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Quaestiones Mathematicae

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/tqma20

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Quaestiones Mathematicae 2024: 1–23.

https://doi.org/10.2989/16073606.2024.2352873

This is the final version of the article that is published ahead of the print and online issue

THIRD-ORDER HANKEL DETERMINANTS FOR q-ANALOGUE ANALYTIC FUNCTIONS DEFINED BY A MODIFIED q-BERNARDI INTEGRAL OPERATOR

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Abstract. In this paper we define a Bernardi type quantum integral operator. It transforms the starlike univalent in the unit disk into a starlike region in it. We show that the upper-bound of the third-order Hankel determinant for classes of q-starlike functions is connected with a q-analogue integral operator, defined by a modified q-Bernardi integral operator. The Fekete-Szegö inequality of these classes is also investigated. Numerous well-known specific instances, examples and graphics are listed in the paper. The computations are done by Mathematica 13.3.

Mathematics Subject Classification (2020): 05A30, 30C45, 30C50.

Key words: Analytic function, q-starlike functions, quantum calculus, Hankel determinants, Fekete-Szegö type inequality.

1. Introduction. Quantum calculus, often known as q-calculus, is a method for studying calculus that is similar to traditional calculus but is focused on finding q-analogous conclusions without the need for limits. The q-derivative is the primary tool. In 1908, Jackson ([20, 21]) introduced and developed the notions

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of q-derivative and q-integral. Lately, many authors have focused on the field of q-calculus. This interest is due to the importance of its applications in various mathematics and quantum physics fields. Also, the geometries for q-analysis have been found in many studies presented on quantum groups. It has also been identified that there is a relationship between q-integral and q-derivative. In addition, certain studies of the fractional q-calculus operators have been investigated by many researchers (for example, see [9, 14, 11, 12, 13, 25, 38, 39, 44]).

The structure of quantum calculus (q-calculus) organizes various classes of operators and particular transformations while developing an intriguing method for calculations. Numerous applications, including physical issues, demonstrated the importance of q-calculus. In the context of geometric function theory, Ismail et al. [19] proposed q-calculus. This work has led to the development of several Ma and Minda classes of analytic functions on the open unit disk, which is associated with the subordination concept. For instance, Seoudy and Aouf's effort [37] to define quantum star-like function sub-classes made use of the idea of q-derivatives. Using a special curve, Zainab et al. [46] newly proposed a sufficient criterion for q-starlikeness. This work is generalized by special functions in [1, 15, 18, 4].

In the study of geometric functions, there are different operators that are used to enhance the class of analytic functions. These operators can be realized as functional expressions, including the differentiation, integration and convolution operators. Most of these operators are generalized by using fractional calculus, quantum calculus of one-dimensional and two-dimensional fractional power parameters, and K-symbol calculus depending on the applications. In this effort, we continue to investigate on the quantum inequalities. For a class of q-starlike functions combined with a q-analogue integral operator, defined by a modified q-Bernardi integral operator (q-BIO), we examine and explore the upper bound of the third-order Hankel determinants (TOHD) in this paper. The study's discussion of several well-known unique examples serves as the inspiration for our work.

2. Methods.

This section deals with the following concepts.

2.1. Geometric concepts.

Suppose that $\mathcal{A}$ is the class of analytic and univalent functions $f(z)$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \tag{1}$$

Two functions in $\mathcal{A}$ are convoluted if they satisfy the following product:

$$(f * \mathcal{F})(z) = z + \sum_{j=2}^{\infty} a_j d_j z^j = (\mathcal{F} * f)(z), \tag{2}$$

where $\mathcal{F}(z) = z + \sum_{j=2}^{\infty} d_j z^j$. Moreover, they are subordinated ($\prec$) if they satisfy the following inequality:

$$f(z) \prec \mathcal{F}(z) \Longleftrightarrow f(0) = \mathcal{F}(0) \text{ and } f(\mathbb{U}) \subset \mathcal{F}(\mathbb{U}), \ (z \in \mathbb{U}).$$

