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The generalized k-connectivity of equally complete bipartite graphs and their line graphs

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Abstract

Generalized k connectivity for graphs and generic graphs is recognized as NP-complete, a parameter that measures the network's ability to connect vertices. Suppose $k_{m,n}$ a completely connected bipartite graph represents the maximum number of internally disjoint Steiner trees (IDSTs) joining a subset $S \subseteq V(G)$ of k vertices in G. In this context, Steiner trees (or "S -trees") T_1, T_2 are considered internally detached if and only if $V(T_1) \cap V(T_2) = S$ and $E(T_1) \cap E(T_2) = \phi$. We define generalized k-connectivity as $\kappa_k(G)$. This study focuses on calculating the precise values of generalized k-connectivity for line- graph of bipartite graphs with $k = 3, 4$ and generalized k-connectivity for bipartite graphs with $k \geq 3$.

Subject Classification: *05C10, 05C25.*

Keywords: *Completed bipartite graph, Generalized k-connectivity, Line graph inwardly disjoint trees, Steiner trees.*

1. Introduction

A desired pair $(V(G), E(G))$ represents (G) graph, where both of $(V(G), E(G))$ symbolize vertices, edges sets, respectively. By the number of vertices which is directly adjacent to $x \in V(G)$ degree of a vertex is determined it that symbolize d by $d_G(x)$. The minimum degrees of G are represented by $\delta = \delta(G)$, respectively. When δ is equal to some natural number q, the graph is

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called q -regular. When the vertices of (G) are separated into the sub-sets X, Y, then each edge has an end vertex at $X$and another at Y [2]. A completed bipartite -graph, symbolized as $(k_{m,n})$ that is a bipartite graph where all vertices in X are adjacent to all vertices in Y, with $m = |X|$ and $n = |Y|$. The application of graphs see [17,18], $L(G)$ is the symbol of the line graph of G where, exactly when the corresponding edges are adjacent that the two vertices in $L(G)$ are adjacent. The vertices of $L(G)$ correspond to the edges of G. Connection is a basic understood in graph theory, measuring the ability G to remain connected after removing vertices or edges. Connectivity for the graph G symbols as $\kappa(G)$, representing the mini number of vertices which have to remove to separate the graph or reduce it to a trivial state The. The$G_\kappa - C$, introduced by Chartrand et al. in [3], is a stronger measure of connectivity used to assess the dependability and fault tolerance in the network. It considers any subset (S) of the vertex set $V(G)$ and defines a Steiner tree as a tree in Gthat contains all vertices in S. S - trees are internally edge-disjoint trees if they share no common edges. The $G_\kappa - C$ for G, symbolize as $\kappa_k(G)$, is the least connectivity value amongst all subsets S of $V(G)$ with k vertices. J.M. Xu [12] established in 2001 that a graph has at least two satisfying edges. Ch.F et al. [10] found a solution for generalized connectivity in graphs for any two integers n and k. The upper and lower bounds of generalized connectivity in a general graph $G$are discussed in [4,6]. Sh. Li et al.[7] introduced the $G_\kappa - C$ of star graphs for $k = 3$, and later Dh. D. Kadhim and A.A. Najim [2]improved it for an equally complete k - partite graph in 2019. The generalized 4-connectivity of hypercube for $CBPG$was studied by Sh. Lin and Q. Zh [9] . In 2022, S. Li, Zhao, et. al. [15] studied internally disjoint trees in the line graph and total graph of $CBPG$. For further information on generalized connectivity, refer to [1,5,8,11,13-16].

