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Estimate a nonparametric copula density function based on probit and wavelet transforms

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ABSTRACT

This study employs wavelet transforms to address the issue of boundary effects. Additionally, it utilizes probit transform techniques, which are based on probit functions, to estimate the copula density function. This estimation is dependent on the empirical distribution function of the variables. The density is estimated within a transformed domain. Recent research indicates that the early implementations of this strategy may have been more efficient. Nevertheless, in this work, we implemented two novel methodologies utilizing probit transform and wavelet transform. We then proceeded to evaluate and contrast these methodologies using three specific criteria: root mean square error (RMSE), Akaike information criterion (AIC), and log-likelihood (LogL). The wavelet transform method works better than the probit transform method at all three levels of correlation, as shown by a simulated study with four types of copulas, five sample sizes, and three levels of correlation. Research has demonstrated that probit transformation methods are most appropriate for linkages involving large and medium sample sizes, as indicated by Frank, Joe, and Tawn Copula. On the other hand, for copula functions for all sample sizes, the wavelet transform method was found to be ideal in cases with low correlation values.

1. Introduction

The nonparametric estimation technique is a common and flexible tool for analyzing data and modeling relationships between variables. The nonparametric estimation is different from the parametric estimation in that it does not take a fixed form or a specific form, but is obtained according to the information derived from the data. All information regarding the phenomena under research is assumed to be regularly distributed in parametric models. Under tight assumptions and circumstances, if the random variables are not normally distributed, we cannot use standard correlation measurements like Kendall's or Spearman's. Separating random variables' effects is extremely challenging, especially when evaluating the degree of positive and negative dependence. As a result, researchers use nonparametric approaches such as the kernel density function to detect dependencies, especially in multivariate distributions.

The problem in the modeling of multivariate functions is the presence of dependency between the observations of the variables of the examined phenomena, which

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can lead to a variety of issues, including boundary effects. In this situation, It is impossible to get the exact estimation for these functions. A suitable statistical tool must be used to characterize the dependence structure between the variables of the examined phenomenon, particularly when the effect extends over a long or medium period of time and the data distribution is unknown. Nonparametric approaches are employed to estimate the copula functions in this research.

Many studies have been published by researchers to help develop ideas for modeling dependency measures in many fields, especially the challenges encountered during the analysis, such as problems of association between study variables and problems of boundary effects. [1] developed the theory of nonparametric estimation of the copula function of a random variable based on the empirical Copula and measuring the sample dependency by means of the empirical copula, and obtained a consistent empirical copula function. [2] clarified and reviewed some parametric, nonparametric, and semi-parametric methods and suggested methods for estimating the probability density function and choosing the appropriate method for estimating smoothing parameter and comparing the mentioned methods in determining the best estimator for the probability density function using the simulation method.

[3] used the copula theory in modelling the survival function of the bivariate variable Weibull distribution and bivariate standard normal distribution cut off at zero point and using simulation experiments for comparison between the estimation of the survival function by using six different copulas [4] presented a paper for inference copula models, based on the rank method. Working in detail on a small imaginary numeric example, illustrate the different steps for checking the dependence between two random variables and modeling it using copulas. It also introduces simple graphical tools and numerical techniques for selecting a suitable model, estimating its parameters, and checking its suitability. An application of the methodology to hydrological data is then presented. [5] investigated kernel methods for obtaining smooth and flexible estimates of the bivariate correlation cumulative distribution function, also discuss the selection of bandwidth parameters. [6] presented a proposal for a new copula by applying the Plackett copula through a mathematical modification that was made on that copula and comparing the Plackett copula with the proposed copula using simulations. [7] introduced the probit transformation of estimating the density of the kernel on the unit interval and he proposed a correct and simple method by combining the concept of transformation with estimating the local likelihood density, resulting in workable density estimations that are free of boundary issues in most cases. [8] investigated the probit transformation of the nonparametric kernel estimation of the copula density. He proposed a kernel type copula density based on the idea of transforming the margin of copula density to normal distributions using the probit function and estimating the density in the transformed domain without boundary bias problems. Thus, obtaining an estimation of the copula density via the back-transformation, and it was then demonstrated that when this method is combined with methods of estimating the local polynomial density. [9] presented a method for estimating the copula density using different kernel density methods, including mirror reflection method, beta kernel method and kernel transformation method, and then comparing the three methods using simulation experiments, the results showed that the transformation kernel estimator is the best among the three methods, and it is proved that the copulas are highly explicitly for high dependency, especially of the Gaussian

type.[10] presented a R package called Kdevine to estimate the density of the multivariate kernel with vine copulas.[11] studied reliability structural analysis methods with multidimensional correlation and when conducting a structural reliability analysis and calculating the probability of structural failure. The techniques that helped analyze structural reliability in light of the correlation problem, include the third-moment technique, the fourth-moment technique, and the D-Vine copula technique. These techniques were based on the first-order reliability method in its basic techniques when transforming the studied random variables into independent standard normal random variables, and iterative algorithms were used to find the probability point of most failures.

These studies were confined to nonparametric kernel functions using a fixed-value smoothing coefficient or a symmetric diagonal matrix.

In addition to many researchers have been studies wavelets. [12] studied the wavelet properties of the sunspot series. [13] employed variable kernel functions to estimate the risk for censored data. [14] used wavelets to estimate the return stock rate of the private banking sector. [15] studied multivariate fractional Brownian motion using discrete wavelets.

The purpose of this research is to estimate the copula density by nonparametric methods through probit transformation depending on the Kernel copula function for the purpose of correcting the boundary effects and wavelet transformation using multi-resolution analysis and comparison. Probit transformation is one of the methods used in boundary correction, and it is the most commonly used method, because this method suffers from biases at boundary points, we used a smoothing coefficient in the form of a full positive matrix.

2. Materials and Methods

2.1 Copula definition

A copula is a function that illustrate modeling the dependency of random variables. Sklar's created and initially utilized the copula [16].

This function has several advantages for modeling dependencies in multivariate data. first, consider the joint distribution's separation into the dependency structure (copula) and the basic marginal distributions.

And which can be viewed as a mathematical tool that is used to represent the relationship structure between two or more random variables. Many articles and studies have been written about nonparametric estimation of copulas. The use of nonparametric methods is more flexible than standard parametric methods, as no assumptions are required.