2.2. Quantum calculus.

DEFINITION 2.1. ([20]) For $0 < q < 1$, the q-derivative operator is formulated by

$$\mathfrak{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0).$$

From above, for the function $f(z)$ in (1), we obtain

$$\mathfrak{D}_q \left\{ z + \sum_{j=2}^{\infty} a_j z^j \right\} = 1 + \sum_{j=2}^{\infty} [j]_q a_j z^{j-1},$$

where the q-number $[j]_q$ is expressed by

$$[j]_q := \begin{cases} \dfrac{1-q^j}{1-q} & (j \in \mathbb{C}) \\ \sum\limits_{j=0}^{n-1} q^j & (j = n \in \mathbb{N}) \\ 0 & (j = 0), \end{cases}$$

and $[j]_q!$ denotes the q-factorial, which is defined as follows:

$$[j]_q! = \begin{cases} [j]_q[j-1]_q \dots [2]_q[1]_q, & j = 1, 2, 3, \dots \\ 1, & j = 0. \end{cases}$$

Then

$$f'(z) = lim_{q \to 1-} \mathfrak{D}_q \left\{ z + \sum_{j=2}^{\infty} [j]_q a_j z^{j-1} \right\} = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}.$$

DEFINITION 2.2. ([20]) Let $\upsilon \in \mathcal{R}$ and $j \in \mathbb{N}$ be positive integers. The symbol for the q-generalized Pochhammer is given by

$$[\upsilon; j]_q = [\upsilon]_q[\upsilon+1]_q[\upsilon+2]_{q \dots} \dots [\upsilon+j-1]_q.$$

Also, the q-gamma function is defined for $\varrho > 0$,

$$\Gamma_q(\upsilon+1) = [\upsilon]_q \Gamma_q(\upsilon) \text{ and } \Gamma_q(1) = 1.$$

Jackson [21] provided the q-integral of a function $f(z)$ as below:

$$\int_0^u f(\nu) d_q \nu = (1-q) u \sum_{j=0}^{\infty} q^j f\left(u q^j\right).$$

DEFINITION 2.3. ([19]) The function $f(z)$ in (1) belongs to the class $\mathcal{S}^*(q)$ if it satisfies the inequality

$$|\frac{z}{f(z)}\left(\mathfrak{D}_q f\right)(z)-\frac{1}{1-q}|\leq\frac{1}{1-q},\quad(z\in\mathbb{U}).\tag{3}$$

When $q\to 1-$, we observe that

$$|\frac{zf'(z)}{f(z)}-\frac{1}{1-q}|\leq\frac{1}{1-q}.$$

Now, the class $\mathcal{S}^*(q)$ reduced to $\mathcal{S}^*$. Alternatively, one can consider the principle of analytic function subordination to obtain

$$\frac{z}{f(z)}\left(\mathfrak{D}_q f\right)(z)\prec\Omega(z),$$

where $\Omega(z)=\frac{1+z}{1-qz}$.

DEFINITION 2.4. For $0<\eta\leq 1$ and $0<\lambda\leq 1$, we define the following subclass of uniformly starlike $S_q^*(\eta,\lambda)$ of order λ of analytic functions

$$|\frac{z\mathfrak{D}_q f(z)}{f(z)}-1|<\lambda|\frac{\eta z\mathfrak{D}_q f(z)}{f(z)}+1|.$$

Equivalently,

$$\frac{z\mathfrak{D}_q f(z)}{f(z)}\prec\Omega(z),$$

where $\Omega(z)=\frac{1+\lambda z}{1-\eta\lambda z}$.

By taking $q\to 1-$, we obtain the class $S^*(\eta,\lambda)$ investigated by Liu et al. [30].

