = k\left[\frac{1}{{\theta_0}^2} + \left(\frac{1}{\theta_0} - \xi\right)^2\right]$$

Similarly, under the opposite assumption,

$$L_1 = k\left[Var(X|\theta_1) + \left(\frac{1}{\theta_1} - \xi\right)^2\right] = k\left[\frac{1}{{\theta_1}^2} + \left(\frac{1}{\theta_1} - \xi\right)^2\right].$$

According to what we discussed so far, by rewriting relation (4) based on L_0 and L_1 we obtain

$$E(IC) = \frac{n^*.L_0}{\lambda} + n^*.L_1\left[h\left(\frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12}\right) + gn + d\right] + \frac{E(T)}{h}(b + nc) + \frac{\alpha y}{\lambda h} + W. \tag{9}$$

Now, the optimization problem can be written as follows:

$$\begin{gathered} MinE(IC) \\ s.t. \\ ARL_0 = 370 \\ \beta \leq 0.75. \end{gathered} \tag{10}$$

This optimization problem was coded in MATLAB and the results were obtained for different values of the input parameters and process shift using the genetic algorithm. The results are presented in Table 5.

Sensitivity Analysis and Numerical Results

In this section, we will examine the effect of each input parameter on the designs, particularly focusing on the rate of process shift and its impact on the results. Additionally, we will demonstrate how to implement the designs using a real numerical example.

Optimization Method

The optimization problem of Equations 2, 7, 8, and 10 can be formulated as a nonlinear constrained optimization problem, which is solved using the Artificial Bee Colony (ABC) algorithm to obtain optimal design parameters. In metaheuristic algorithms, only a range of input parameters is considered, and within those ranges, the algorithm seeks to find the best solution for the objective. The algorithm randomly selects values within the specified range and calculates the objective parameter based on those values. Karaboga (2005) introduced ABC, which is inspired by the intelligent behavior of honeybee swarms. ABC consists of different types of bees: employed bees, onlooker bees, and scout bees. Employed bees remain on a food source and store information about its neighborhood. Onlooker bees receive information from employed bees and choose a food source to gather nectar from. Scout bees are responsible for discovering new food sources. The ABC algorithm follows the following procedures:

(1) Initialization by moving the scouts.
(2) Movement of the onlookers.
(3) Scouts move only when the counters of employed bees reach their limits.
(4) Updating the memory.
(5) Checking the termination condition.

For more information on this topic, refer to the works of (Karaboga 2005; Karaboga and Akay 2009; Karaboga and Basturk 2008).

In this article, to solve the optimization problem, the parameters that should be determined in the algorithm are as follows;

The number of colony size (employed bees+onlooker bees) = 20,

The number of food sources equals the half of the colony size = 10,

The number of cycles for foraging (stopping criteria) = 30,

Sensitivity Analysis

To study the effect of each design parameter (c, b, W, g, d, λ, θ_0, δ, k, C_0, C_1, and y) on the designs introduced in the previous sections (statistical, economic, economic-statistical with Taguchi's loss function, and without a loss function), we considered 48 different combinations of levels for the design parameters. These values are presented in Table 2.

In rows 1 to 26 of Table 2, different values of input parameters are given. According to these scenarios, statistical and economic results are determined based on statistical design (the average run length approach), economic design (Duncan's modified economic model), economic-statistical design (based on combining the average run length approach and modified Duncan's economic model), and the economic-statistical approach integrated with the loss function (Taguchi's loss function). In rows 27 to 33, we examine the effect of more cases of θ_0 values on the results. In rows 34 to 42, several cases of process shift (different values of the process shift coefficient δ) are examined to evaluate the performance of each method under different shifts. In rows 43 to 48, the effect

Table 2. Different scenarios considered for the input parameters of the design.