According to Sklar theorem 1959, every joint cumulative distribution function F of continuous random quantities (X, Y) can be written as $F(x, y) = C(F_X(x), F_Y(y))$, for all $(x, y) \in R^2$, where F_X and F_Y are continuous marginal distributions and C: $[0, 1]^2 \to [0,1]$ is a unique corresponding to this joint distribution. Therefore, the copula is the joint cumulative distribution function with uniformly distributed marginal distributions on [0, 1] [17][18].

Therefore, every multivariate CDFs with standard uniform marginal that show the dependence structure of random variables X and Y, and their marginal cumulative distribution functions are described by

$$U = F_X(X) \; and \; V = F_Y(Y) \qquad (1)$$

Where U and V are uniformly distributed variables and $(U, V) \in [0,1]$.The probability of two random variables, $X \leq x$ and $Y \leq y$, is described by the joint CDF $F_{XY}(X, Y) = P(X \leq x, Y \leq y)$.$C(u, v) = \Pr(U \leq u, V \leq v)$

Where $C(u, v)$ is called a copula and can be uniquely determined when u and v are continuous [19].

The following is the formula for a Gaussian copula: [20]

$$C_\theta^{Ga}(u_1,u_2) = \frac{1}{2\pi\sqrt{1-\theta^2}} \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \exp[-\frac{s^2-2\theta st+t^2}{2(1-\theta^2)}]dtds$$

Φ Represents the standard normal distribution function, while Φ^{-1} represents the inverse of standard normal distribution function.

A Frank copula is given by [21].

$$C(u_1,u_2) = \frac{1}{\theta} log\left(1+\frac{(e^{\theta u_1}-1)(e^{\theta u_2}-1)}{e^\theta - 1}\right), \theta \in (-\infty,+\infty)$$

Joe copula is provided by

$C_\alpha(u,v) = 1-[(1-u)^\alpha + (1-v)^\alpha - (1-u)^\alpha(1-v)^\alpha]^{\frac{1}{\alpha}}$ as well as its density

$c_\alpha(u,v) = [w^\alpha + z^\alpha - wz^\alpha]^{\frac{1}{\alpha}-2} wz^{\alpha-1}[\alpha - 1 + w^\alpha + z^\alpha - wz^\alpha]\ , \alpha \in [1,\infty)$

Where $w = 1-u\ and\ z = 1-v$. It is distinguished by upper tail dependency. moreover , $\lambda_U = 2-2^{\frac{1}{\alpha}}$ [22].

Tawn copula is

$$C = exp\left\{\left(log(u_1)+log(u_2)\right)A\left(\frac{log(u_2)}{log(u_1u_2)}\right)\right\}, where$$

$$A(x) = (1-\alpha_1)x + (1-\alpha_2)(1-x) + \left(\left(\alpha_1(1-x)\right)^\theta + (\alpha_2 x)^\theta\right)^{\frac{1}{\theta}}$$

$and\ (\theta,\alpha_1,\alpha_2) \in (1,\infty)\times[0,1]^2\ \ , for\ \alpha_1=\alpha_2=1$, we recover the Gumbel copula.

$At\ any\ time\ \alpha_1 \neq \alpha_2$ it will be asymmetric in its components.

2.2 Kernel and probit estimation:

There are numerous nonparametric methods for estimating the dependence structure between two random variables, such as polynomial approximation copulas and kernel smoothing copulas [8].

2.2.1 Kernel density function estimation:

The d-dimensional multivariate kernel density estimator in its general form is [23][24].

$$\hat{f}(x,H) = \frac{1}{n|H|^{1/2}}\sum_{i=1}^{n} K(H^{-1/2}(x-X_i)) \qquad (3)$$

$$= \frac{1}{n}\sum_{i=1}^{n} K_H(x-X_i).$$

$$K_H(x) = |H|^{-1/2}K\left(H^{-1/2}x\right) \qquad (4)$$

Where H is positive and symmetric definite bandwidth matrix and K is kernel function, and $|H| \to 0\ , n|H| \to \infty\ as\ n \to \infty$

There are several nonparametric techniques to estimate the dependence structure between two random variables, such as empirical [1]. polynomial approximation copula [17] and kernel smoothing copulas [25].

In the classical statistics texts, a kernel is a nonparametric method for estimating the probability density function (pdf) of a continuous random variable. Any probability density can be used for the kernel [26]

In this study we use kernel type copula estimators because this method is the most commonly used in nonparametric estimation of copulas, Although its flexible [7]. But is not appropriate for the unit squared copula densities, essentially because it is heavily influenced by boundary bias issues for estimation function. In addition, most common copulas permit unbounded densities, and kernel methods are not consistent in that case. Therefore, many researchers study and provide solutions to the boundary bias, including [27] [28] [8].

The standard kernel estimator for c, denoted by $\hat{c}^*$

$$\hat{c}^*(u,v) = \frac{1}{n|H_{UV}|^{1/2}} \sum_{i=1}^{n} K\left(H_{UV}^{-1/2}\begin{pmatrix} u - Ui \\ v - Vi \end{pmatrix}\right) \qquad (5)$$

where $(u,v) \in [0,1]$and k: $R^2 \to R$. H_{UV} *is bandwidth matrix*

The use of kernel techniques to estimate an unknown bivariate copula density we will see that the boundedness of a copula density's support necessitates the use of more advanced techniques than the one considered. U, V ~U[0, 1] are random variables with the joint distribution C and the corresponding density c: $[0, 1]^2 \to$R. We assume that the copula C has i.i.d variables $\{U_i = F_X(X_i), V_i = F_Y(Y_i)\ , i = 1, \ldots, n\}$, and our goal is to estimate the density c [7].

2.3 probit Transformation Estimation Method (PTE) :

Data transformations are commonplace, and widely used to enhance the application and performance of classical estimating methods, this procedure, deals almost skewed data, heavy tails, or bounded supsport.

Several studies have investigated the transformation density estimation technique in the context of kernel density estimation, and they have presented a number of transformation families and transformation selection criteria. These studies created parametric families of transformations that approximate normality in a range of non-normal distribution. Although our essential goal of simple density estimation does not necessitate normality, Transformations can serve a variety of purposes in statistical analysis [29].

To solve the problems that caused boundary bias by transforming the data so that its distribution is supported on the full R^2.In other words, this method can be correct the boundaries in a natural way, and this method is characterized by being able to deal with boundary copula densities [25].