The classical integral operator is generalized in complex analysis as the q-analogue integral operator. It is superior to the classical operator in a number of ways. There are uses for the q-analog integral operator in physics, especially in quantum field theory and statistical mechanics studies. Furthermore, we can propose general families of first-order differential superordination by employing the q-Bernardi integral operator that superordination-preserving angle, following the methods previously described in the works examined by Bulboacă [6]. Based on these discoveries and motivated by earlier outcomes achieved through the use of q-calculus, our study aims to introduce a new q-analogue integral operator $\mathcal{K}^{\mu}_{q,\tau,u}f(z)$ related to the q-derivative of the q-Bernardi integral operator (q-BIO). Then, we employ the operator $\mathcal{K}^{\mu}_{q,\tau,u}f(z)$ to establish the classes $\mathcal{KS}^*(q;\tau,\mu)$ and $\mathcal{US}^*(q;\eta,\lambda)$. Moreover, we investigate the Hankel determinants of second and third orders for these classes. Additionally, we determine the upper-bound $|a_3-\gamma a_2^2|$ of Fekete-Szegö type inequality.

The modified q-Bernardi integral operator for univalent functions is defined by Definition 2.5 below:

DEFINITION 2.5. For $f(z) \in \mathcal{A}$, the modified q-Bernardi integral operator $\mathcal{J}_{\nu,q}^{k} f(z) : \mathcal{A} \to \mathcal{A}$ for univalent functions is defined by

$$\mathcal{J}_{\nu,q}^{k} f(z) := \begin{cases} \mathcal{J}_{\nu,q}^{1}\big(\mathcal{J}_{\nu,q}^{k-1} f(z)\big) & (k \in \mathbb{N}) \\ \\ f(z) & (k = 0), \end{cases} \tag{4}$$

where $\mathcal{J}_{\nu,q}^{1}$ is given by (see, [36])

$$\begin{aligned} \mathcal{J}_{\nu,q}^{1} f(z) =& \frac{[1+\nu]_q}{z^{\nu}} \int_0^z t^{\nu-1} f(t) d_q t \\ =& \, z + \sum_{j=2}^{\infty} \frac{[1+\nu]_q}{[j+\nu]_q} a_j z^j, \ (\nu \in \mathbb{N}, z \in \mathbb{U}). \end{aligned} \tag{5}$$

For $\mathcal{J}_{\nu,q}^{1} f(z)$, we consider

$$\mathcal{J}_{\nu,q}^{2} f(z) = \mathcal{J}_{\nu,q}^{1}(\mathcal{J}_{\nu,q}^{1} f(z)) = z + \sum_{j=2}^{\infty} \Big(\frac{[1+\nu]_q}{[j+\nu]_q}\Big)^2 a_j z^j, \quad (\nu \in \mathbb{N}, z \in \mathbb{U}) \tag{6}$$

and

$$\mathcal{J}_{\nu,q}^{k} f(z) = z + \sum_{j=2}^{\infty} \Big(\frac{[1+\nu]_q}{[j+\nu]_q}\Big)^k a_j z^j, \ (k \in \mathbb{N}_0, \nu \in \mathbb{N}, z \in \mathbb{U}). \tag{7}$$

Now, we define the q-derivative of the generalized q-BIO $\mathcal{J}_{\nu,q}^{k} f(z)$, as follows:

$$\mathfrak{D}_q \mathcal{J}_{\nu,q}^{k} f(z) := \frac{\mathcal{J}_{\nu,q}^{k} f(qz) - \mathcal{J}_{\nu,q}^{k} f(z)}{(q-1)z}, \ (z \in \mathbb{U}).$$

Hence, we obtain

$$\mathfrak{D}_q \mathcal{J}_{\nu,q}^{k} f(z) = 1 + \sum_{j=2}^{\infty} \Big(\frac{[1+\nu]_q}{[j+\nu]_q}\Big)^k [j]_q \ a_j z^{j-1}. \tag{8}$$

Consequently, we have

$$z \mathfrak{D}_q \mathcal{J}_{\nu,q}^{k} f(z) = z + \sum_{j=2}^{\infty} \Big(\frac{[1+\nu]_q}{[j+\nu]_q}\Big)^k [j]_q \ a_j z^{j}.$$

