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THE HOMOGENEOUS q-SHIFT OPERATOR AND THE GENERALIZED AL-SALAM-CARLITZ POLYNOMIALS

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Abstract The homogeneous q-shift operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$ is established. As a generalization of the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$, new polynomials $h_n(x, y, a, b|q)$ are described. We present an operator proof of the generating function and its extension, Rogers formula, and Mehler's formula to the polynomials $h_n(x, y, a, b|q)$ by using the operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$. By giving specific values to parameters of a new polynomial $h_n(x, y, a, b|q)$, the generating function and its extension, Rogers formula and Mehler's formula for Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ are determined. The inverse linearization formula for $h_n(x, y, a, b|q)$ may be deduced from the Rogers formula for $h_n(x, y, a, b|q)$, from which the inverse linearization formulas for $U_n(x, y, a; q)$ may be deduced.

1 Introduction

The concepts and terminologies used in [13] are followed in this paper, and we assume that $|q| < 1$. The q-shifted factorial is defined as [10, 13]

$$(a;q)_k = \begin{cases} 1, & \text{if } k = 0, \\ (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k = 1, 2, 3, \cdots. \end{cases}$$

We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

For the multiple q-shifted factorials, we will use the following notation:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The generalized basic hypergeometric series ${}_r\phi_s$ is defined by

$$\begin{aligned} {}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) &= {}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, x\right) \\ &= \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k\,(b_1;q)_k\cdots(b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} x^k, \end{aligned} \tag{1.1}$$

where $q \neq 0$ when $r > s+1$. Note that [23]

$$ {}_{r+1}\phi_r \left(\begin{array}{c} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{array} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} x^n, \quad |x| < 1. $$

The q-binomial coefficients are given by

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. $$

The q-analog of the Chu-Vandermonde summation is [5]

$$ \sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} n+m \\ h \end{bmatrix}. \tag{1.2} $$

The following identities will be used often in this paper [13]:

$$ (q/a;q)_k = a^{-k} (-1)^k q^{\binom{k}{2}+k} \frac{(aq^{-k};q)_\infty}{(a;q)_\infty}. \tag{1.3} $$

$$ \binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + nk. \tag{1.4} $$

For basic concepts of q-calculus, see [16, 17, 18, 28].

The classical Rogers-Szegö polynomials are defined as follows [4, 6, 7]:

$$ h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{1.5} $$

The Cauchy polynomials $P_n(x,y)$ are defined by [14, 15]

$$ P_n(x,y) = \begin{cases} (x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y), & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases} \tag{1.6} $$

In 1965, Al-Salam and Carlitz [3] defined the following polynomials:

$$ u_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} \, {}_2\phi_1 \left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} ; q, qx/a \right), $$

The polynomials $u_n^{(a)}(x;q)$ can be rewritten as

$$ u_n^{(a)}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a^k P_{n-k}(x,1). $$

The q-differential operator is defined as [9, 19]:

$$ D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}. \tag{1.7} $$

Theorem 1.1. [9, 19]. *For $n \geqslant 0$, we have The Leibniz rule for D_q is given by:*

$$ D_q^n \{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{f(a)\} D_q^{n-k} \{g(aq^k)\}. \tag{1.8} $$

In 1997, Chen and Liu [9] defined the q-exponential operator $T(bD_q)$ as follows:

$$ T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}, \tag{1.9} $$

It is easy to prove that [9]

$$T(D_q)\left\{x^n\right\} = h_n(x|q). \tag{1.10}$$

In 2003, Chen *et al.* [8] gave the generating function for Cauchy polynomials $P_n(x,y)$ as, also see [1, 24]

$$\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1 \tag{1.11}$$

and they introduced the homogenous q-difference operator D_{xy}, which is suitable for the study of the Cauchy polynomials, acting on function in two variables x and y, also see [25, 2]:

$$D_{xy}\{f(x,y)\} = \frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}. \tag{1.12}$$

Based on the homogeneous q-difference operator, Chen *et al.* [8] built up the homogeneous q-shift operator as follows:

$$\mathbb{E}(D_{xy}) = \sum_{n=0}^{\infty} \frac{D_{xy}^n}{(q;q)_n},$$

and they have the following results:

Theorem 1.2. [8]. *Let D_{xy} be defined as in (1.12). Then*

$$D_{xy}^k\left\{P_n(x,y)\right\} = \frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(x,y). \tag{1.13}$$

$$D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t^k\frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{1.14}$$

Chen *et al.* [8] also introduced the bivariate Rogers-Szegö polynomials as follows:

$$h_n(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(x,y).$$

In 2007, Chen *et al.* [11] provide an operator technique to construct Mehler's formula and the Rogers formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$, also see [20, 21, 22]. These findings are proven by using the q-exponential operator $T(D_q)$ and the homogeneous q-shift operator $\mathbb{E}(D_{xy})$.