Row	c	b	W	g	d	λ	θ_0	δ	k	C_0	C_1	y
1	5	50	500	50	80	0.01	1	1	500	500	2000	100
2	5	50	500	50	80	0.01	1	1.2	500	500	2000	100
3	2.5	50	500	50	80	0.01	1	1.2	500	500	2000	100
4	10	50	500	50	80	0.01	1	1.2	500	500	2000	100
5	5	25	500	50	80	0.01	1	1.2	500	500	2000	100
6	5	100	500	50	80	0.01	1	1.2	500	500	2000	100
7	5	50	250	50	80	0.01	1	1.2	500	500	2000	100
8	5	50	1000	50	80	0.01	1	1.2	500	500	2000	100
9	5	50	500	25	80	0.01	1	1.2	500	500	2000	100
10	5	50	500	100	80	0.01	1	1.2	500	500	2000	100
11	5	50	500	50	40	0.01	1	1.2	500	500	2000	100
12	5	50	500	50	160	0.01	1	1.2	500	500	2000	100
13	5	50	500	50	80	0.005	1	1.2	500	500	2000	100
14	5	50	500	50	80	0.02	1	1.2	500	500	2000	100
15	5	50	500	50	80	0.1	1	1	500	500	2000	100
16	5	50	500	50	80	10	1	1	500	500	2000	100
17	5	50	500	50	80	50	1	1	500	500	2000	100
18	5	50	500	50	80	100	1	1	500	500	2000	100
19	5	50	500	50	80	200	1	1	500	500	2000	100
20	5	50	500	50	80	0.01	0.5	1.2	500	500	2000	100
21	5	50	500	50	80	0.01	2	1.2	500	500	2000	100
22	5	50	500	50	80	0.01	1	0.6	500	500	2000	100
23	5	50	500	50	80	0.01	1	2.4	500	500	2000	100
24	5	50	500	50	80	0.01	1	1.2	250	500	2000	100
25	5	50	500	50	80	0.01	1	1.2	1000	500	2000	100
26	5	50	500	50	80	0.01	1	1.2	500	250	2000	100
27	5	50	500	50	80	0.01	1	1.2	500	1000	2000	100
28	5	50	500	50	80	0.01	1	1.2	500	500	1000	100
29	5	50	500	50	80	0.01	1	1.2	500	500	4000	100
30	5	50	500	50	80	0.01	1	1.2	500	500	2000	50
31	5	50	500	50	80	0.01	1	1.2	500	500	2000	200
32	5	50	500	50	80	0.01	0.2	1.2	500	500	2000	100
33	5	50	500	50	80	0.01	0.6	1.2	500	500	2000	100
34	5	50	500	50	80	0.01	0.8	1.2	500	500	2000	100
35	5	50	500	50	80	0.01	1.4	1.2	500	500	2000	100
36	5	50	500	50	80	0.01	1.6	1.2	500	500	2000	100
37	5	50	500	50	80	0.01	1.8	1.2	500	500	2000	100
38	5	50	500	50	80	0.01	2	1.2	500	500	2000	100
39	5	50	500	50	80	0.01	1	0.2	500	500	2000	100
40	5	50	500	50	80	0.01	1	0.4	500	500	2000	100
41	5	50	500	50	80	0.01	1	0.8	500	500	2000	100
42	5	50	500	50	80	0.01	1	1	500	500	2000	100
43	5	50	500	50	80	0.01	1	1.5	500	500	2000	100
44	5	50	500	50	80	0.01	1	1.8	500	500	2000	100
45	5	50	500	50	80	0.01	1	2	500	500	2000	100
46	5	50	500	50	80	0.01	1	4	500	500	2000	100
47	5	50	500	50	80	0.01	1	6	500	500	2000	100
48	5	50	500	50	80	0.01	1	1.2	100	500	2000	100
49	5	50	500	50	80	0.01	1	1.2	200	500	2000	100
50	5	50	500	50	80	0.01	1	1.2	400	500	2000	100
51	5	50	500	50	80	0.01	1	1.2	700	500	2000	100
52	5	50	500	50	80	0.01	1	1.2	1000	500	2000	100
53	5	50	500	50	80	0.01	1	1.2	2000	500	2000	100

of the cost coefficient of the loss function (k) on the outputs is investigated. In the three initial designs that are not integrated with the loss function, rows 43 to 48 are not evaluated.

It should be noted that in the first row, which is considered as a control, the fixed cost coefficients of the economic model (C_0 and C_1) and the cost

Table 3. The optimal values, statistical and economic parameters under the statistical design.