For $\mu > -1$, we introduce a q-analogue integral operator $\mathcal{K}_{q,\nu,k}^{\mu} f(z) : \mathcal{A} \to \mathcal{A}$ as follows:

$$\mathcal{K}_{q,\nu,k}^{\mu} f(z) * \mathcal{F}_q^{\mu+1}(z) = z \mathfrak{D}_q \mathcal{J}_{\nu,q}^{k} f(z),$$

where

$$\mathcal{F}_q^{\mu+1}(z) = z + \sum_{j=2}^{\infty} \frac{[\mu+1; j]_q}{[j-1]_q!} z^j, \ (z \in \mathbb{U}).$$

From the above operator, we conclude that

$$\begin{aligned}\mathcal{K}^{\mu}_{q,\nu,k}f(z) &= z + \sum_{j=2}^{\infty}\Big(\frac{[1+\nu]_q}{[j+\nu]_q}\Big)^k \frac{[j]_q[j-1]_q!}{[\mu+1;j]_q} a_j z^j \\ &= z + \sum_{j=2}^{\infty}\chi_j a_j z^j,\end{aligned} \tag{9}$$

where

$$\chi_j = \Big(\frac{[1+\nu]_q}{[j+\nu]_q}\Big)^k \frac{[j]_q!}{[\mu+1;j]_q}. \tag{10}$$

$$(k \in \mathbb{N}_0, \nu \in \mathbb{N}, \mu > -1, \text{ and } z \in \mathbb{U}).$$

We note from (9) that

$$q^{\mu} z\mathfrak{D}_q \mathcal{K}^{\mu+1}_{q,\nu,k}f(z) = [\mu+1,q]\mathcal{K}^{\mu}_{q,\nu,k}f(z) - [\mu,q]\mathcal{K}^{\mu+1}_{q,\nu,k}f(z). \tag{11}$$

Next, we present the next classes $\mathcal{KS}^*(q;k,\mu)$ and $\mathcal{US}^*(q;\eta,\lambda)$ of q-starlike functions using the q-analogue integral operator $\mathcal{K}^{\mu}_{q,\nu,k}f(z)$.

Definition 2.6. We call $f(z)$ in $\mathcal{KS}^*(q;k,\mu)$, if and only if

$$\Big|\frac{z\mathfrak{D}_q(\mathcal{K}^{\mu}_{q,\nu,k}f(z))}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)} - \frac{1}{1-q}\Big| \leq \frac{1}{1-q},$$

or equivalently

$$\frac{z\mathfrak{D}_q(\mathcal{K}^{\mu}_{q,\nu,k}f(z))}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)} \prec \Omega(z), \qquad (\Omega(z) = \frac{1+z}{1-qz}). \tag{12}$$

$$(k \in \mathbb{N}_0, \nu \in \mathbb{N}, \mu > -1, \text{ and } z \in \mathbb{U}).$$

Definition 2.7. For $0 < \eta \leq 1$ and $0 < \lambda \leq 1$, we call $f(z)$ in $\mathcal{US}^*(q;\eta,\lambda)$, if and only if

$$\frac{z\mathfrak{D}_q(\mathcal{K}^{\mu}_{q,\nu,k}f(z))}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)} \prec \Omega(z), \qquad (\Omega(z) = \frac{1+\lambda z}{1-\eta\lambda z}). \tag{13}$$

This subordination class is equivalent to

$$\Big|\frac{z\mathfrak{D}_q(\mathcal{K}^{\mu}_{q,\nu,k}f(z))}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)} - 1\Big| < \lambda\Big|\frac{\eta z\mathfrak{D}_q(\mathcal{K}^{\mu}_{q,\nu,k}f(z))}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)} + 1\Big|,$$

where $k \in \mathbb{N}_0, \nu \in \mathbb{N}, \mu > -1,$ and $z \in \mathbb{U}$.

Remark 1. We note from (12) and (13) that

(1) If $k = 0$ and $\mu = 1$, then the class $\mathcal{KS}^*(q;k,\mu)$ would reduce the class $\mathcal{S}^*(q)$ defined by Ismail et al. [19].