In 2010, Chen *et al.* [12] extended the definition of Al-Salam-Carlitz polynomials as follows:

$$U_n(x,y,a;q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}a^kP_{n-k}(x,y). \tag{1.15}$$

Based on the operator D_{xy}, Chen *et al.* [12] construct the following homogeneous q-shift operator:

$$\mathbb{F}(aD_{xy}) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(aD_{xy})^n}{(q;q)_n}, \tag{1.16}$$

and they gave the following idetities for $U_n(x,y,a;q)$:

Theorem 1.3. [12]. *Let $U_n(x,y,a;q)$ be defined as in (1.15), then*

- *The generating function for $U_n(x,y,a;q)$ is*

$$\sum_{n=0}^{\infty}U_n(x,y,a;q)\frac{t^n}{(q;q)_n} = \frac{(at,yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.17}$$

- *The Rogers-type formula for* $U_n(x,y,a;q)$ *is*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}U_{n+m}(x,y,a;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{(as,ys;q)_\infty}{(xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(xs;q)_k(at)^k}{(q;q)_k(as,ys;q)_k}\,{}_2\phi_1\left(\begin{matrix}y/x,0\\ ysq^k\end{matrix};q,xt\right), \quad (1.18)$$

provided that max$\{|xs|,|xt|\}<1$.

- *The inverse linearization formula for* $U_n(x,y,a;q)$ *is*

$$U_{n+m}(x,y,a;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^kq^{\binom{k}{2}}(aq^m)^kP_{n-k}(x,y)U_m(x,yq^{n-k},a;q). \quad (1.19)$$

- *The Mehler's formula for* $U_n(x,y,a;q)$ *is*

$$\sum_{n=0}^{\infty}(-1)^nq^{-\binom{n}{2}}U_n(x,y,a;q)U_n(u,v,b;q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(abt,ybt,avt;q)_\infty}{(xbt,aut;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}y/x,v/u,q/abt\\ q/aut,q/xbt\end{matrix};q,q\right), \quad (1.20)$$

where $y/x=q^{-r}$ *or* $v/u=q^{-r}$ *for a non-negative integer r and max*$\{|xtbq^{-r}|,|autq^{-r}|\}<1$.

In 2013, Saad and Sukhi [26] defined the q-exponential operator $R(bD_q)$ as follows:

$$R(bD_q)=\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}}{(q;q)_k}(bD_q)^k,$$

and they presented an operator approach for deriving Mehler's formula for Cauchy polynomials $P_n(x,y)$.

$$\sum_{n=0}^{\infty}P_n(x,y)P_n(z,w)\frac{t^n}{(q;q)_n}=\frac{(xwt;q)_\infty}{(xzt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}w|z\\ xwt\end{matrix};q,ys\right), \quad |xzt|<1. \quad (1.21)$$

Theorem 1.4. [27]. *Let* D_q *be defined as in* (1.7), *then*

$$D_q^k\left\{\frac{(vx;q)_\infty}{(tx;q)_\infty}\right\} = t^k(v/t;q)_k\frac{(vxq^k;q)_\infty}{(tx;q)_\infty}. \quad (1.22)$$

$$D_q^k\left\{{}_1\phi_1\left(\begin{matrix}a\\ 0\end{matrix};q,bx\right)\right\} = (-1)^kb^k(a;q)_kq^{\binom{k}{2}}\,{}_1\phi_1\left(\begin{matrix}aq^k\\ 0\end{matrix};q,bxq^k\right). \quad (1.23)$$

2 The Generating Function for $h_n(x,y,a,b|q)$ and its Extension

This section introduces the homogeneous q-shift operator $\widetilde{\mathbb{F}}(a,b;D_{xy})$ and the polynomials $h_n(x,y,a,b|q)$. The generating function and its extension for the polynomials $h_n(x,y,a,b|q)$ are obtained by using the operator $\widetilde{\mathbb{F}}(a,b;D_{xy})$. We provide some specific values for the parameters in the generating function, as well as its extension for the polynomials $h_n(x,y,a,b|q)$, to deduce the generating function and its extensions for $U_n(x,y,a;q)$.

Definition 2.1. Let D_{xy} be defined as in (1.12). We establish the following homogeneous q-shift operator:

$$\widetilde{\mathbb{F}}(a,b;D_{xy})=\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(bD_{xy})^k. \quad (2.1)$$

- Setting $a = 0$ and then $b = a$ in (2.1), we get the operator $\mathbb{F}(aD_{xy})$, given in (1.16), defined by Chen *et al.* [12]. This means that the operator $\mathbb{F}(aD_{xy})$ is a special case of the operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$.

The following theorem is easy to verfy:

Theorem 2.2. *Let the operator* $\widetilde{\mathbb{F}}(a, b; D_{xy})$ *be defined as in* (2.1)*, then*

$$\widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\} = \frac{(yt; q)_\infty}{(xt; q)_\infty} \, {}_1\phi_1 \left(\begin{matrix} a \\ 0 \end{matrix} ; q, bt \right), \quad |xt| < 1. \tag{2.2}$$

Definition 2.3. Let the polynomials $P_n(x, y)$ be defined as in (1.6). We introduce the generalized Al-Salam–Carlitz polynomials $h_n(x, y, a, b|q)$ as follows:

$$h_n(x, y, a, b|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} (a; q)_k b^k P_{n-k}(x, y). \tag{2.3}$$

- Setting $a = 0$ and then $b = a$, the polynomials $h_n(x, y, a, b|q)$ reduce to the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$, which were previously explored by Chen *et al.* [12].