Abbreviations

Abbreviations	Meaning
$G_\kappa - C$	Generalized κ -connectivity
IDSTs	internally disjoint $S -$ trees
MNIDST	maximum number of IDSTs
$CBPG$	complete bipartite graph

2. Preliminary Results

Lemma 2.1: [6] *Any connected G graph have order n arranged by minimum degree δ, if there are two adjacent vertices with same δ, the system have $k_k(G) \le \delta - 1$ of $3 \le k \le n$.*

Lemma 2.2: [6] *If n and k are integers where $(2 \le k \le n)$, G is connected with n vertices so, vertex connectivity $(\kappa_k(G)$) is less or equal than edge connectivity $(\lambda_k(G))$, which is in turn less or equal than the less degree of $(\delta(G))$.*

Lemma 2.3: [8] *For any edge $e = xy \in E(G)$, the grade of e in the line graph $L(G)$ is equal to the sum of the degrees of (x , y) in simple undirected graph,* *minus* 2. *Specifically, whether G is q -regular, so $L(G)$ is $(2q-2)$ -regular. In fact, the IDSTs with the maximum number of connections, denoted as MNIDST for convenience, are of interest. The count of cases for an equally $CBPG$ can be calculated as the ceiling of $(k+1)/2$, whereas for a non-equally $CBPG$, it is simply $k+1$Furthermore, any vertex with a non-positive index is not part of the vertex set $V(G)$, and consequently, any edge incident to such a vertex is not included in $E(G)$.*

3. Principal findings

This section's goal is to assess the $G_\kappa - C$ of both the equally $CBPG$, considering a range of values for k from 3 to 9. Moreover, we will examine the non-equally $CBPG$ along with its line, focusing specifically on the cases when k takes on the values 3, 4, and 5.

Theorem 3.1: *Any integer n and k, where $n \geq 2$ and k takes the values 3, 4, 5, the following statement holds true: the rank of CBPG $k_{n,n}$ is equal to $n-1$.*

Proof: Consider the *CBPG* $k_{n,n}$ with vertex sets ($X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$) where each vertex in X is adjacent to every vertex in Y, and $|X| = |Y| = n$. Since $k_{n,n}$ is a regular graph, the minimum degree is $\delta = n$. By applying Lemma (2.1) , we obtain the inequality. $\kappa_k(k_{n,n}) \geq n-1$. To establish the lower bound $\kappa_k(k_{n,n}) \geq n-1$, we require to demonstrate for every k, $S \subseteq V(k_{n,n})$ at least there is$n-1$ IDSTs connecting the vertices in S within $k_{n,n}$, where $|S| = k$. we discuss the case define as:

Case(r) $S_r = \{y_{r-3}, y_{r-2}, \ldots, y_{r-1}, x_1, x_2, \ldots, x_{k-r+1}\}$, where $r = 1,2,3$, then MNIDST in$k_{n,n}$ is $n-1 \leq \kappa_k(S) \leq n$, as explained in: $T_i = x_1 y_i x_2 \cup x_3 y_i x_4 \cup \ldots \cup x_{q-1} y_i x_h \cup x_j y_1 x_h y_{3-m}$ (unless $k=5, r=3$) and $i=3$) and $T_4 = y_1 x_1 y_3 x_2 \cup y_3 x_3 y_2$ at ($k=5$,$r=3$and $i=3$ Where

$$i = \begin{cases} 1,2,\ldots,n \text{ if } r=1 \\ 2,\ldots,n \text{ if } r=2,3 \end{cases}, q = \begin{cases} k-r+1 \text{ if } r=1,2 \\ k-3 \text{ if } r=3 \end{cases}, h = \begin{cases} 0 \text{ if } r=1 \\ 1 \text{ if } r=2,3 \end{cases},$$

$$m = \begin{cases} 3 \text{ if } r=2 \\ 1 \text{ if } r=3 \end{cases} \text{ and } j = \begin{cases} 0 \text{ if } k=3,4,5, r=2 \text{ and } i=2,3,\ldots,n \text{ and } \\ k=5, r=3 \text{ and } \forall i \geq 4 \\ 3 \text{ if } k=5, r=3 \text{ and } \forall i=2 \end{cases}.$$

When $h=0$ then the branch $y_i x_h$ is not exist. From the argument above, we get $n-1 \leq \kappa_k(S) \leq n$. Thus, we get $\kappa_k(k_{n,n}) = n-1$.

Lemma 3.1: *Let $G = k_{n,n}$ where $n \geq 3$, we have $\kappa_k(k_{n,n}) \leq \delta - 2$ where* $k \geq 6$.