The difficulty in the copula density estimation of (U, V) is primarily due to the constrained nature of its support $I = [0,1]^2$ .Now define

$$S = \Phi^{-1}(U) \quad \text{and} \quad T = \Phi^{-1}(V) \qquad (6)$$

Where Φ is the standard normal cumulative distribution function and Φ^{-1}its quantile function or the probit transformation [7]. Given that both U and V are uniform distribution [0,1], S and T have standard normal distributions, but this does not imply that the vector (S, T) is bivariate normal. If the joint CDF of (S,T) is the Gaussian ,then F_{ST} is the Gaussian copula because copulas are invariant for increasing transformations. [18] has unconstrained support R^2, and estimating its density f_{ST} cannot be affected by boundary problems. Furthermore, due to its normal margins, one can expect f_{ST} to be well-behaved and easy to estimate. Under mild assumptions, f_{ST} and its partial derivatives up to the second order are found to be bounded on R^2, in this case copula density is unbounded. If F_{ST} refer to copula C, and the variables (S,T) are standard normal distribution , then we can write Sklar's theorem as equation below :

$$F_{ST}(s,t) = C\big(\Phi(s), \Phi(t)\big) \qquad (7)$$

When differentiate F_{ST} with respect to s and t , we get the joint density of (s,t)

$$f_{ST}(s,t) = c\big(\Phi(s), \Phi(t)\big)\varphi(s)\varphi(t) \qquad (8)$$

Where φ is standard normal density Inverting this equation yields.

$$c(u,v) = \frac{f_{ST}(\Phi^{-1}(u), \Phi^{-1}(v))}{\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))} \qquad (9)$$

For any $(u,v) \in [0,1]^2$, therefore, any estimator $\hat{f}_{ST}\ on\ R^2$automatically generates a Copula density estimate on the interior of I .

$$\hat{c}^{(\tau)}(u,v) = \frac{\hat{f}_{ST}(\Phi^{-1}(u), \Phi^{-1}(v))}{\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))} \qquad (10)$$

Where the symbol (τ) is refer to the transformation idea. When appropriate, $\hat{c}^{(\tau)}$can alternatively be defined by continuity at the limits of I. This transformation-based estimator has a number of appealing qualities. Because $(\Phi^{-1}(u), \Phi^{-1}(v))$ is not defined for $(u,v) \notin I$ cannot allocate any probability outside I. Also, if f_{ST} is a true density function, in the sense that $f_{ST}\ (s,t) \geq 0\ for\ all\ (s,t)$and

$$\int\int_{R^2} \hat{f}_{ST}(s,t)dsdt = 1$$

Then, through transformation in variables $u = \Phi(s)$ and $v = \Phi(t)$,

$$\hat{c}^{(\tau)}(u,v) \geq 0\ \ forall\ \ u,v \in I \quad ; \int\int_I \hat{c}^{(\tau)}(u,v)dudv = 1$$

According the bivariate kernel density estimator, which we shall denote by $\hat{f}_{ST}$. when apply to the copula:

$$c(u,v) = \frac{f_{ST}\big(\Phi^{-1}(u), \Phi^{-1}(v)\big)}{\varphi\big(\Phi^{-1}(u)\big)\varphi\big(\ \Phi^{-1}(v)\big)} \qquad (11)$$

for all $u,v \in [0,1]^2$

The first basic idea we should use the standard kernel density estimator such as $\hat{f}_{ST}$.Specifically, we use the estimate as:

$$\hat{f}^{*}_{ST}(s,t) = \frac{1}{n|H_{ST}|^{1/2}} \sum_{i=1}^{n} K(H_{ST}^{-1/2} \begin{pmatrix} s - S_i \\ t - T_i \end{pmatrix}) \qquad (12)$$

Where K is a bivariate kernel function and $H_{ST}$is symmetric positive –definite matrix, and

$$\{S_i = \Phi^{-1}(U_i), T_i = \Phi^{-1}(V_i); \qquad i = 1, \ldots, n\} \qquad (13)$$

Is the transform domain sample but (U_i, V_i) not Available, and (S_i, T_i) as well. Instead, one must make use of

$$\Big\{\Big(\hat{S}_i = \Phi^{-1}\big(\hat{U}_i\big), \hat{T}_i = \Phi^{-1}\big(\hat{V}_i\big)\Big); i = 1, \ldots, n\Big\} \qquad (14)$$

That pseudo-transformed sample as a result, the feasible form $\hat{f}^{*}_{ST}(s,t)$ is

$$\hat{f}_{ST}(s,t) = \frac{1}{n|H_{ST}|^{1/2}} \sum_{i=1}^{n} k(H_{ST}^{-1/2} \begin{pmatrix} s - \hat{S}_i \\ t - \hat{T}_i \end{pmatrix}) \qquad (15)$$

Based on equation (11), this leads to a "probit transform kernel copula density estimator". [7][10]

$$\hat{c}^{(\tau)}(u,v) = \frac{1}{n|H_{ST}|^{1/2}\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))} \sum_{i=1}^{n} K\left(H_{ST}^{-1/2} \begin{pmatrix} \Phi^{-1}(u) - \Phi^{-1}(\hat{U}_i) \\ \Phi^{-1}(v) - \Phi^{-1}(\hat{V}_i) \end{pmatrix} \right) \qquad (16)$$

As a result, the asymptotic equation for the parameter of probit transformation is also obtained. The bias and variance of this method for copula density estimator are in the following form, respectively

$$Bias[\hat{c}^{\tau}(u,v)] = \frac{1}{2} m_2(K) \Big\{ h_{11} \Big[c_{uu}(u,v)\varphi^2\big(\Phi^{-1}(u)\big) - 3c_u(u,v)\varphi\big(\Phi^{-1}(u)\big)\Phi^{-1}(u) + c(u,v)\Big(\big(\Phi^{-1}(u)\big)^2 - 1\Big)\Big] + h_{22}\Big[c_{vv}(u,v)\Phi^{-1}(u)\Phi^{-1}(v) - 3c_v(u,v)\varphi\big(\Phi^{-1}(v)\big)\Phi^{-1}(v) + c(u,v)\Big(\big(\Phi^{-1}(v)\big)^2 - 1\Big)\Big] + 2h_{12}\big[c(u,v)\Phi^{-1}(u)\Phi^{-1}(v) - c_u(u,v)\varphi\big(\Phi^{-1}(u)\big)\Phi^{-1}(v) - c_v(u,v)\Phi^{-1}(u)\varphi\big(\Phi^{-1}(v)\big)\big]\Big\} + o\{tr(H)\} \qquad (17)$$