The homogeneous q-shift operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$ can be used to describe the polynomials $h_n(x, y, a, b|q)$. The following theorem is straightforward to prove:

Theorem 2.4. *Let the operator* $\widetilde{\mathbb{F}}(a, b; D_{xy})$ *be defined as in* (2.1)*, then*

$$h_n(x, y, a, b|q) = \widetilde{\mathbb{F}}(a, b; D_{xy}) \{P_n(x, y)\}. \tag{2.4}$$

The following theorem is simple to demonstrate:

Theorem 2.5 (The generating function for $h_n(x, y, a, b|q)$). *Let the polynomials* $h_n(x, y, a, b|q)$ *be defined as in* (2.3)*, then*

$$\sum_{n=0}^{\infty} h_n(x, y, a, b|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty} \, {}_1\phi_1 \left(\begin{matrix} a \\ 0 \end{matrix} ; q, bt \right), \quad |xt| < 1. \tag{2.5}$$

- Setting $a = 0$ and then $b = a$ in the generating function for the polynomials $h_n(x, y, a, b|q)$ (2.5), we recover the generating function for polynomials $U_n(x, y, a; q)$ obtained by Chen *et al.* [12] (equation (1.17)).

Theorem 2.6. *For* $|xt| < 1$*, we have*

$$\widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \frac{P_j(x, y)}{(yt; q)_j} \right\} = \frac{(ytq^j; q)_\infty}{(xt; q)_\infty} \sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} (y/x; q)_{j-i} (xt; q)_i x^{j-i} b^i (a; q)_i$$
$$\times \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (aq^i; q)_n (btq^i)^n}{(q; q)_n}. \tag{2.6}$$

Proof. Let us try to solve the following sum in two different ways:

$$\sum_{n=0}^{\infty} h_n(x, y, a, b|q) h_n(z|q) \frac{t^n}{(q; q)_n} \tag{2.7}$$

In the first way, we express $h_n(z|q)$ as $T(D_q)\{z^n\}$ by (1.10), the sum (2.7) equals

$$\sum_{n=0}^{\infty} h_n(x, y, a, b|q) T(D_q) \{z^n\} \frac{t^n}{(q; q)_n}$$
$$= T(D_q) \left\{ \sum_{n=0}^{\infty} h_n(x, y, a, b|q) \frac{(zt)^n}{(q; q)_n} \right\}$$

$$= T(D_q)\left\{\frac{(yzt;q)_\infty}{(xzt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix};q,bzt\right)\right\}, \quad |xzt|<1 \quad \text{(by using (2.5))}$$
$$= \sum_{n=0}^{\infty}\frac{D_q^n}{(q;q)_n}\left\{\frac{(yzt;q)_\infty}{(xzt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix};q,bzt\right)\right\} \quad \text{(by using (1.9))}$$
$$= \sum_{n=0}^{\infty}\frac{1}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\left\{\frac{(yzt;q)_\infty}{(xzt;q)_\infty}\right\}$$
$$\times D_q^{n-k}\left\{{}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix};q,bztq^k\right)\right\} \quad \text{(by using (1.8))}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{q^{k(k-n)}}{(q;q)_k(q;q)_{n-k}}(tx)^k(y/x;q)_k\frac{(yztq^k;q)_\infty}{(xzt;q)_\infty}$$
$$\times D_q^{n-k}\left\{{}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix};q,bztq^k\right)\right\} \quad \text{(by using (1.22))}$$
$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{-kn}(tx)^k(y/x;q)_k}{(q;q)_k(q;q)_n}\frac{(yztq^k;q)_\infty}{(xzt;q)_\infty}D_q^n\left\{{}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix};q,bztq^k\right)\right\}$$
$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{-kn}(tx)^k(y/x;q)_k}{(q;q)_k(q;q)_n}\frac{(yztq^k;q)_\infty}{(xzt;q)_\infty}$$
$$\times(-1)^n q^{\binom{n}{2}}(a;q)_n(btq^k)^n\,{}_1\phi_1\left(\begin{matrix} aq^n \\ 0 \end{matrix};q,bztq^{k+n}\right) \quad \text{(by using (1.23))}$$
$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(tx)^k(y/x;q)_k(-1)^n q^{\binom{n}{2}}(a;q)_n(bt)^n}{(q;q)_k(q;q)_n}\frac{(yztq^k;q)_\infty}{(xzt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} aq^n \\ 0 \end{matrix};q,bztq^{k+n}\right)$$
$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(tx)^k(y/x;q)_k(-1)^n q^{\binom{n}{2}}(a;q)_n(bt)^n}{(q;q)_k(q;q)_n}$$
$$\times\sum_{j=0}^{\infty}h_j(x,yq^k,aq^n,bq^{k+n}|q)\frac{(zt)^j}{(q;q)_j}. \quad \text{(by using (2.5))} \tag{2.8}$$