Proof: Let $k_{n,n}$ be a $CBPG$,$G(X \cup Y, E)$, such that $|X| = |Y| = n$. Let $S \subseteq V(k_{n,n})$ where $\delta = k$, $S = \{u_1, u_2, \ldots, u_k\}$ where $u_1, u_2 \in Y$ and $u_i \in X$; $i = 3,4,\ldots,k$. We want to prove that the MNIDSTS connecting in G are $\delta - 2$, which is $T_1, T_2, \ldots, T_{\delta-2}$. Clearly, the Steiner tree containing the set S has a maximum degree of two or more. Assuming that $d_G(u_1) \geq 3$ in one particular Steiner tree, denoted as T_1, implies the existence of $\delta - 3$ Steiner trees connecting S in the graph G, distinct from T_1. Consequently, the minimum number of IDSTs (MNIDSTs) connecting S in G. Alternatively, if we suppose that $d_G(u_1) = 2$ in two Steiner trees, say T_1 and T_2, then there exist $\delta - 4$ additional Steiner trees connecting S, resulting in $\delta - 2$ MNIDSTs connecting S in G. In the case where the maximum degree of vertices in S is two or less in a single tree, denoted as T_1, and the vertices in S in other trees have a degree of one, T_1 requires at least $k - 5$ vertices from the set Y (excluding u_1 and u_2). Therefore, the remaining number of vertices from the set Y, which are different from u_1, u_2, and the $k - 5$ vertices, is $n - k - 3$. In this scenario, the MNIDSTs connecting S in G amount to $n - k - 3$. Based on the above results, we can conclude that $\kappa_k(k_{n,n}) \leq \delta - 2$.

Theorem 3.2: *Any integer* n , k *where* $n \geq 3$ *and* $k = 6,7,8,9$. *Then* $\kappa_k(k_{n,n}) = n - 2$.

Proof: Let $k_{n,n}$ be a $CBPG$,$G(X \cup Y, E)$ in which each vertex of X is adjacent with each vertex of $Y = \{y_1, y_2, \dots, y_n\}$ such that $|X| = |Y| = n$. Since $k_{n,n}$ is an regular graph this means that $\delta = n$ then by lemma (3.1) we get $\kappa_k(k_{n,n}) \leq n - 2$. To prove $\kappa_k(k_{n,n}) \geq n - 2$, the suggested system require that for any k −subset $S \subseteq V(k_{n,n})$ at least there is an $n - 2$ IDSTs connecting S in $k_{n,n}$. we discuss five cases as define:

Case(r) $S_r = \{y_{r-3}, y_{r-2}, y_{r-1}, x_1, x_2, \dots, x_{k-r+1}\}$,where $r = 1,2,3$, then the $MNIDST$connecting$S$in$k_{n,n}$is $n - 2 \leq S \leq n$, as explained in:

$$T_j = x_1 y_i x_2 \cup x_3 y_i x_4 \cup \dots \cup x_q y_i x_h \cup y_1 x_h y_{3-m}$$

Where $i = \begin{cases} 1,2,\dots,n \text{ if } r = 1 \\ 2,3,\dots,n \text{ if } r = 2 \\ 3,4,\dots,n \text{ if } r = 3 \end{cases} j = \begin{cases} i \text{ if } r = 1 \\ i - 1 \text{ if } r = 2, \\ i - 2 \text{ if } r = 3 \end{cases} q = \begin{cases} k \text{ if } r = 1 \\ k - r \text{ if } r = 2,3 \end{cases}$,

$$h = \begin{cases} 0 \text{ if } r = 1 \\ i \text{ if } r = 2,3 \end{cases} \text{ and } m = \begin{cases} 3 \text{ if } r = 2 \\ 1 \text{ if } r = 3 \end{cases}.$$

When $h = 0$ or negative the branch $y_i x_h$ is not exist.