Row	h	k_1	k_2	$MinARL_1$
1	31	3.22	3.259	1.001530592
2	17	3.805	3.844	1.000475709
3	42	3.922	3.961	1.00041337
4	7	3.61	3.649	1.000601202
5	40	3.493	3.532	1.000691886
6	22	3.727	3.766	1.00052241
7	46	3.922	3.961	1.00041337
8	6	3.688	3.727	1.000547453
9	42	3.532	3.571	1.000660231
10	9	3.922	3.961	1.00041337
11	9	3.922	3.961	1.00041337
12	12	3.922	3.961	1.00041337
13	47	3.649	3.688	1.000573698
14	34	3.688	3.727	1.000547453
15	31	3.688	3.727	1.000957991
16	24	3.454	3.493	1.001210865
17	8	3.805	3.844	1.000852124
18	44	3.454	3.493	1.001210865
19	5	3.766	3.805	1.000886043
20	30	3.922	3.961	1.002203529
21	38	3.493	3.532	1.000020433
22	13	3.688	3.727	1.002536518
23	19	3.493	3.532	1.000020433
24	22	3.688	3.727	1.000547453
25	44	3.766	3.805	1.000498512
26	43	3.922	3.961	1.00041337
27	32	3.883	3.922	1.000433184
28	18	3.844	3.883	1.000453948
29	14	3.883	3.961	1.000846912
30	47	3.805	3.883	1.000930089
31	32	3.883	3.922	1.000433184
32	14	3.649	3.688	1.003895793
33	37	3.727	3.766	1.001895541
34	14	3.922	3.961	1.000851982
35	28	3.844	3.883	1.000099455
36	37	3.766	3.805	1.000052236
37	31	3.688	3.727	1.000028038
38	25	3.883	3.922	1.000008014
39	18	3.61	3.649	1.003788642
40	12	3.922	3.961	1.003234617
41	45	3.142	3.181	1.002493654
42	15	3.922	3.961	1.000757966
43	15	3.922	3.961	1.000158354
44	28	3.727	3.805	1.000159881
45	34	3.805	3.844	1.000037179
46	42	3.883	3.961	1.000000048
47	35	3.727	3.766	1
48	32	3.649	3.688	1.000573698
49	13	3.649	3.688	1.000573698
50	42	3.922	3.961	1.00041337
51	40	3.727	3.766	1.00052241
52	24	3.688	3.727	1.000547453
53	45	3.922	3.961	1.00041337

coefficient of the loss function (k) were determined in such a way that when no changes were made in the process ($\delta = 1$), the cost of the process was equal in the two approaches (the integrated and non-integrated economic models).

Table 4. The optimal values, statistical and economic parameters under the economic design.

Row	h	k_1	k_2	$LOSS$
1	6	3.844	3.883	1368.327
2	6	3.727	3.805	1368.33
3	6	3.922	3.961	1367.904
4	6	3.805	3.844	1369.155
5	4	3.727	3.766	1363.612
6	8	3.727	3.766	1375.316
7	6	3.103	3.142	1367.259
8	6	3.883	3.922	1370.466
9	5	3.61	3.649	1291.251
10	7	3.649	3.688	1481.27
11	5	3.922	3.961	1235.17
12	8	3.61	3.649	1531.536
13	6	3.922	3.961	1110.787
14	7	3.571	3.61	1602.568
15	11	3.766	3.805	1906.239
16	50	0.1	3.961	2001.234
17	50	0.1	3.961	2001.143
18	50	0.1	3.961	2001.123
19	50	0.1	3.961	2001.112
20	6	3.727	3.766	1368.354
21	6	3.337	3.376	1368.314
22	6	3.454	3.493	1368.362
23	6	3.883	3.922	1368.314
24	6	3.766	3.805	1368.322
25	6	3.454	3.493	1368.326
26	6	3.454	3.493	1261.045
27	7	3.532	3.571	1582.625
28	10	3.805	3.844	796.0891
29	4	3.61	3.649	2507.852
30	6	3.922	3.961	1367.973
31	6	3.844	3.883	1369.016
32	6	3.922	3.961	1368.374
33	6	3.922	3.961	1368.341
34	6	3.532	3.571	1368.334
35	6	3.922	3.961	1368.315
36	6	3.571	3.61	1368.315
37	6	3.883	3.922	1368.314
38	6	3.727	3.766	1368.314
39	6	3.766	3.805	1368.374
40	6	3.922	3.961	1368.367
41	6	3.454	3.493	1368.346
42	6	3.766	3.805	1368.328
43	6	3.883	3.922	1368.316
44	6	3.844	3.883	1368.315
45	6	3.883	3.922	1368.314
46	6	3.688	3.727	1368.314
47	6	3.844	3.883	1368.314
48	6	3.415	3.454	1368.326
49	6	3.571	3.61	1368.324
50	6	3.103	3.142	1368.332
51	6	3.805	3.844	1368.321
52	6	3.688	3.727	1368.323
53	6	3.922	3.961	1368.32

Using the ABC algorithm for Equations 2, 7, 8, and 10, we computed the optimal values of statistical, economic, and economic-statistical combined with Taguchi's loss function, both with and without applying

Table 5. The optimal values, statistical and economic parameters under the economic-statistical design.