On the second way, by using (2.4), we express $h_n(x,y,a,b|q)$ as $\widetilde{\mathbb{F}}(a,b;D_{xy})\{P_n(x,y)\}$, the sum in (2.7) equals

$$\sum_{n=0}^{\infty}\widetilde{\mathbb{F}}(a,b;D_{xy})\{P_n(x,y)\}\,h_n(z|q)\frac{t^n}{(q;q)_n}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty}P_n(x,y)\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}z^k\frac{t^n}{(q;q)_n}\right\} \quad \text{(by using (1.5))}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}P_{n+k}(x,y)\frac{z^k t^{n+k}}{(q;q)_k(q;q)_n}\right\}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty}P_k(x,y)\frac{(zt)^k}{(q;q)_k}\left(\sum_{n=0}^{\infty}P_n(x,yq^k)\frac{t^n}{(q;q)_n}\right)\right\}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty}P_k(x,y)\frac{(zt)^k}{(q;q)_k}\frac{(ytq^k;q)_\infty}{(xt;q)_\infty}\right\}, \quad |xt|<1 \quad \text{(by using (1.11))}$$
$$= \sum_{k=0}^{\infty}\frac{(zt)^k}{(q;q)_k}\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_k(x,y)}{(yt;q)_k}\right\}. \tag{2.9}$$

Equating the coefficients of $(zt)^j$ on equations (2.8) and (2.9), we get

$$\begin{aligned}
&\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_j(x,y)}{(yt;q)_j}\right\}\\
&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(tx)^k(y/x;q)_k(-1)^nq^{\binom{n}{2}}(a;q)_n(bt)^n}{(q;q)_k(q;q)_n}h_j(x,yq^k,aq^n,bq^{k+n}|q)\\
&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(tx)^k(y/x;q)_k(-1)^nq^{\binom{n}{2}}(a;q)_n(bt)^n}{(q;q)_k(q;q)_n}\\
&\quad\times\sum_{i=0}^{j}\begin{bmatrix}j\\ i\end{bmatrix}(-1)^iq^{\binom{i}{2}}(aq^n;q)_i(bq^{k+n})^iP_{j-i}(x,yq^k)\quad\text{(by using (2.3))}\\
&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{j}\begin{bmatrix}j\\ i\end{bmatrix}(-1)^iq^{\binom{i}{2}}b^i\frac{(txq^i)^k(y/x;q)_k}{(q;q)_k}\frac{(-1)^nq^{\binom{n}{2}}(a;q)_{n+i}(btq^i)^n}{(q;q)_n}(yq^k/x;q)_{j-i}x^{j-i}\\
&=\sum_{i=0}^{j}\begin{bmatrix}j\\ i\end{bmatrix}(-1)^iq^{\binom{i}{2}}b^i(y/x;q)_{j-i}(a;q)_ix^{j-i}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(aq^i;q)_n(btq^i)^n}{(q;q)_n}\\
&\quad\times\sum_{k=0}^{\infty}\frac{(txq^i)^k(yq^{j-i}/x;q)_k}{(q;q)_k}\\
&=(ytq^j;q)_\infty\sum_{i=0}^{j}\begin{bmatrix}j\\ i\end{bmatrix}(-1)^iq^{\binom{i}{2}}\frac{b^i(y/x;q)_{j-i}(a;q)_ix^{j-i}}{(xtq^i;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(aq^i;q)_n(btq^i)^n}{(q;q)_n}\\
&=\frac{(ytq^j;q)_\infty}{(xt;q)_\infty}\sum_{i=0}^{j}\begin{bmatrix}j\\ i\end{bmatrix}(-1)^iq^{\binom{i}{2}}(y/x;q)_{j-i}(xt;q)_ix^{j-i}b^i(a;q)_i\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(aq^i;q)_n(btq^i)^n}{(q;q)_n}.
\end{aligned}$$

□

- Setting $a=0$ and then $b=a$ in equation (2.6), we recover Theorem 3.4 obtained by Chen *et al.* [12].

Theorem 2.7 (Extended generating function for $h_n(x,y,a,b|q)$). *Let the polynomials $h_n(x,y,a,b|q)$ be defined as in (2.3), then*