Where $m_2(K) = \int z^2 \, K(z)dz$

The variance is

$$var\left(\hat{f}(s,t)\right) = n^{-1}|H|^{-1/2} R(K) f(s,t) + o\left(n^{-1}|H|^{-1/2}\right) \qquad (18)$$

Where $R(K) = \int K(z)^2 \, dz$

Then the variance of probit transformation copula density as below

$$var\big(\hat{c}^{\tau}(u,v)\big) = \frac{R(K)}{n|H|^{1/2}} \times \frac{c(u,v)}{\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))} + o((n|H|)^{-1}) \qquad (19)$$

When we using standard normal distribution of kernel density and normal distribution for density function then $m_2(K) = 1 \; and \; R(k) = (4\pi)^{-d/2}$. Where d represents a number of variables

Observe that

$$H_{AMISE} = \left[\frac{4}{(d+2)n}\right]^{\frac{2}{d+4}} \hat{\Sigma}$$

3. Wavelet Copula Density Estimation (WCDE):

3.1Wavelets:

Wavelets are an extension of Fourier analysis in that both seek to express complex functions using the sum of simple ones. Wavelet theory, on the other hand, came considerably later than Fourier analysis. [30] [31]

Wavelets have accomplished impressive acceptance in earth sciences [32] [33]. Wavelets have been used successfully in a variety of application, including numerical analysis, engineering, signal and image processing, statistics, and geophysics. We will use mathematical creation of wavelets discrete type transformations, first, provide a detailed of the space $L^2(R)$ from the standpoint of the multi-resolution analysis.

Multi-resolution is a method for describing the building of spaces and providing an analytical explanation of the components and bases of these spaces. Let us first construct the square-integrable function, often known as the space of Lebesgue measurable functions, which is written as $L^2(R)$ and defined as [34]

$L^2(R) = \{f: R \to R; \int_{-\infty}^{\infty} |f(x)|^2 < \infty\}$ [35].

A wavelet is a mathematical function tool used to divide a given function into compounds of different frequencies and explore each configuration using the appropriate solution for each measurement. These tiny waves display information and data in time and frequency domains. The continuity of their signal is limited in two variables: Unlike the sine function, which extends between $(-\infty, \infty)$, the wavelet function is irregular and asymmetric. Wavelet is defined mathematically as a real value function on the real axis that fluctuates up and down consistently around zero. [36] [37] in other words it is defined as a signal of limited time length (continuity) with an average value of zero. The wavelet transform is based on the pressure of the wavelet to be processed with two functions: the first is the mother wavelet function Ψ(x) to obtain a set of coefficients characterized by the wavelet coefficients or detailed coefficients D(s,t), and the second is the scaling function ∅(x), also called the father's function, to obtain the approximate coefficients A(s,t) [38].

Then , we approximate the signal using wavelets and find a group of wavelet subgroups that are constructed from expansion or compression and shifting of the original wavelet and represent the signal or data to be analyzed. In other words, the process is the transformation of large-scale measurements into precise measurements by aggregating these data or signals. The main result of the transformation process is the mother wavelet function defined as: [39] [40]

$$\Psi_{a,b}(x) = \frac{1}{\sqrt{a}} \Psi\left(\frac{x-b}{a}\right) \quad a, b \in R, a \neq 0 \qquad (20)$$

Where a and b are dilation and translation parameters,

Ψ refer to mother wavelet

$\Psi_{a,b}$ refer to daughter wavelet

There are two types of wavelet transforms: continuous wavelet transforms and discrete wavelet transforms.

To approximate the probability density function, the probability density function is decomposed into a set of infinite functions (daughter wavelets) in the time domain on an orthonormal basis by a scaling function (father wavelet) and a wavelet function (mother wavelet) [39].

The approximation is defined as:

$$\varphi_{jk}(t) = 2^{j/2} \varphi(2^j t - k) \qquad (21)$$

and

$$\psi_{jk}(t) = 2^{j/2} \psi(2^j t - k) \qquad (22)$$

In this study, we use mother and father Daubechies wavelets [40].

3.2. ***Wavelet – Copula Estimation*:**

In this section, it will be referred to as "wavelet- copula", and the procedure can be easily performed in two steps:

The first step involves using wavelet analysis to decompose variables. The second step uses the decomposed input variables to estimate the copula density function. Since modeling dependence by copula is sensitive to the marginal model, a major innovation of the procedure is the combination of wavelet analysis with copula models.

Copula density estimates are constructed using wavelet analysis. This process is easy to implement using out-of-the-box wavelet tools and is based on algorithms that automatically deal with boundary effects. Pseudo-samples $(R_i/n, S_i/n)$, measured on arbitrary divisions of the unit square, are a more promising approach [41].

Wavelet-based estimation of copula density helps explain the underlying dependence structure. In general, the wavelet analysis of the second - order function is $h(x,y)$ a hat allows you to analyze this mapping infinitely simultaneously number of resolution levels j = 0, 1, ...

The decomposition at any level $j_0 \in N$ is given by

At every level $j_0 \in N$, the decomposition is given by

$$h(x,y) = h_{j0}\,(x,y) + D_{j0}\,h(x,y), x,y \in R \tag{23}$$

so that

$$h_{j0}\,(x,y) = \sum_{k \in z^2} \alpha_{j0k}\varphi_{j0k}(x,y) \tag{24}$$

is a trend (or approximation) and

$$D_{j0}h(x,y) = \sum_{j=j0}^{\infty}\left(\sum_{k \in z^2} \beta_{jk}^{(1)}\,\psi_{jk}^{(1)}\,(x,y) + \sum_{k \in z^2} \beta_{jk}^{(2)}\,\psi_{jk}^{(2)}\,(x,y) + \sum_{k \in z^2} \beta_{jk}^{(3)}\,\psi_{jk}^{(3)}\,(x,y)\right) \tag{25}$$

is a collection of three sorts of details: vertical edges, horizontal edges, and oblique (corner of the square). In this form, the coefficients α_{j_0k} and $\beta_{jk}^{(1)}, \beta_{jk}^{(2)}$ and $\beta_{jk}^{(3)}$ with $j \geq j_0$ are unique for each choice of $j_0 \in N$. For all j ∈ N and $k = (k_1, k_2) \in z^2$, the functions φ_{j0k} and $\psi_{jk}^{(1)}$, $\psi_{jk}^{(2)}$ and $\psi_{jk}^{(3)}$ are defined as follow:

$$\begin{aligned}
\varphi_{jk_1k_2}\,(x,y) &= \varphi_{jk_1}\,(x)\varphi_{jk_2}\,(y) \\
\psi_{jk_1k_2}^{(1)}\,(x,y) &= \varphi_{jk_1}\,(x)\psi_{jk_2}\,(y) \\
\psi_{jk_1k_2}^{(2)}\,(x,y) &= \psi_{jk_1}\,(x)\varphi_{jk_2}\,(y) \\
\psi_{jk_1k_2}^{(3)}(x,y) &= \psi_{jk_1}(x)\psi_{jk_2}\,(y)
\end{aligned} \tag{26}$$

in terms of a certain scaling function, a corresponding wavelet, and their location-scale transformations provided by .