Row	h	k_1	k_2	$ARL1$	$LOSS$
1	6	3.883	3.922	1.000788134	1368.326562
2	6	3.883	3.922	1.000433184	1368.320705
3	6	3.766	3.805	1.000498512	1367.905117
4	6	3.298	3.337	1.000874458	1369.16132
5	4	3.727	3.766	1.00052241	1363.61177
6	8	3.415	3.454	1.000759826	1375.321125
7	6	3.922	3.961	1.00041337	1367.247567
8	6	3.922	3.961	1.00041337	1370.466002
9	5	3.571	3.61	1.000630025	1291.251865
10	7	3.22	3.259	1.000960342	1481.27528
11	5	3.532	3.571	1.000660231	1235.17519
12	8	3.805	3.844	1.000475709	1531.534747
13	6	3.883	3.922	1.000433184	1110.78772
14	7	3.454	3.493	1.00072506	1602.569162
15	11	3.532	3.571	1.001119906	1906.240543
16	50	1.348	2.362	1.19836211	2001.25627
17	50	1.621	3.454	1.199158659	2001.144442
18	50	1.621	3.454	1.199158659	2001.123104
19	50	1.66	3.727	1.19914725	2001.111775
20	6	3.883	3.922	1.002255817	1368.350778
21	6	3.844	3.883	1.0000088	1368.313703
22	6	3.844	3.883	1.002309349	1368.351661
23	6	3.688	3.727	1.000012796	1368.313769
24	6	3.883	3.922	1.000433184	1368.320705
25	6	3.61	3.649	1.000601202	1368.323478
26	6	3.922	3.961	1.00041337	1261.039196
27	7	3.142	3.181	1.001054671	1582.630174
28	10	3.688	3.727	1.000547453	796.0897756
29	4	3.688	3.727	1.000547453	2507.85076
30	6	3.493	3.532	1.000691886	1367.977385
31	6	3.766	3.805	1.000498512	1369.016964
32	6	3.805	3.844	1.003752094	1368.375463
33	6	3.493	3.532	1.002244162	1368.350585
34	6	3.922	3.961	1.000851982	1368.327616
35	6	3.649	3.766	1.000388433	1368.319967
36	6	3.571	3.61	1.000075959	1368.314811
37	6	3.571	3.61	1.000036099	1368.314153
38	6	3.61	3.649	1.000015431	1368.313812
39	6	2.908	2.947	1.004362211	1368.385528
40	6	3.922	3.961	1.003234617	1368.366926
41	6	3.532	3.571	1.00182409	1368.343655
42	6	3.532	3.571	1.001119906	1368.332036
43	6	3.766	3.805	1.000200112	1368.31686
44	6	3.922	3.961	1.000058246	1368.314519
45	6	3.571	3.61	1.000059369	1368.314537
46	6	3.571	3.883	1.000000446	1368.313565
47	6	3.649	3.727	1	1368.313558
48	6	3.571	3.61	1.000630025	1368.323953
49	6	3.844	3.883	1.000453948	1368.321048
50	6	3.532	3.571	1.000660231	1368.324452
51	6	3.337	3.376	1.000834443	1368.327326
52	6	3.727	3.766	1.00052241	1368.322178
53	6	3.922	3.961	1.00041337	1368.320378

a loss function, for each row of Table 2. The results are presented in Tables 3–6, respectively.

Table 6. The optimal values, statistical and economic parameters under the economic-statistical design integrated with Taguchi's loss function.

Row	h	k_1	k_2	*ARL1*	*LOSS*
1	24	1.27	2.167	1.199487998	506.9344079
2	50	1.348	2.869	1.199614967	421.1441162
3	50	1.387	3.142	1.199416645	421.0963045
4	50	.841	1.348	1.199236679	421.2482904
5	50	1.426	3.532	1.199354586	420.6469893
6	50	1.348	2.869	1.199614967	422.1441162
7	50	1.387	3.142	1.199416645	420.2101648
8	50	1.426	3.532	1.199354586	423.0192904
9	50	1.426	3.532	1.199354586	427.234122
10	50	1.348	2.869	1.199614967	411.8505682
11	50	1.426	3.532	1.199354586	431.5298305
12	50	1.348	2.869	1.199614967	407.5597649
13	50	.685	1.075	1.196574287	446.8798337
14	50	1.387	3.142	1.199416645	399.262937
15	50	1.27	2.167	1.199487998	504.3429771
16	50	1.621	3.454	1.199158659	501.3229817
17	50	1.27	2.167	1.199487998	501.1472771
18	50	1.27	2.167	1.199487998	501.1238179
19	50	1.621	3.454	1.199158659	501.1119543
20	50	1.855	3.025	1.198658257	1660.533444
21	32	.49	.802	1.194180111	111.1069625
22	7	3.181	3.22	1.003441455	1146.199871
23	50	.724	1.933	1.199438752	355.9850523
24	50	1.348	2.869	1.199614967	214.5856463
25	50	1.387	3.142	1.199416645	834.2651754
26	50	1.348	2.869	1.199614967	421.1441162
27	50	1.426	3.532	1.199354586	421.1469893
28	50	1.348	2.869	1.199614967	421.1441162
29	50	.997	1.66	1.198836875	421.1527025
30	50	1.426	3.532	1.199354586	418.6193828
31	50	1.348	2.869	1.199614967	426.1990828
32	50	2.44	3.922	1.199966709	10335.8646
33	50	1.738	2.947	1.199478623	1155.578096
34	50	1.738	3.961	1.199356919	653.5266555
35	50	1.036	2.791	1.199384749	218.8025676
36	50	.88	2.089	1.199725361	169.400418
37	38	.763	1.66	1.197173189	135.4578227
38	33	.763	3.961	1.190684295	111.1136232
39	2	3.727	3.766	1.003700693	11872.29182
40	4	3.883	3.922	1.00328564	2650.175521
41	12	3.883	3.922	1.0013769	688.234142
42	24	1.66	3.727	1.19914725	506.9345615
43	50	1.036	2.05	1.197907566	369.0431025
44	50	.919	2.05	1.199433371	353.5717817
45	50	.841	1.972	1.199948676	351.6313872
46	50	.451	3.961	1.197087078	390.7715684
47	50	.334	3.961	1.155795478	421.6315976
48	31	1.387	3.142	1.199416645	90.29367881
49	50	1.348	2.869	1.199614967	173.2739523
50	50	1.348	2.869	1.199614967	338.5207283
51	50	1.387	3.142	1.199416645	586.3938528
52	50	1.426	3.532	1.199354586	834.2664644
53	50	1.387	3.142	1.199416645	1660.502917