$$\begin{aligned}
\sum_{n=0}^{\infty}h_{n+j}(x,y,a,b|q)\frac{t^n}{(q;q)_n}&=\frac{(ytq^j;q)_\infty}{(xt;q)_\infty}\sum_{i=0}^{j}\begin{bmatrix}j\\ i\end{bmatrix}(-1)^iq^{\binom{i}{2}}(y/x;q)_{j-i}(xt;q)_ix^{j-i}b^i(a;q)_i\\
&\quad\times\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(aq^i;q)_n(btq^i)^n}{(q;q)_n},\qquad |xt|<1.
\end{aligned}\tag{2.10}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty}h_{n+j}(x,y,a,b|q)\frac{t^n}{(q;q)_n}&=\sum_{n=0}^{\infty}\widetilde{\mathbb{F}}(a,b;D_{xy})\{P_{n+j}(x,y)\}\frac{t^n}{(q;q)_n}\quad\text{(by using (2.4))}\\
&=\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty}P_{n+j}(x,y)\frac{t^n}{(q;q)_n}\right\}\\
&=\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty}P_j(x,y)P_n(x,yq^j)\frac{t^n}{(q;q)_n}\right\}\\
&=\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{P_j(x,y)\left(\sum_{n=0}^{\infty}P_n(x,yq^j)\frac{t^n}{(q;q)_n}\right)\right\}
\end{aligned}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ P_j(x,y) \frac{(ytq^j;q)_\infty}{(xt;q)_\infty} \right\}, \quad |xt|<1$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{P_j(x,y)}{(yt;q)_j} \right\}$$

The proof of Theorem 2.7 is now completed using equation (2.6). □

- Setting $j=0$ in (2.10), we get the generating function for the polynomials $h_n(x,y,a,b|q)$ (2.5).
- Letting $a=0$ and then $b=a$ in the extended generating function for the polynomials $h_n(x,y,a,b|q)$ (2.10), we obtain the extended generating function for polynomials $U_n(x,y,a;q)$.

Corollary 2.8. *For* $|xt|<1$, *we have*

$$\sum_{n=0}^{\infty} U_{n+j}(x,y,a;q)\frac{t^n}{(q;q)_n} = \frac{(at,ytq^j;q)_\infty}{(xt;q)_\infty} \sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} \frac{(-1)^i q^{\binom{i}{2}} (xt;q)_i a^i P_{j-i}(x,y)}{(at;q)_i}.$$

3 The Rogers formula for $h_n(x,y,a,b|q)$

We want to offer an operator approach to Rogers formula for the generalized Al-Salam–Carlitz polynomials $h_n(x,y,a,b|q)$ in this section. The Rogers formula for the Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ is obtained by incorporating special variable values in the Rogers formula for $h_n(x,y,a,b|q)$. The Rogers formula for $h_n(x,y,a,b|q)$ allows us to deduce the inverse linearization formula for $h_n(x,y,a,b|q)$, from which we can deduce the inverse linearization formulas for $U_n(x,y,a;q)$.

Theorem 3.1 (Rogers formula for $h_n(x,y,a,b|q)$). *Let the polynomials* $h_n(x,y,a,b|q)$ *be defined as in* (2.3)*, then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,a,b|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(xs;q)_i(a;q)_i(bt)^i}{(q,ys;q)_i}$$
$$\times\ {}_1\phi_1\left(\begin{matrix} aq^i \\ 0 \end{matrix};q,bsq^i\right)\ {}_2\phi_1\left(\begin{matrix} y/x,0 \\ ysq^i \end{matrix};q,xt\right), \quad \max\{|xs|,|xt|\}<1. \tag{3.1}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,a,b|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \widetilde{\mathbb{F}}(a,b;D_{xy})\{P_{n+m}(x,y)\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} \quad \text{(by using (2.4))}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x,y)P_m(x,q^ny)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\right\}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\left(\sum_{m=0}^{\infty} P_m(x,yq^n)\frac{s^m}{(q;q)_m}\right)\right\}$$
$$= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{P_n(x,y)\frac{(ysq^n;q)_\infty}{(xs;q)_\infty}\right\}, \quad |xs|<1$$
$$= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(ys;q)_\infty}{(xs;q)_\infty}\frac{P_n(x,y)}{(ys;q)_n}\right\}.$$

From Theorem 2.6, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,a,b|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{(ysq^n;q)_\infty}{(xs;q)_\infty}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(y/x;q)_{n-i}(xs;q)_i x^{n-i}b^i(a;q)_i\\
&\quad\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(aq^i;q)_m(bsq^i)^m}{(q;q)_m}\\
&=\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n(ys;q)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(xs;q)_i P_{n-i}(x,y)b^i(a;q)_i\\
&\quad\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(aq^i;q)_m(bsq^i)^m}{(q;q)_m}\\
&=\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{t^n}{(q;q)_i(q;q)_{n-i}(ys;q)_n}(-1)^i q^{\binom{i}{2}}(xs;q)_i P_{n-i}(x,y)b^i(a;q)_i\\
&\quad\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(aq^i;q)_m(bsq^i)^m}{(q;q)_m}\\
&=\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^{n+i}}{(q;q)_i(q;q)_n(ys;q)_{n+i}}(-1)^i q^{\binom{i}{2}}(xs;q)_i P_n(x,y)b^i(a;q)_i\\
&\quad\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(aq^i;q)_m(bsq^i)^m}{(q;q)_m}\\
&=\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(xs;q)_i(a;q)_i bt^i}{(q,ys;q)_i}\sum_{n=0}^{\infty}\frac{P_n(x,y)t^n}{(q;q)_n(ysq^i;q)_n}\\
&\quad\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(aq^i;q)_m(bsq^i)^m}{(q;q)_m}\\
&=\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(xs;q)_i(a;q)_i bt^i}{(q,ys;q)_i}\sum_{n=0}^{\infty}\frac{(y/x;q)_n(xt)^n}{(q;q)_n(ysq^i;q)_n}\\
&\quad\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(aq^i;q)_m(bsq^i)^m}{(q;q)_m}\\
&=\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(xs;q)_i(a;q)_i(bt)^i}{(q,ys;q)_i}\,{}_1\phi_1\left(\begin{matrix}aq^i\\0\end{matrix};q,bsq^i\right)\\
&\quad\times{}_2\phi_1\left(\begin{matrix}y/x,0\\ysq^i\end{matrix};q,xt\right),\quad |xt|<1.
\end{aligned}$$