(2) If $q \to 1-, k = 0$, and $\mu = 1$, then the class $\mathcal{US}^*(q; \eta, \lambda)$ would reduce the class $\mathcal{US}^*(\eta, \lambda)$ defined by Liu et al. [30].

The l^{th} Hankel determinant for integers $j, l \in \mathbb{N}$ was studied and investigated by Noonan and Thomas [33] in 1976, which is given by

$$\mathcal{H}_l(j) = \begin{bmatrix} a_j & a_{j+1} & \cdots & a_{j+l-1} \\ a_{j+1} & a_{j+2} & \cdots & a_{j+l} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j+l-1} & a_{j+l} & \cdots & a_{j+2(l-1)} \end{bmatrix}.$$

We find that

$$\mathcal{H}_2(1) = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = a_1 a_3 - a_2^2 = a_3 - a_2^2, \ (a_1 = 1),$$

$$\mathcal{H}_2(2) = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix} = a_2 a_4 - a_3^2$$

and the TOHD is given as:

$$\mathcal{H}_3(1) = a_3\left(a_2 a_4 - a_3^2\right) - a_4\left(a_4 - a_2 a_3\right) + a_5\left(a_3 - a_2^2\right).$$

Researching the sharp boundaries of Hankel determinants for a certain class of complex valued functions has attracted the attention of several experts in the area; for example, the function $f(z)$ given by (1), the average of growth of $\mathcal{H}_l(j)$ *as* $j \to 0$, was determined by Noor [34]. In particular, the second order (SOHD) introduced by many authors (see, [3, 23, 17, 7, 28, 40]). Janteng et al. [22] studied the functional $|a_2 a_4 - a_3^2|$ and discovered a sharp bound for functions in $\mathcal{S}$ and $\mathcal{C}$. It is known that the Fekete-Szegö inequality is a special case of the Hankel determinant when $\mathcal{H}_2(1) = |a_3 - a_2^2|$. For bi-univalent functions engaging the symmetric q-derivative operator, Srivastava et al. [43] recently found the estimate of the SOHD (similarly in [42]). In 2010, Babalola [5] released the effort on $\mathcal{H}_3(1)$, in which the upper bound on $\mathcal{H}_3(1)$ for $\mathcal{S}^*, \mathcal{C}$ and $\mathcal{K}$ is calculated. Subsequently, numerous studies presented this determinant (see, [32, 27, 35, 26]).

3. Main lemmas. Assume that $\omega(z) = 1 + \sum_{j=1}^{\infty} \omega_j z^j$, is an analytic function in $\mathbb{U}$ such that $\mathcal{R}(\omega(z)) > 0$. The class of function $\omega(z)$ is denoted by $\mathcal{P}$. To obtain our main results, we need the following lemmas:

LEMMA 3.1. ([8]) *If the function* $\omega(z) \in \mathcal{P}$, *then*

$$|\omega_j| \le 2, \quad (j \ge 2).$$

LEMMA 3.2. ([29]) *If the function* $\omega(z) \in \mathcal{P}$, *then*

$$2\omega_2 = \omega_1^2 + \xi\left(4 - \omega_1^2\right)$$

and

$$4\omega_3 = \omega_1^3 + 2\left(4 - \omega_1^2\right)\omega_1 \xi - \left(4 - \omega_1^2\right)\omega_1 \xi^2 + 2\left(4 - \omega_1^2\right)\left(1 - |\xi|^2\right) z$$

with $|\xi| \le 1$ *and* $|z| \le 1$, *for some* ξ *and* z.