$$\varphi_{jk_3}(t) = 2^{j/2}\varphi(2^j\,t - k_3) \tag{27}$$

$$\text{and} \quad \psi_{jk_3}\,(t) = 2^{j/2}\psi(2^jt - k_3) \tag{28}$$

for any $t \in R$, and $k_3 \in Z$. The functions φ and ψ (the father and mother wavelet functions respectively) are defined by Many technical limitations have to be achieved. To ensure that the family of position scales they create constitutes an orthonormal system of L^2, the set of square-integrable functions. The selection of each pair (φ, ψ) yields a separate multiresolution analysis with the required degree of regularity. In this study assumed to have compact support [0, L] as is the case for the widely utilized and provides an overview of this viewpoint. [40][41] A wavelet representation is distinguished by the fact that the trend at level j_0+1 Consistent with the trend at level j_0, highlighted by horizontal, vertical, and diagonal features corresponding to level j_0.in other words

$$h_{j_0+1} = h_{j_0} + \left(\sum_{k\in z^2} \beta^{(1)}_{j_0k}\, \psi^{(1)}_{j_0k} + \sum_{k\in z^2} \beta^{(2)}_{j_0k}\, \psi^{(2)}_{j_0k} + \sum_{k\in z^2} \beta^{(3)}_{j_0k}\, \psi^{(3)}_{j_0k} \right) \tag{29}$$

The actual copula C was detected with $h_5$by setting

$$H(x,y) = C\big(F\,(x), G(y)\big), \tag{30}$$

Assume that $(X_1,Y_1),\dots,(X_n,Y_n)$ is a random sample from the unknown distribution H. The empirical are represented by F_n and G_n distributions related to F and G

$$\left(\frac{R_i}{n},\frac{S_i}{n}\right) = \big(F_n(X_i), G_n(Y_i)\big), i = 1,\dots,n. \tag{31}$$

W

here R_i and S_i are the ranks of X_i and Y_i respectively .

Let φ and ψ be the corresponding wavelet for a given scaling function. Both functions are considered real-valued and compactly support [0, L] for some L > 0. For each j ∈ N, define $\varphi_{jk}, \psi^{(1)}_{jk}, \psi^{(2)}_{jk}, and\ \psi^{(3)}_{jk}$ as in (4) for each $k = (k_1,k_2) \in z^2$. The set $\{\varphi_{j_0k}, \psi^{(1)}_{jl}, \psi^{(2)}_{jl}, \psi^{(3)}_{jl} : j \geq j_0, k \in z^2, l \in Z^2\}$ is the orthonormal basis of $L(R^2)$ for any arbitrary$j_0 \in N$. Given a copula density c, it may be expanded as (3) with

$$\alpha_{j_0k} = \int_0^1\int_0^1 c(u,v)\,\varphi_{j_0k}(u,v)dvdu, k \in z^2 \tag{32}$$

According to Eq. (15), the change in variables $u = F\,(x)\ and\ v = G(y)$ yields

$$\alpha_{j_0k} = \int_0^1\int_0^1 \varphi_{j_0k}(F\,(x), G(y))\,h(x,y)dydx = E_h\{\varphi_{j_0k}\big(F\,(X), G(Y)\big)\}, \tag{33}$$

where E_h is the expectation based on the original observations' common distribution $(X_1,Y_1),\dots,(X_n,Y_n)$ If $F\ and\ G$ are unknown, a nonparametric is generated by substituting F and G with their empirical distribution function, $F_n\ and\ G_n$. The estimator is therefore rank-based, i.e.

$$\tilde{\alpha}_{j_0k} = \frac{1}{n}\sum_{i=1}^{n} \varphi_{j_0k}\big(F_n(X_i), G_n(Y_i)\big) = \frac{1}{n}\sum_{i=1}^{n} \varphi_{j_0k}\left(\frac{R_i}{n},\frac{S_i}{n}\right) \tag{34}$$

A wavelet-based estimate of c is then given by:

$$\tilde{c}_{j_0}(u,v) = \textstyle\sum_{k\in z^2} \tilde{\alpha}_{j_0k}\ \varphi_{j_0k}(u,v), \qquad u,v \in [0,1] \tag{35}$$

where the smoothing index of the technique is denoted by the number j_0. It is worth noting that, is not always the copula density, $\tilde{c}_{j_0}$ just as an empirical copula is not a copula. [1] $\tilde{c}_{j_0}$ In particular, it can be negative in the section of the domain, so that it cannot be merged into 1. When you want an estimate of the intrinsic copula density, it can be obtained by truncating and normalizing $\tilde{c}_{j_0}$

From a numerical standpoint, it is crucial to notice that the sum over k in (18) is finite since the wavelet is supported by compact support. Consequently, in reality. Only $[L^2(R)]$ c terms must be computed in the special situation when the copula density must be estimated at a single point $(u_0, v_0) \in [0,1]^2$. For these reasons, the procedure's performance is determined by the level j_0 selected. The latter should be determined in the most efficient method possible [39].

5. Discussion and results:

(1) We simulate five different random samples (n= 32,64,128,256, 512) with replication (r=1000).

(2) Generate X, Y variables from a uniform distribution.

(3) The marginal distributions of the random variables X and Y (F and G) are uniformly simulated.]

(4) Finding the probit transformation of the observations of the variables that were generated in step 2.

(5) Determine the number of vanishing moments at 4 degree.