Case(4) $S_4 = \{y_1, y_2, y_2, x_1, x_2, \dots, x_{k-3}\}$. Then the $MNIDST$in $k_{n,n}$ is $n - 2$, as explained in : $T_1 = y_2 x_1 y_1 x_2 y_3 x_{m+2} \cup y_1 x_m \cup x_3 y_2 x_{m+1}$,$T_2 = x_1 y_4 x_2 y_2 \cup x_3 y_4 x_4 \cup \dots \cup x_{k-4} y_4 x_{k-3} \cup y_1 x_3 y_2$, where $x_m, x_{m+1}, x_{m+2} \in S$ and

$$m = \begin{cases} 0 \text{ if } k = 6 \\ 4 \text{ if } k = 7,8,9 \end{cases}, T_3 = x_1 y_5 x_2 \cup x_3 y_5 x_4 y_2 \cup y_3 x_5 y_5 x_6 y_1$$

(used only at $k = 8,9$), $T_4 = x_1 y_6 x_2 \cup x_3 y_6 x_4 y_3 \cup y_1 x_5 y_6 x_6 y_2$ (used only at $k = 9$) and $T_{i-2} = x_1 y_i x_2 \cup x_3 y_i x_4 \cup \dots \cup x_{k-3} y_i x_i y_1 \cup y_2 x_i y_3$ Where

$$i = \begin{cases} k - 2, k - 1, \dots, n \text{ if } k = 7,8,9 \\ k - 1, k, \dots, n \text{ if } k = 6 \end{cases}.$$

Case (5) $S_4 = \{y_1, y_2, y_2, y_4, x_1, x_2, \ldots, x_{k-4}\}$, (used only at $k = 8,9$). Then the $MNIDST$ in$k_{n,n}$ is $n-2$, as explained in: $T_1 = x_4y_1x_1y_3x_2x_m \cup y_3x_3y_4$, $T_2 = x_3y_2x_1y_4x_2y_1x_m \cup y_4x_4y_3$ $T_3 = x_1y_5x_2 \cup y_1x_3y_5x_4y_2 \cup y_3x_5 \cup x_5y_4$ and $T_{i-2} = x_1y_ix_2 \cup x_3y_ix_4 \cup x_my_ix_iy_1 \cup y_2x_iy_3 \cup y_4x_i$ where $i = 6,7,\ldots,n$ and

$$m = \begin{cases} 0 \text{ if } k = 8 \\ 5 \text{ if } k = 9 \end{cases}. \text{ Thus } \kappa_k(k_{n,n}) = n-2.$$

Theorem 3.4: *The extended 4-connectivity for line G graph of CBPG is* $\kappa_4(L(k_{m,n})) = m+n-3$ *where* $1 \le n \le m$.

Prove: Spouse $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ be two parts of $k_{m,n}$such ;$V(K_{m,n}) = V(X \cup Y)$ and let $L(k_{m,n})$ be the line graph of a *CBPG* $k_{m,n}$ with $1 \le n \le m$. Since $L(k_{m,n})$ is a $m+n-2$-regular graph, we get by lemma (2.1)$\kappa_4(L(k_{m,n})) \le m+n-3$. Next we prove that $\kappa_4(L(k_{m,n})) \ge m+n-3$ to achieve this we need to proof that there is IDSTs connecting in for subset. We can write $L(k_{m,n})$ as n sets and m vertices in each set $x_iy_j \in V(L(k_{m,n}))$ where $(i = 1,2,\ldots,m) and (j = 1,2,\ldots,n))$ is a vertex in $L(k_{m,n})$ and its edge in $k_{m,n}$, also, the edge in $L(k_{m,n})$ are $(x_iy_j, x_iy_{j'})$ and $(x_iy_j, x_{i'}y_j)$ where $(i = 1,2,\ldots,m)$, $(j = 1,2,\ldots,n)$, $(i' = 1,2,\ldots,m)$ and $(j' = 1,2,\ldots,n$.