LEMMA 3.3. ([31]) *If $\omega(z) \in \mathcal{P}$ then*

$$|\omega_2 - \varkappa\omega_1^2| \leq \begin{cases} -4\varkappa + 2, & if \ \varkappa \leq 0, \\ 2, & \text{if } \ 0 \leq \varkappa \leq 1, \\ 4\varkappa - 2 & \text{if } \ \varkappa \geq 1. \end{cases} \tag{14}$$

(1) *When $\varkappa < 0$ or $\varkappa > 1$, the equality in (14) is true if and only if $\omega(z) = \frac{1+z}{1-z}$ or one of its rotations.*

(2) *When $0 < \varkappa < 1$, the equality in (14) is true if and only if $\omega(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations.*

(3) *When $\varkappa = 0$, the equality in (14) is true if and only if*

$$\omega(z) = \left(\frac{1}{2} + \frac{\tau}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\tau}{2}\right)\frac{1-z}{1+z}, \quad \text{with } \ 0 \leq \tau \leq 1$$

or one of its rotations.

(4) *When $\varkappa = 1$, the equality in (14) is true if and only if*

$$\frac{1}{\omega(z)} = \left(\frac{1}{2} + \frac{\tau}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\tau}{2}\right)\frac{1-z}{1+z}, \ \text{with } \ 0 \leq \tau \leq 1.$$

Although the above upper bound in (2) is sharp, as mentioned in [31], the interval can be partitioned into two regions $0 < \varkappa \leq \frac{1}{2}$ and $\frac{1}{2} \leq \varkappa < 1$, as follows

$$|\omega_2 - \varkappa\omega_1^2| + \varkappa|\omega_1|^2 \leq 2, \ \text{if } 0 < \varkappa \leq \frac{1}{2}$$

and

$$|\omega_2 - \varkappa\omega_1^2| + (1-\varkappa)\,|\omega_1|^2 \leq 2, \ \text{if } \ \frac{1}{2} \leq \varkappa < 1.$$

In the following section, we investigate initial coefficient estimates $|a_j|$ $(j = 2, 3, 4)$, Hankel determinants and Fekete-Szegö type inequality.

4. Main results. To determine the upper-bound of the TOHD $\mathcal{H}_3(1)$, we must begin by solving the second-order Hankel determinant problem $\mathcal{H}_2(1)$ of first kind for the classes $\mathcal{KS}^*(q; k, \mu)$ and $\mathcal{US}^*(q; \eta, \lambda)$.

THEOREM 4.1. *If $f(z) \in \mathcal{KS}^*(q; k, \mu)$, where $f(z)$ in (1) then for some $k \in \mathbb{N}_0, \mu > 0, \nu \in \mathbb{N}_0$ (or $k \in \mathbb{N}, \mu > -1, \nu \in \mathbb{N}_0$) and $q \in (0,1)$, the following coefficients inequality is valid*

$$\mathcal{H}_2(1) = |a_3 - a_2^2| \leq \frac{1}{q\chi_3},$$

where χ_3 have been included in (10) with $j = 3$.

Proof. In view of the subordination condition (12), we get

$$\frac{z\mathfrak{D}_q\left(\mathcal{K}^{\mu}_{q,\nu,k}f(z)\right)}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)}=\Omega\left(\upsilon\left(z\right)\right). \tag{15}$$

We proceed to illustrate the function $\omega(z)$, as follows:

$$\omega\left(z\right)=\frac{1+\upsilon\left(z\right)}{1-\upsilon\left(z\right)}=1+\omega_1z+\omega_2z^2+\omega_3z^3+\omega_4z^4+\dots.$$

Clearly, $\omega\in\mathcal{P}$, then we obtain

$$\upsilon\left(z\right)=\frac{\omega\left(z\right)-1}{\omega\left(z\right)+1}$$

and

$$\Omega\left(\upsilon\left(z\right)\right)=\frac{2\omega(z)}{1+\left(1-q\right)\omega\left(z\right)+q}.$$