In Table 3, $MinARL_1$ is calculated using Equation 2, and the parameter values of the model for which ARL_1 is minimized (i.e., $ArgMin_{(h,k_1,k_2)}ARL_1$) are also given in Table 3.

Table 4 displays the values of $LOSS = MinE(CH)$ for economic design, which is calculated using Equation 7. Furthermore, this table also provides the values of economic design parameters for which $E(CH)$is minimized (i.e.,$ArgMin_{(h,k_1,k_2)}E(CH)$).

In Tables 5 and 6 the value of ARL_1 is calculated using Equation 1.

Table 5 displays the values of $LOSS = MinE(CH)$ for economic-statistical design, which is calculated using Equation 8. Furthermore, this table also provides the values of the design parameters for which $E(CH)$is minimized (i.e.,$ArgMin_{(h,k_1,k_2,ARL1)}E(CH)$).

Table 6 displays the values of $LOSS = MinE(IC)$ for economic-statistical design integrated with Taguchi's loss function, which is calculated using Equation 10. Furthermore, this table also provides the values of the design parameters for which $E(IC)$is minimized (i.e.,$ArgMin_{(h,k_1,k_2,ARL1)}E(IC)$).

According to the obtained values, by drawing the results of the four considered approaches against each other, we compared the performance of these approaches.

Figure 2 shows the values obtained for the statistical parameter (ARL_1) in the statistical, economic and economic-statistical schemes. As can be seen, the value of ARL_1 in the statistical design (which was about one, and indeed very desirable) was less than the economic and statistical-economic designs in all the 42 studied cases. This means that in the first step, after the shift to out of control, an alarm will be received. The value of ARL_1 in the economic-statistical design was close to the considered limit (equal to 4), a desirable value in control charts. This statistical feature was close to 9 in the economic

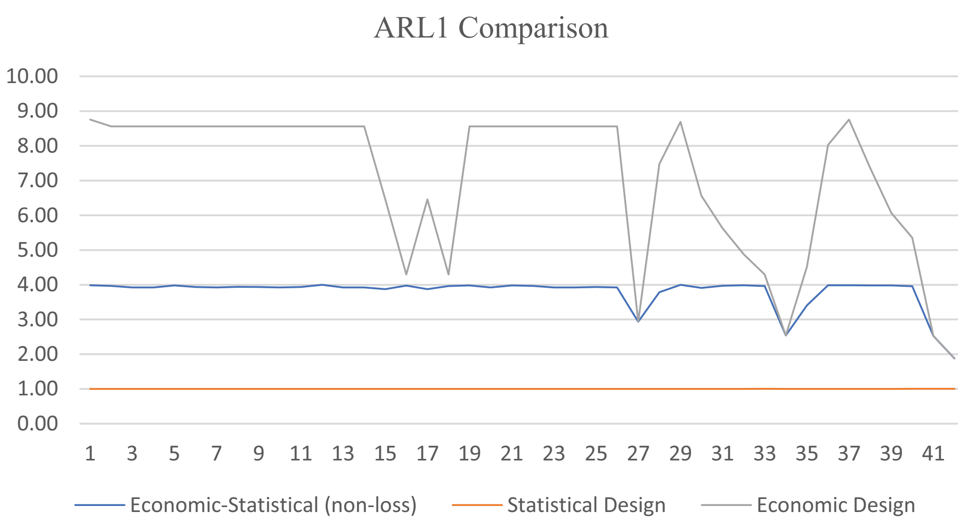

Figure 2. Values obtained for the statistical parameter (ARL_1) in the statistical, economic and economic-statistical designs.

design, which changed in cases related to the different values of θ_0. It will be decreased by decreasing θ_0.