Case (1): If $u = x_1y_1$, $v = x_2y_1$, $r = x_3y_1$ and $w = x_4y_1$. Then the $MNIDST$in $L(k_{m,n})$ is $m+n-3$, selected as: $T_1 = wvur$, $T_2 = uwrv$,$T_{i+1} = ux_1y_ix_2y_iv \cup x_iy_ix_3y_ir \cup x_1y_ix_4y_iw$where $(i = 1,2,\ldots,n)$ and $T_{n+j-3} = ux_jy_1v \cup x_jy_1r \cup x_jy_1w$ where $(j = 5,6,\ldots,m)$.

Case (2): If $u = x_1y_1$, $v = x_2y_1$, $r = x_3y_1$ and $w = x_1y_2$. Then the $MNIDST$ in $L(k_{m,n})$ is $m+n-3$, selected as : $T_1 = wuvr$, $T_2 = vx_2y_2wx_3y_2ru$, $T_{i-1} = ux_iy_1v \cup x_iy_1r \cup x_iy_1x_iy_2w$ where $(i = 4,5,\ldots,m)$ and $T_{m+j-3} = ux_1y_jw \cup x_1y_jx_2y_jv \cup x_1y_jx_3y_jr$ where $(j = 2,3,\ldots,n)$.

Case (3): If $u = x_1y_1$, $r = x_1y_2$ and $w = x_2y_2$.Then the $MNIDST$in $L(k_{m,n})$ is $m+n-3$, selected as : $T_1 = vurw$, $T_{i-2} = ux_iy_1v \cup x_3y_1x_iy_2r \cup x_iy_2w$ where $(i = 3,4,\ldots,m)$ and $T_{m+j-3} = ux_1y_jr \cup x_1y_jx_2y_jv \cup x_2y_jw$ where $(j = 3,4,\ldots,n)$.

Case (4): If $u = x_1y_1$, $v = x_2y_1$, $r = x_1y_2$ and $w = x_1y_3$. Then the $MNIDST$ in $L(k_{m,n})$ is $m+n-3$, selected as : $T_1 = rwuv$,$T_2 = ux_2y_2v \cup x_2y_2x_2y_3w$ $T_i = ux_iy_1v \cup x_iy_1x_iy_2r \cup x_iy_1x_iy_3w$ where $(i = 3,4,\ldots,m)$ and $T_{m+j-3} = ux_1y_jr \cup x_1y_jw \cup x_1y_jx_2y_jv$ where $(j = 4,5,\ldots,n)$.

Case (5): If $u = x_1y_1$, $v = x_1y_2$, $r = x_1y_3$ and $w = x_1y_4$.Then the $MNIDST$ in $L(k_{m,n})$ is $m+n-3$, (see Figure 1) selected as : $T_1 = wvur$, $T_2 = uwrv$,

$T_{i+1} = ux_iy_1x_iy_2v \cup x_iy_1x_iy_3r \cup x_iy_1x_iy_4w$ where $(i = 2,3,\ldots,m)$ and $T_{m+j-3} = ux_1y_jv \cup x_1y_jr \cup x_1y_jw$ where $(j = 5,6,\ldots,n)$.

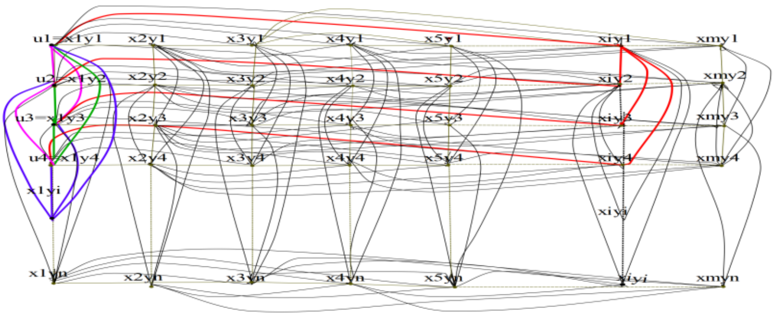

Figure 1

By the five cases above, deduce any sub-set (if IDSTs are connecting). Thus $\boldsymbol{\kappa_4(L(k_{m,n})) = m + n - 3}$.

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