A computation yields

$$\frac{2\omega\left(z\right)}{1+\left(1-q\right)\omega\left(z\right)+q}=1+\frac{(1+q)\omega_1}{2}z+\left\{\frac{(1+q)\omega_2}{2}-\frac{\left(1-q^2\right)\omega_1^2}{4}\right\}z^2$$
$$+\left\{\frac{(1+q)\omega_3}{2}-\frac{\left(1-q^2\right)\omega_1\omega_2}{2}+\frac{(1+q)(1-q)^2\omega_1^3}{8}\right\}z^3$$
$$+\left\{\frac{\left(1+q\right)\omega_4}{2}+\frac{\left(1-q^2\right)\omega_2^2}{4}-\frac{\left(1-q^2\right)\omega_1\omega_3}{2}+\frac{3(1+q)(q-1)^2\omega_1^2\omega_2}{8}\right.$$
$$\left.+\frac{(1+q)(1-q)^3\omega_1^4}{16}\right\}z^4+\cdots.$$

From the left-hand side of equation (15), we obtain

$$\begin{aligned}&\frac{z\mathfrak{D}_q\left(\mathcal{K}^{\mu}_{q,\nu,k}f(z)\right)}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)}\\&=1+q\chi_2a_2z+\left\{q\left(1+q\right)\chi_3a_3-q\chi_2^2a_2^2\right\}z^2\\&+\left\{q\left(1+q+q^2\right)\chi_4a_4-q\left(2+q\right)\chi_2\chi_3a_2a_3+q\chi_2^3a_2^3\right\}z^3\\&+\left\{q\left(1+q+q^2+q^3\right)\chi_5a_5-q\left(2+q+q^2\right)\chi_2\chi_4a_2a_4-q\left(1+q\right)\chi_3^2a_3^2\right.\\&\left.+q\left(3+q\right)\chi_2^2\chi_3a_2^2a_3-q\chi_2^4a_2^4\right\}z^4+\cdots.\end{aligned} \tag{16}$$

By comparison, we get

$$a_2=\frac{(1+q)}{2q\chi_2}\omega_1, \tag{17}$$

$$a_3=\frac{1}{2q\chi_3}\omega_2+\frac{\left(1+q^2\right)}{4q^2\chi_3}\omega_1^2 \tag{18}$$

and

$$a_4 = \frac{(1+q)}{2q\left(1+q+q^2\right)\chi_4}\omega_3 - \frac{(1+q)(q-2)(2q+1)}{4q^2\left(1+q+q^2\right)\chi_4}\omega_1\omega_2 + \frac{(1+q)\left(1+q^2\right)\left(1-q+q^2\right)}{8q^3\left(1+q+q^2\right)\chi_4}\omega_1^3. \tag{19}$$

Put the values of a_2 and a_3 from above in the functional $|a_3 - a_2^2|$, we obtain

$$|a_3 - a_2^2| = \frac{1}{2q\chi_3}|\omega_2 - \left(\frac{(1+q)^2\chi_3 - \left(1+q^2\right)\chi_2^2}{2q\chi_2^2}\right)\omega_1^2|.$$

Making use of Lemma 3.2 with $\omega_1 \leq 2$, then we have

$$|a_3 - a_2^2| = \frac{1}{2q\chi_3}|\frac{\xi(4-\omega_1^2)}{2} - \left(\frac{(1+q)^2\chi_3 - \left(1+q+q^2\right)\chi_2^2}{2q\chi_2^2}\right)\omega_1^2|.$$

If we take $\omega_1 = \omega$, where $0 \leq \omega \leq 2$ with $|\xi| = \delta$, we have

$$|a_3 - a_2^2| \leq \frac{1}{2q\chi_3}\left(\frac{\delta\left(4-\omega^2\right)}{2} + \left(\frac{(1+q)^2\chi_3 - \left(1+q+q^2\right)\chi_2^2}{2q\chi_2^2}\right)\omega^2\right) = F_1(\omega,\delta). \tag{20}$$

Now, by the partially differentiating of the function $F_1(\omega,\delta)$ with respect to δ, we observe

$$\frac{\partial F_1(\omega,\delta)}{\partial\delta} > 0.$$

This leads to the fact that the function $F_1(\omega,\delta)$ is an increasing function of δ, when $\delta \in [0,1]$. Thus, the maximum value of $F_1(\omega,\delta)$ at $\delta = 1$ achieves the relation

$$\max\{F_1(\omega,\delta)\} = F_1(\omega,1) = G_1(\omega),$$

where

$$G_1(\omega) := \frac{1}{2q\chi_3}\left(2 + \left(\frac{(1+q)^2\chi_3 - (1+q)^2\chi_2^2}{2q\chi_2^2}\right)\omega^2\right).$$