□

- Placing $a=0$ and then $b=a$ in the Rogers-type formula for the polynomials $h_n(x,y,a,b|q)$ (3.1), we recover the Rogers-type formula for polynomials $U_n(x,y,a;q)$ obtained by Chen *et al.* [12] (equation (1.18)).

Theorem 3.2 (The inverse linearization formula for $h_n(x,y,a,b|q)$). *Let the polynomials $h_n(x,y,a,b|q)$ be defined as in* (2.3)*, then*

$$h_{n+m}(x,y,a,b|q)=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(a;q)_i(bq^m)^i P_{n-i}(x,y)h_m(x,yq^{n-i},aq^i,b|q). \tag{3.2}$$

Proof. Rewrite the Rogers-type formula (3.1) as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,a,b|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(xs;q)_i(a;q)_i b^i P_n(x,y)t^{n+i}}{(q;q)_i(q;q)_n(ys;q)_{i+n}}\frac{(ys;q)_\infty}{(xs;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} aq^i \\ 0\end{matrix};q,bsq^i\right)$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(a;q)_i b^i P_n(x,y)t^{n+i}}{(q;q)_i(q;q)_n}\frac{(ysq^{i+n};q)_\infty}{(xsq^i;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} aq^i \\ 0\end{matrix};q,bsq^i\right)$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(a;q)_i b^i P_n(x,y)t^{n+i}}{(q;q)_i(q;q)_n}\sum_{m=0}^{\infty} h_m(x,yq^n,aq^i,b|q)\frac{(sq^i)^m}{(q;q)_m}\quad\text{(by using (2.5))}$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(-1)^i q^{\binom{i}{2}}(a;q)_i b^i P_{n-i}(x,y)t^{n}}{(q;q)_i(q;q)_{n-i}}\sum_{m=0}^{\infty} h_m(x,yq^{n-i},aq^i,b|q)\frac{(sq^i)^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix} n \\ i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(a;q)_i(bq^m)^i P_{n-i}(x,y)h_m(x,yq^{n-i},aq^i,b|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}.$$

Equating the coefficients of $t^n s^m$ in the above equation, we obtain the desired identity. □

- Setting $a=0$ and $b=a$ in the inverse linearization formula for the polynomials $h_n(x,y,a,b|q)$ (3.2), we recover the inverse linearization formula for polynomials $U_n(x,y,a;q)$ obtained by Chen *et al.* [12] (equation (1.19)).

4 The Mehler's formula for $h_n(x,y,a,b|q)$

In this section, we will demonstrate an operator approach to Mehler's formula for the generalized Al-Salam–Carlitz polynomials $h_n(x,y,a,b|q)$. Miller's formula for Al-Salam–Carlitz polynomials $U_n(x,y,a;q)$ is produced by substituting special values for variables in Mehler's formula for $h_n(x,y,a,b|q)$.

Theorem 4.1 (The Mehler's formula for $h_n(x,y,a,b|q)$). *Let the polynomials $h_n(x,y,a,b|q)$ be defined as in (2.4), then*

$$\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}} h_n(x,y,a,b|q)h_n(u,v,c,d|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(ydt;q)_\infty}{(xdt;q)_\infty}\sum_{n=0}^{\infty}\frac{(y/x,v/u;q)_n(uq/d)^n}{(q,q/xdt;q)_n}\sum_{i=0}^{\infty}P_i(u,vq^n)\frac{(a;q)_i(tbq^{-n})^i}{(q;q)_i}$$
$$\times\sum_{k=0}^{\infty}\frac{(xdtq^{-n},aq^i;q)_k(bcdtq^{-n})^k q^{k^2-k}}{(q,ydt;q)_k}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(yq^n/x;q)_j(cxdtq^{-n+k})^j}{(q,ydtq^k;q)_j}$$
$$\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(aq^{i+k};q)_m(bdtq^{-n+k})^m}{(q;q)_m},\tag{4.1}$$

where $y/x=q^{-r}$ or $v/u=q^{-r}$ for a non-negative integer r and $|xdtq^{-r}|<1$.