(6) For the dependence structure, we consider four copula function (Gaussian , Frank ,Tawn, and Joe) ,with Kendall's tau τ = 0.7,0.5,0.3. as shown in Tables from 1 to 12.

Tables from 1 into 12 represent the estimated root mean squares error of the copula density functions for nonparametric estimation methods and Akaike criteria and logarithm maximum likelihood criteria (LogL) at a correlation level tau = 0.7,0.5,0.3 respectively with 1000 repetitions for each experiment that were used to determine the performance of the best estimation method it was found that the best estimation method for the copula density function in the case of strong and weak correlations and for all sample sizes and for four copulas(Gaussian, Frank, Tawn, and Joe).The method was probit transformation for all sample sizes and for all four copulas is the best at tawn and Joe copulas when tau is strong but at Frank copula at small sample size. While the method (WCDE) is the Best at Gaussian copula in all level correlation ant at all sample size.

In medium and weak correlation, it was the method (WCDE) at all Frank, Tawn and Joe copulas function.

The 3D plot of the real copula functions (Gaussian, Frank, Tawn, and Joe) are illustrated in Figures (1, 2, 3, and 4) below, in addition to the preface shapes for each of them using the (PTE, WCDE) methods. It can be noted, through 3D figures, that the distribution of the observations of the copula function estimated by the (WCDE) method was accurate at the edges while it was less accurate at the center for all functions. It is also evident from the three-dimensional figures that the probability density function of the real (Gaussian) copula function is characterized by the similar concentration of observations at the center and at the edges, with the withdrawal of observations towards the tail and its relatively little expansion at the center. Through the three-dimensional figure, the (WCDE) smoothing of the Gaussian function was more flat at the center and more congruent at the tails (extremities) when compared to the real probability density function. Besides, when estimating the copula function (Frank, tawn, and Joe), the smoothing of the probability density functions was less flat at the center, but it was more withdrawn towards the tails despite the presence of a great match between the smoothed and the real functions. Additionally, despite having observed that the smoothed and real functions had a significant match, the smoothing of the probability density functions while estimating the copula function (Frank, Tawn, and Joe) was less flat. In general, it can be said that the smoothing when estimating the copula function (Gaussian) is slightly better than the smoothing when estimating the copula functions (Frank,Tawn, Joe).

Table 1: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Gaussian copula when $\tau = 0.7$

Gaussian Sample size	Method	RMSE	AIC	LOGL
32	PTE	0.29933	-38.599	22.71667
	WCDE	**0.18843**	**-57.4329**	**29.8099**
64	PTE	0.23146	-97.1761	49.99241
	WCDE	**0.16175**	**-116.785**	**59.63232**
128	PTE	0.21907	-215.913	109.6572
	WCDE	**0.13695**	**-414.451**	**208.0832**
256	PTE	0.22168	-374.616	194.7401
	WCDE	**0.0767**	**-494.185**	**248.3973**
512	PTE	0.18511	-771.349	387.8594
	WCDE	**0.03925**	**-1314.73**	**658.6745**

Table 2: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Frank copula when $\tau = 0.7$

Frank Sample size	Method	RMSE	AIC	LOGL
32	PTE	**0.15321**	**-81.6971**	**41.67683**
	WCDE	0.16953	-73.1424	37.44939
64	PTE	**0.15021**	**-95.0123**	**48.9454**
	WCDE	0.16593	-122.257	62.16542
128	PTE	**0.14841**	**-280.421**	**141.4846**
	WCDE	0.22525	-260.514	131.355
256	PTE	0.14168	-458.297	230.3296
	WCDE	**0.05901**	**-485.975**	**244.9677**
512	PTE	0.14168	-485.297	244.3296
	WCDE	**0.05179**	**-1260.37**	**631.4375**

Table 3: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Tawn copula when $\tau = 0.7$

Tawn Sample size	Method	RMSE	AIC	LOGL
32	PTE	**0.15321**	**-81.6971**	**41.67683**
	WCDE	0.19771	-70.7703	40.02418
64	PTE	**0.15021**	**-95.0123**	**48.9454**
	WCDE	0.18102	-83.0877	42.97897
128	PTE	**0.14841**	**-280.421**	**141.4846**
	WCDE	0.17395	-240.125	127.4758
256	PTE	**0.14168**	**-485.297**	**244.3296**
	WCDE	0.17055	-414.966	215.1595
512	PTE	**0.14168**	**-1000.297**	**501.3296**
	WCDE	0.14552	-485.56	244.807

Table 4: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Joe copula when $\tau = 0.7$

Joe Sample size	Method	RMSE	AIC	LOGL
32	PTE	**0.17041**	**-86.6256**	**43.92094**
	WCDE	0.59031	-60.2817	30.79977
64	PTE	**0.15969**	**-117.411**	**59.98481**
	WCDE	0.42713	-63.8449	32.46232
128	PTE	**0.15494**	**-214.156**	**108.7846**
	WCDE	0.38647	-67.469	36.21485
256	PTE	**0.14883**	**-453.003**	**228.3562**
	WCDE	0.37677	-248.321	126.1468
512	PTE	**0.14246**	**-872.756**	**438.4459**
	WCDE	0.20222	-435.536	220.0326

Table 5: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Gaussian copula when $\tau = 0.5$

Gaussian Sample size	Method	RMSE	AIC	LOGL
32	PTE	0.64513	-14.0259	9.10046
	WCDE	**0.55886**	**-18.9267**	**11.26035**
64	PTE	0.51196	-50.1634	27.14061
	WCDE	**0.46729**	**-66.2573**	**34.80296**
128	PTE	0.49618	-102.509	53.73536
	WCDE	**0.45667**	**-113.008**	**58.77383**
256	PTE	0.44895	-180.244	95.50428
	WCDE	**0.35513**	**-216.31**	**111.0381**
512	PTE	0.42957	-342.784	174.5219
	WCDE	**0.25761**	**-358.82**	**182.4047**

Table 6: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Frank copula when $\tau = 0.5$

Frank Sample size	Method	RMSE	AIC	LOGL
32	PTE	**0.42059**	**-16.5343**	**10.4112**
	WCDE	0.48065	-25.2846	14.28811
64	PTE	**0.41134**	**-81.704**	**42.66972**
	WCDE	0.44904	-59.8056	31.32155
128	PTE	0.39342	-145.273	74.18602
	WCDE	**0.36845**	**-148.747**	**76.70922**
256	PTE	0.38824	-221.718	113.3838
	WCDE	**0.27853**	**-249.719**	**127.3506**
512	PTE	0.38815	-484.543	245.4771
	WCDE	**0.24437**	**-644.141**	**324.3424**