In Figure 3, we compare the amount of cost in the three considered designs. As can be seen, unlike Figure 2, the economic design had a better performance than the statistical design. The decrease and increase of costs in points 23 and 24 were related to the decrease and increase of in-control costs in the two cases examined in Table 1. The amount of cost in the economic-statistical design in almost all cases was consistent with the values of the economic design. This means that there is no significant difference between the economic and statistical-economic designs in terms of economic characteristics.

Therefore, considering the acceptable statistical results in the economic-statistical design and the desirable economic results in this plan, the economic-statistical design can be a desirable design for constructing the desired control chart, which is recommended.

Figures 4 and 5 compare the statistical and economic performance of the economic-statistical design combined with Taguchi's loss function with those of the non-integrated design.

As can be seen in Figure 4, the difference between the statistical performance of the two cases under study is not considerable, and the results are almost identical. Therefore, as expected, the application of the loss function does not change the statistical property of the control chart.

Figure 5 compares the economic performance of the two approaches, and reveals a significant difference in this property. At almost all levels, the cost of the process was reduced by applying the loss function, and in a small number of cases, the cost was increased.

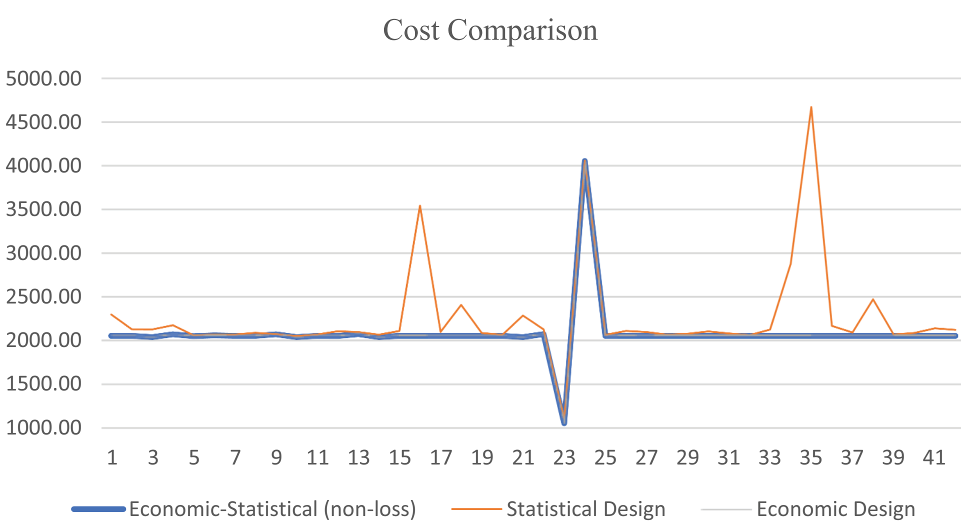

Figure 3. Values obtained for the economic parameter (cost) in the statistical, economic and economic-statistical designs.

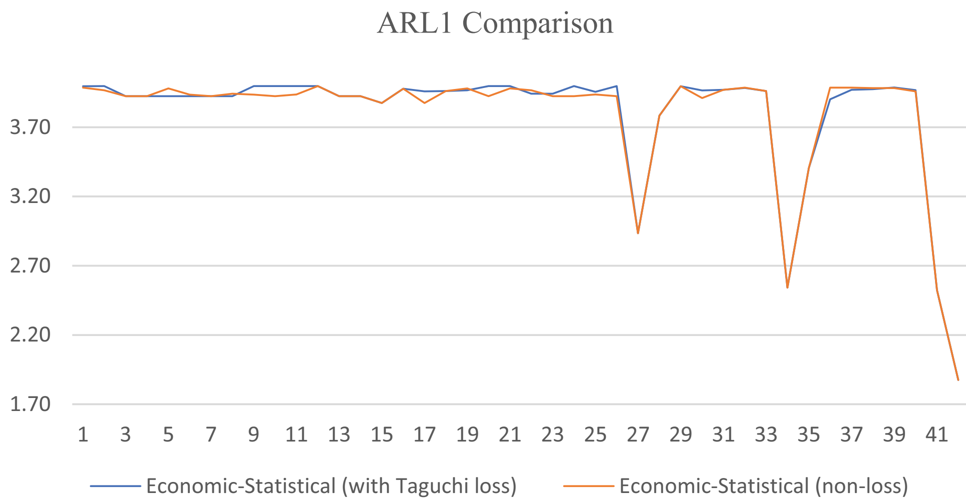

Figure 4. Values obtained for the statistical parameter (ARL_1) in the economic-statistical design combined with Taguchi's loss function and the design without application of the loss function.