Proof. By using (2.3) and (2.4), we get

$$\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}} h_n(x,y,a,b|q)h_n(u,v,c,d|q)\frac{t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}\widetilde{\mathbb{F}}(a,b;D_{xy})\{P_n(x,y)\}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(c;q)_k d^k P_{n-k}(u,v)\frac{t^n}{(q;q)_n}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n+k}q^{-\binom{n}{2}+\binom{k}{2}}(c;q)_k d^k t^n}{(q;q)_k(q;q)_{n-k}}P_n(x,y)P_{n-k}(u,v)\right\}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n+2k}q^{-\binom{n+k}{2}+\binom{k}{2}}(c;q)_k d^k t^{n+k}}{(q;q)_k(q;q)_n}P_{n+k}(x,y)P_n(u,v)\right\}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)P_n(x,y)t^n}{(q;q)_n}\sum_{k=0}^{\infty}P_k(1,c)P_k(x,yq^n)\frac{(dtq^{-n})^k}{(q;q)_k}\right\}. \tag{4.2}$$

The terminating condition $v/u = q^{-r}$ or $y/x = q^{-r}$ implies that the first sum in (4.2) is finite. Also, we observe that when n approaches infinity, the second sums in (4.2) do not converge. To avoid this issue, we can focus on the scenario when $v/u = q^{-r}$ or $y/x = q^{-r}$, where r is a positive integer.

Using equation (1.21), equation (4.2) is equal

$$\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}}h_n(x,y,a,b|q)h_n(u,v,c,d|q)\frac{t^n}{(q;q)_n}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\Bigg\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)P_n(x,y)t^n}{(q;q)_n}\frac{(ydt;q)_\infty}{(xdtq^{-n};q)_\infty}$$
$$\times\ {}_1\phi_1\left(\begin{matrix} yq^n/x \\ ydt\end{matrix};q,cxdtq^{-n}\right)\Bigg\} \quad \text{(by using (1.21))}$$
$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\Bigg\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)P_n(x,y)t^n}{(q;q)_n}\frac{(ydt;q)_\infty}{(xdtq^{-n};q)_\infty}$$
$$\times\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(yq^n/x;q)_j(cxdtq^{-n})^j}{(q,ydt;q)_j}\Bigg\}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(cxdtq^{-n})^j}{(q;q)_j}$$
$$\times\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(ydt;q)_\infty}{(xdtq^{-n};q)_\infty}\frac{P_n(x,y)(yq^n/x;q)_j x^j}{(ydt;q)_j}\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(cxdtq^{-n})^j}{(q;q)_j}$$
$$\times\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(ydtq^j;q)_\infty}{(xdtq^{-n};q)_\infty}P_n(x,y)P_j(x,q^n y)\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(cxdtq^{-n})^j}{(q;q)_j}$$
$$\times\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(ydtq^{-n}q^{j+n};q)_\infty}{(xdtq^{-n};q)_\infty}P_{n+j}(x,y)\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(cxdtq^{-n})^j}{(q;q)_j}$$
$$\times\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(ydtq^{-n};q)_\infty}{(xdtq^{-n};q)_\infty}\frac{P_{n+j}(x,y)}{(ydtq^{-n};q)_{n+j}}\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}P_n(u,v)t^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(cxdtq^{-n})^j}{(q;q)_j}\frac{(ydtq^{-n}q^{n+j};q)_\infty}{(xdtq^{-n};q)_\infty}$$

$$\times \sum_{i=0}^{n+j} \begin{bmatrix} n+j \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} (xdtq^{-n};q)_i p_{n+j-i}(x,y) b^i (a;q)_i$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^i;q)_m (bdtq^{-n+i})^m}{(q;q)_m} \qquad \text{(by using (2.6))}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(u,v) t^n}{(q;q)_n} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (cxdtq^{-n})^j}{(q;q)_j} \frac{(ydtq^j;q)_\infty}{(xdtq^{-n};q)_\infty}$$

$$\times \sum_{i=0}^{\infty} \sum_{k=0}^{i} \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} n \\ i-k \end{bmatrix} q^{(j-k)(i-k)} (-1)^i q^{\binom{i}{2}} (xdtq^{-n};q)_i (a;q)_i b^i P_{n+j-i}(x,y)$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^i;q)_m (bdtq^{-n+i})^m}{(q;q)_m}. \qquad \text{(by using (1.2))}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(u,v) t^n}{(q;q)_n} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (cxdtq^{-n})^j}{(q;q)_j} \frac{(ydtq^j;q)_\infty}{(xdtq^{-n};q)_\infty}$$

$$\times \sum_{k=0}^{j} \sum_{i=0}^{n} \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} q^{(j-k)i} (-1)^{i+k} q^{\binom{i+k}{2}} (xdtq^{-n};q)_{i+k} (a;q)_{i+k} b^{i+k} P_{n+j-i-k}(x,y)$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^{i+k};q)_m (bdtq^{-n+i+k})^m}{(q;q)_m} \qquad \text{(by setting } i \to i+k\text{)}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{n+i} q^{-\binom{n+i}{2}} P_{n+i}(u,v) t^{n+i}}{(q;q)_i (q;q)_k (q;q)_n (q;q)_j}$$