Table 7: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Tawn copula when $\tau = 0.5$

Tawn Sample size	Method	RMSE	AIC	LOGL
32	PTE	**0.42059**	**-16.5343**	**10.4112**
	WCDE	0.54244	-8.03816	6.34288
64	PTE	**0.41134**	**-59.704**	**31.66972**
	WCDE	0.49021	-45.6272	25.02905
128	PTE	**0.39342**	**-148.273**	**76.18602**
	WCDE	0.48712	-121.104	62.55991
256	PTE	0.38824	-195.718	100.3838
	WCDE	**0.18847**	**-221.834**	**113.3944**
512	PTE	0.38815	-484.543	245.4771
	WCDE	**0.15348**	**-986.863**	**494.9124**

Table 8: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Joe copula when $\tau = 0.5$

Joe Sample size	Method	RMSE	AIC	LOGL
32	PTE	**0.42297**	**-20.1695**	**11.94385**
	WCDE	0.53243	-19.6327	11.63855
64	PTE	**0.41975**	**-55.5584**	**29.87232**
	WCDE	0.48461	-45.4424	25.01778
128	PTE	**0.47773**	**-138.883**	**71.77869**
	WCDE	0.41914	-154.231	79.1561
256	PTE	0.45859	-217.66	111.6068
	WCDE	**0.37908**	**-250.979**	**127.8033**
512	PTE	0.42731	-399.153	202.6058
	WCDE	**0.22915**	**-445.204**	**225.2306**

Table 9: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Gaussian copula when $\tau = 0.3$

Gaussian Sample size	Method	RMSE	AIC	LOGL
32	PTE	0.90599	-8.0097	6.17241
	WCDE	**0.71163**	**-10.79708**	**7.51427**
64	PTE	0.80242	-25.8767	15.42052
	WCDE	**0.55352**	**-48.6918**	**26.3745**
128	PTE	0.73139	-27.7824	17.43187
	WCDE	**0.42452**	**-109.671**	**56.80845**
256	PTE	0.71448	-86.0686	46.49385
	WCDE	**0.41359**	**-271.17**	**137.6521**
512	PTE	0.66743	-112.706	60.65348
	WCDE	**0.34671**	**-694.429**	**349.3544**

Table 10: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Frank copula when $\tau = 0.3$

Frank Sample size	Method	RMSE	AIC	LOGL
32	PTE	0.90599	-10.0097	7.17241
	WCDE	**0.7231**	**-10.2541**	**7.32794**
64	PTE	0.80242	-25.8767	15.42052
	WCDE	**0.66538**	**-26.0737**	**15.78511**

128	PTE	0.73139	-27.7824	17.43187
	WCDE	**0.59807**	**-65.7989**	**35.52795**
256	PTE	0.71448	-86.0686	46.49385
	WCDE	**0.44442**	**-253.352**	**128.7044**
512	PTE	0.66743	-112.706	60.65348
	WCDE	**0.3717**	**-415.229**	**209.8027**

Table 11: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Tawn copula when $\tau = 0.3$

Tawn Sample size	Method	RMSE	AIC	LOGL
32	PTE	0.7151	-9.30299	6.94363
	WCDE	**0.66441**	**-11.8305**	**8.07624**
64	PTE	0.71314	-21.1549	13.43301
	WCDE	**0.64268**	**-21.6373**	**13.47802**
128	PTE	0.6932	-56.5897	31.11079
	WCDE	**0.60952**	**-63.3696**	**33.94843**
256	PTE	0.69094	-104.666	55.57158
	WCDE	**0.46535**	**-223.006**	**113.7815**
512	PTE	0.67853	-171.691	90.04039
	WCDE	**0.15607**	**-854.443**	**428.785**

Table 12: Root-mean square error,(AIC)criterion and logarithm likelihood criteria for Joe copula when $\tau = 0.3$

Joe Sample size	Method	RMSE	AIC	LOGL
32	PTE	0.71841	-5.35054	5.12134
	WCDE	**0.62913**	**-12.0969**	**8.09585**
64	PTE	0.68521	-21.1653	13.1342
	WCDE	**0.49163**	**-42.0206**	**23.06409**
128	PTE	0.6379	-42.2662	24.05792
	WCDE	**0.41235**	**-113.394**	**58.62762**
256	PTE	0.72761	-115.09	60.9177
	WCDE	**0.39105**	**-409.191**	**206.0382**
512	PTE	0.71878	-232.741	119.9393
	WCDE	**0.39068**	**-334.387**	**169.9307**

The figures 1,2,3 and 4 are explain the behavior for all four copula

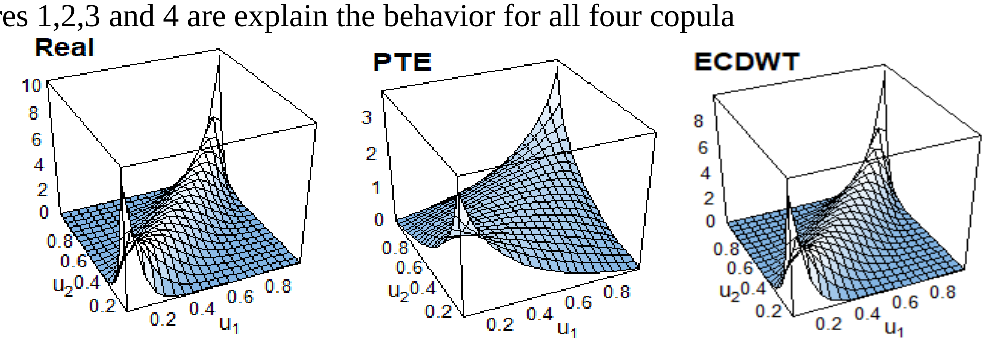

Figure (1) three dimension Gaussian copula density when (n=128, tau=0.7)