Figure 5. Values obtained for the economic parameter (cost) in the economic-statistical design combined with Taguchi's loss function and the design without application of the loss function.

Our prediction is that the increase in costs is due to changes in the rate of process shift (the shift coefficient δ) and the value of the distribution parameter (θ_0). To investigate this issue and find the main causes of these changes, we drew the economic results in the two cases (the economic-statistical design combined with Taguchi's loss function and the design without application of the loss function) against different values of δ and θ_0 in Figure 6.

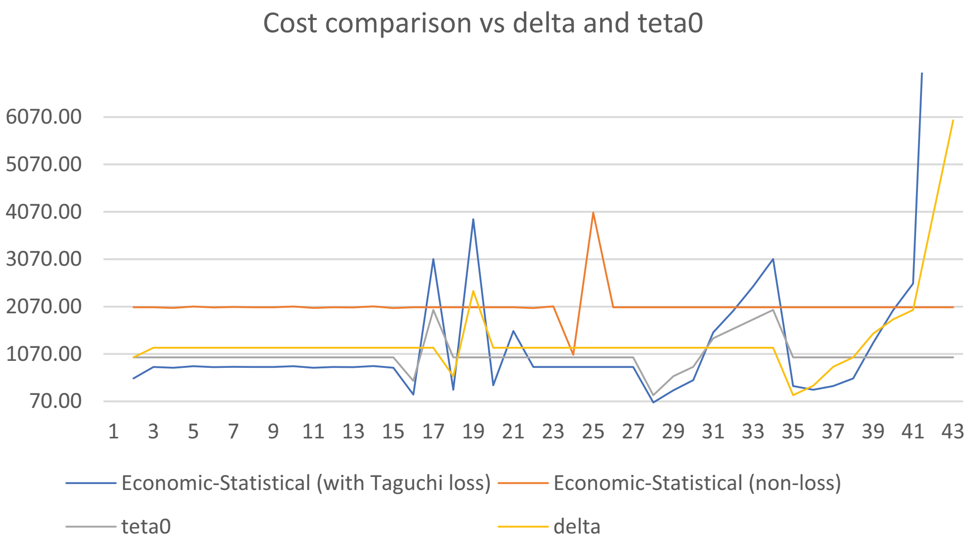

Figure 6. The effect of changes in θ_0 and δ on the values obtained for the economic parameter (cost) in the economic-statistical design combined with Taguchi's loss function and the design without application of the loss function.

First, it was observed that the economic-statistical design without application of the loss function was insensitive to changes in the parameters of quality characteristic distribution (θ_0) and process shift (δ). However, in practice, what is expected and desirable is a direct relationship between the cost and the deviation of the quality characteristic from a target. This was considered in the definition of the loss function (L_0 and L_1) and Equation (5). In the integrated model, increasing θ_0 and δ had a direct effect on the cost of the process.

An Illustrative Example

Lifetime may be regarded as a quality characteristic that requires control. An instance of this is when the occurrence of defects in a process is small. In such scenarios, the C-chart and U-chart could not be used for defect counts. Therefore, instead of employing them, the intervals between successive defect occurrences are taken into account, and a control chart is constructed for these durations (for further examination, refer to (Montgomery 2020)). To fit such data, one of the best distributions that can be considered is the exponential distribution.

In this section, a distinct illustration of lifetime (regarded as a quality characteristic) is introduced. The durability of certain items like car seats, office chairs, and baby inflatable chairs relies on the applied pressure. As weight increases, which is linked to pressure, the lifespan of a seat cushion

Table 7. Lifespan of the child inflatable seat's resistance to the weight placed on it.

Lifetime	0.58	0.09	1.16	0.31	1.20	1.50	1.05	2.11	0.20	0.33	0.34	1.97	1.83	0.37	1.43
Weight	140	180	120	163	116	106	133	93	170	160	153	96	100	150	110
Lifetime	0.17	3.74	1.83	0.11	0.33	1.07	1.28	0.29	1.05	1.07	0.42	0.41	1.09	2.68	0.06
Weight	173	80	103	176	156	126	113	166	136	130	143	146	123	90	185

Table 8. Results obtained for numerical example.

method	h	k_1	k_2	ARL_1	$LOSS$
statistical design	21.31	3.12	3.74	1.0002	-
economic design	5.78	3.21	3.94	-	1351.18
economic-statistical design	6.28	3.01	3.68	1.18	1351.21
economic-statistical design integrated with Taguchi's loss function	41.15	2.17	3.09	1.18	412.85

decreases. The subsequent data pertains to the duration of child's inflatable seats produced in a workshop, measured in hours for weights exceeding 80 kg (Table 7).