$$\times (-1)^{j+k} q^{\binom{j+k}{2}} (cdtq^{-n-i})^{j+k} q^{ji} (-1)^{i+k} q^{\binom{i+k}{2}} (a;q)_{i+k} b^{i+k} P_{n+i+j+k-i-k}(x,y)$$

$$\times \frac{(ydtq^{j+k};q)_\infty}{(xdtq^{-n-i+i+k};q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^{i+k};q)_m (bdtq^{-n-i+i+k})^m}{(q;q)_m}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{n+i} q^{-\binom{n+i}{2}} P_{n+i}(u,v) t^{n+i}}{(q;q)_i (q;q)_k (q;q)_n (q;q)_j}$$

$$\times (-1)^{j+k} q^{\binom{j+k}{2}} (cdtq^{-n-i})^{j+k} q^{ji} (-1)^{i+k} q^{\binom{i+k}{2}} (a;q)_{i+k} b^{i+k} P_{n+j}(x,y)$$

$$\times \frac{(ydtq^{j+k};q)_\infty}{(xdtq^{-n+k};q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^{i+k};q)_m (bdtq^{-n+k})^m}{(q;q)_m}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(x,y) P_n(u,v) t^n}{(q;q)_n} \sum_{i=0}^{\infty} P_i(u,vq^n) \frac{(a;q)_i (btq^{-n})^i}{(q;q)_i}$$

$$\times \sum_{k=0}^{\infty} \frac{(aq^i;q)_k (bcdtq^{-n})^k q^{k^2-k}}{(q;q)_k} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} P_j(x,yq^n) (cdtq^{-n+k})^j}{(q;q)_j}$$

$$\times \frac{(ydtq^{j+k};q)_\infty}{(xdtq^{-n+k};q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^{i+k};q)_m (bdtq^{-n+k})^m}{(q;q)_m} \qquad \text{(by using(1.4))}$$

$$= \frac{(ydt;q)_\infty}{(xdt;q)_\infty} \sum_{n=0}^{\infty} \frac{(y/x, v/u;q)_n (uq/d)^n}{(q,q/xdt;q)_n} \sum_{i=0}^{\infty} P_i(u,vq^n) \frac{(a;q)_i (btq^{-n})^i}{(q;q)_i}$$

$$\times \sum_{k=0}^{\infty} \frac{(xdtq^{-n}, aq^i;q)_k (bcdtq^{-n})^k q^{k^2-k}}{(q,ydt;q)_k} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (yq^n/x;q)_j (cxdtq^{-n+k})^j}{(q,ydtq^k;q)_j}$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^{i+k};q)_m (bdtq^{-n+k})^m}{(q;q)_m}. \qquad \text{(by using (1.3) and (1.6))} \tag{4.3}$$

□

- Setting $a = c = 0$ and then $b = a$ and then $d = b$ in Mehler's formula for the polynomials $h_n(x, y, a, b|q)$ (4.1), we recover Mehler's formula for Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ obtained by Chen et al. [12] (equation (1.20)).

Proof.

$$\sum_{n=0}^{\infty}(-1)^n q^{-\binom{n}{2}} U_n(x,y,a;q) U_n(u,v,c;q) \frac{t^n}{(q;q)_n}$$

$$= \frac{(abt, ybt;q)_\infty}{(xbt;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (y/x, v/u, q/abt;q)_n (aut)^n}{(q, q/xbt;q)_n} \sum_{i=0}^{\infty} P_i(u, vq^n) \frac{(atq^{-n})^i}{(q;q)_i}$$

$$= \frac{(abt, ybt, avt;q)_\infty}{(xbt, aut;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (y/x, v/u, q/abt;q)_n}{(q, q/xbt, autq^{-n};q)_n} (aut)^n, \quad |autq^{-n}| < 1$$

$$= \frac{(abt, ybt, avt;q)_\infty}{(xbt, aut;q)_\infty} \sum_{n=0}^{\infty} \frac{(y/x, v/u, q/abt;q)_n}{(q, q/xbt, q/aut;q)_n} q^n$$

$$= \frac{(abt, ybt, avt;q)_\infty}{(xbt, aut;q)_\infty} {}_3\phi_2 \left(\begin{matrix} y/x, v/u, q/abt \\ q/xbt, q/aut \end{matrix} ; q, q \right),$$

where $y/x = q^{-r}$ or $v/u = q^{-r}$ for a nonnegative integer r, and $\max\{|xbtq^{-r}|, |autq^{-r}|\} < 1$. □

5 Conclusion remarks

The homogeneous q-shift operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$ and the generalized Al-Salam-Carlitz polynomials $h_n(x, y, a, b|q)$ are an extension of the operator $\mathbb{F}(aD_{xy})$ and the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ defined by Chen *et al.* [12], respectively. The identities for the generalized Al-Salam-Carlitz polynomials $h_n(x, y, a, b|q)$ are an extension of the identities of the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$.

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