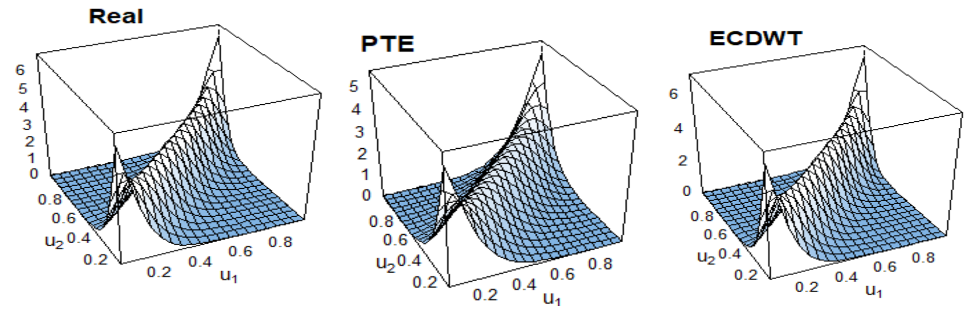

Figure (2) three dimension for Frank copula density when (n=128, tau=0.7)

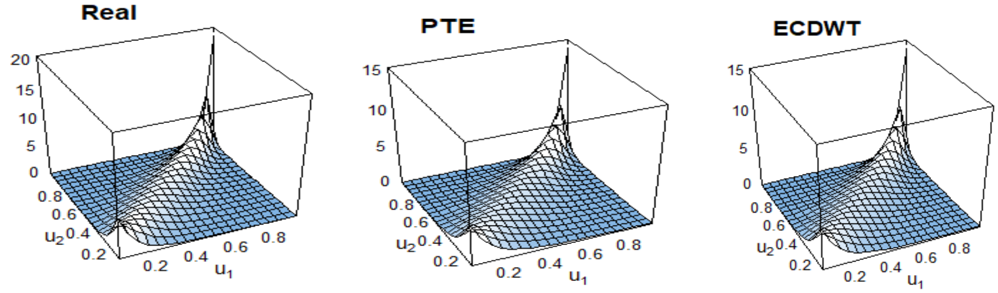

Figure (3) three dimension for Tawn copula density when (n=128, tau=0.7)

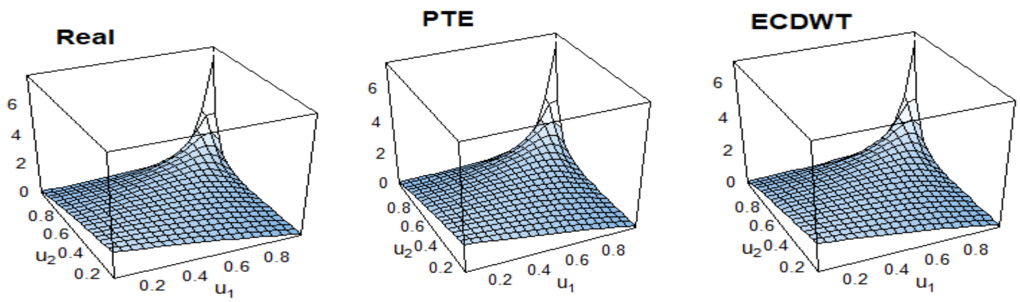

Figure (4) three dimension for Joe copula density when (n=128, tau=0.7)

A copula functions were also drawn for the data that were generated at several levels of correlation. There are many drawing methods to describe, interpret and analyze the nature of the associative functions, but the circular form, which is based on a normal distribution, and the three-dimensional form were chosen because they are considered one of the most common and used shapes in this field. The normal of the probability density functions of the assumed copula at the correlation level (0.7) can be more clearly understood through three-dimensional drawings.

Figure (1) above represents the assumed and estimated three-dimensional shapes of the Gaussian function when tau = 0.7 and n = 128. It is clear from it that the Gaussian function is characterized by similar dependency at the center and at the edges, and that the observations of the probability density function estimated by the (WCDE) method are characterized by flatness. Clearly at the center, but at the edges, the smoothing was identical to the assumed copula function.

Figure (2) above represents the estimates of the Frank function when (n = 128 and tau=0.7) and it shows that the Frank function is characterized by similar dependency at the center and at the edges, noting that the difference in the distribution of observations between the Gaussian and Frank functions It is that the observations at the center in the Frank function are less flat than in the Gaussian function

As for smoothing using the WCDE method, we notice that the distribution of observations fluctuates at the edges, but it is better at the center.

Figure (3)above represents the assumed and estimated probability density function of the copula (Tawn) at the high level of correlation and the sample size (128), and it is clear from it that the copula function (Tawn) is characterized by a large concentration of observations at the right side. the (WCDE) method are characterized by flatness. Clearly at the center, but at the edges, the smoothing was identical to the assumed copula function.

Figure (4) above represents the probability density function for the Joe association when (tau=0.7 and n=128), and it is clear from it (that the assumed Joe copula function has a right tail and that the concentration of observations was clearly on the left side, while the distribution of observations in the middle appears flat) Estimation using the PTE method: It is clear that there is instability in the flatness of the observations at the center, and that the flatness of the observations at the right tail and the left edge was more identical. As for the (WCDE) method, the performance was not good at the center, which was characterized by instability because the observations were too flat, or at the right tail, where the concentration of observations was greater, but the concentration of observations at the left end was more similar to the assumed form of the association.

6.Conclusion:

This study introduced copula estimation using probit and wavelet transforms, specifically employing Daubechies wavelets of four degrees. The simulation results, were obtained by employing four copulas (Gaussian, Frank, Tawn, and Joe), for five different sample sizes (n = 32, 64, 128, 256, 512) and evaluated based on three criteria (RMSE, AIC, and LOGL), provide a statistical measure for selecting the copula that exhibits the best performance when wavelets are used to estimate the copula density function at high, medium, and low correlation levels (tau = 0.7, 0.5, 0.3).

1- For all the copula functions that have been studied for all nonparametric estimation methods referred to in the theoretical part and for all sample sizes and at correlation levels, the value of the square root of the mean square error (RMSE) decreases as the sample size increases, while the (LogL) criterion is as maximum as possible, As for the Akaike criteria as minimum as possible.

2-Estimation and identification of Copula density functions based on rank-dependent wavelets.

3- Presented the root mean square error and developed a linear wavelet estimator.

4-Wavelet algorithms are quick to compute, and simple to update and adapt to your model.

5-The suggested linear wavelet density estimator's numerical performance was shown on simulated datasets.

6-Comparisons of generated data for various sample sizes were also explained. However, wavelet-based copula function estimators for the underlying dependency structure, do not cover the basic criteria of parametric models.

7- The method of estimating the copula density function using (PTE) is the best method for the Frank and Joe copulas function.

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