We performed the Anderson–Darling goodness-of-fit test on the Lifetime dataset from Table 7. Based on the Anderson–Darling statistic and p-values of the goodness-of-fit test, it can be observed that, at a significance level of .05, the values conform to an Exponential distribution (Anderson–Darling test statistic = .512, for $N = 30$, and p-value = .473 > .05). However, the assumption of Normality is rejected (Anderson–Darling test statistic = 1.116, for $N = 30$, and p-value = .005 < .05). Therefore, for these data, using conventional control charts that are based on the assumption of a normal distribution of the quality characteristic is by no means appropriate. Hence, it is better to use the proposed designs in this article for this purpose.

Given that the mean and standard deviation of the lifetime values are nearly 1 and .87, respectively, we can proceed to fit an Exponential distribution to the data using a mean of $\theta_0 = 1$.

If we assume that the values of the model parameters are as follows: $c = 2.3$, $b = 30$, $W = 400$, $g = 20$, $d = 50$, $\lambda = 0.01$, $\theta_0 = 1$, $\delta = 0.8$, $k = 100$, $C_0 = 50$, $C_1 = 1000$, and $y = 100$, according to this information, the statistical, economic, economic-statistical with Taguchi's loss function, and control charts without a loss function designed on an exponential base would be as follows (refer to Table 8).

Based on Table 8, if a person does not want to use economic design for any reason, then the alternative is to use statistical design. The best option in this case would be to take samples of size 27 from the process every 5.35 hours, assuming the LCL = .11 and the UCL = 1.54. In this scenario, if the process distribution parameter (θ_0), changes from 1 to .8, this chart will detect the change on average after approximately 1.03 sampling iterations.

Also, if a person wants to use economic design, in this case, considering that the economic statistical design with the Taguchi loss function has the minimum $LOSS$ compared to other designs, the best option is to take 36 samples of the process every 7.92 hours. Consider LCL equal to .18 and UCL equal to 2.44. In this case, if θ_0 changes from 1 to .8, this chart will detect the change after an average of 4 sampling rounds, and the average cost per hour of the process will be \$554.29.

Conclusion

In the construction of control charts, the distribution of quality characteristics is usually considered to be normal. However, this assumption is not valid for many datasets, especially those with high skewness. In such cases, a suitable alternative is the exponential distribution. This distribution is used in many fields, including reliability engineering, queueing theory, telecommunications, physics, finance, biology, and epidemiology.

Therefore, given the importance and applications of the subject, contrary to what has been done in the past (using approximations to normalize the characteristics, which leads to errors), this paper constructs a control chart based on economic-statistical design directly, without using any approximations.

According to the useful results of applying loss functions to economic models, Taguchi's loss function was combined with the economic model and the output parameters were obtained under four approaches, namely, statistical, economic, economic-statistical integrated with the aforementioned loss function, and economic-statistical without application of the loss function. The utilized statistical design was based on controlling the average run length of the process, and Duncan's modified economic model was used to construct the economic and economic-statistical designs. The results showed that the economic-statistical design had quite favorable economic and at the same time statistically acceptable results. Additionally, the results confirmed the initial assumption that the application of the loss function makes the calculation of cost in the control chart more accurate and actual, as it is clear that the greater the offset of the quality characteristic from the ideal value, the higher the cost of the process. This feature can be established only by applying the loss function. Utilizing the contents of this article and building a control process based on it can be useful for industries with exponential characteristics.

Also, there are other distributions that can be considered for further improvements. Some of these distributions include:

- Gamma distribution: This distribution is often used to model the time between events or the size of certain objects.
- Log-normal distribution: This distribution is often used to model variables that are positive and skewed, such as income or prices.
- Weibull distribution: This distribution is often used to model failure times in reliability analysis.
- Beta distribution: This distribution is often used to model proportions or probabilities.

By considering these different distributions, one may be able to improve the accuracy of control charts and better detect any deviations from the expected process behavior. However, it is important to note that selecting the appropriate distribution requires careful consideration of the underlying process and the characteristics of the data being analyzed.

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Author Contributions

Conceptualization, M.T. and A.A.H.; Data curation, S.H.Z.A.-T.; Formal analysis, M.T.; Investigation, A.A.H.; Methodology, M.T. and A.A.H.; Software, M.T. and A.A.H.; Visualization, S.H.Z.A.-T.; Writing – original draft, S.H.Z.A.-T.; Writing – review and editing, M.T. and A.A.H. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement

The authors confirm that the data supporting the findings of this study are available within the article.

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