Obviously, $G_1(\omega)$ admits a maximum record at $\omega = 0$, which implies that

$$|a_3 - a_2^2| \leq G_1(\omega) = \frac{1}{q\chi_3}. \qquad \square$$

EXAMPLE 4.2. Consider the normalized analytic function $f(z)$. Then it has the following coefficients inequality

$$\begin{aligned}
|a_3 - a_2^2| \leq \frac{1}{q\chi_3} &= \frac{1}{q\left(\left(\frac{[1+\nu]_q}{[j+\nu]_q}\right)^k \frac{[j]_q!}{[\mu+1;j]_q}\right)} \\
&= \frac{1}{q\left(\left(\frac{1-q^{1+\nu}}{1-q^{3+\nu}}\right)^k\left(\frac{(1-q^3)(1-q^2)(1-q)}{(1-q^{3+\mu})(1-q^{2+\mu})(1-q^{1+\mu})}\right)\right)} \\
&= \frac{1}{q}\left(\frac{1-q^{3+\nu}}{1-q^{1+\nu}}\right)^k\left(\frac{(1-q^{3+\mu})(1-q^{2+\mu})(1-q^{1+\mu})}{(1-q^3)(1-q^2)(1-q)}\right).
\end{aligned}$$

It is well known that $|a_3 - a_2^2| \leq 1$. Thus, it is sufficient to show that

$$\left(\frac{1-q^{3+\nu}}{1-q^{1+\nu}}\right)^k \left(\frac{(1-q^{3+\mu})(1-q^{2+\mu})(1-q^{1+\mu})}{(1-q^3)(1-q^2)(1-q)}\right) \leq q. \tag{21}$$

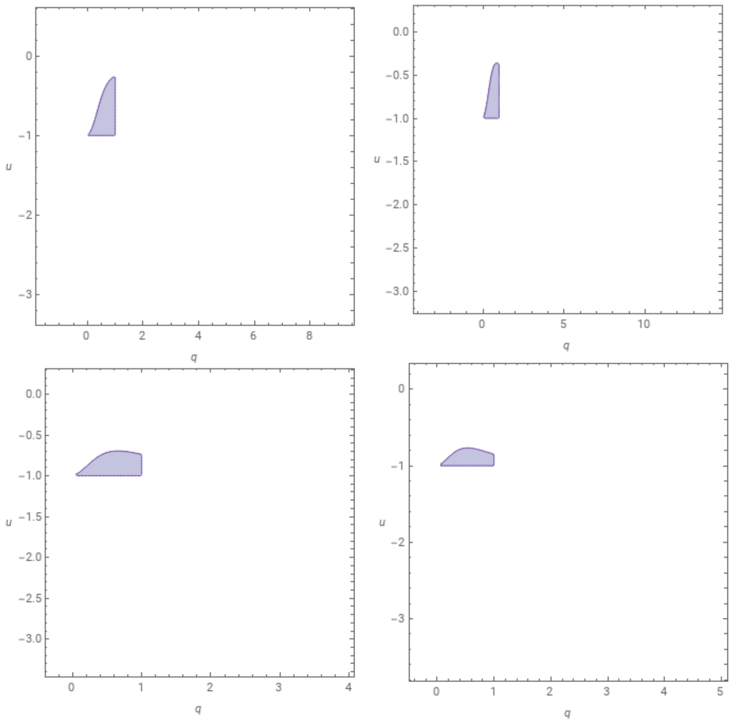

Figure 1: Inequality plot of (21) shows the relation between $q \in (0,1)$ and μ some $(k,\nu) = (1,2), (2,3), (3,1)$ and $(4,1)$ respectively.

Figure 1 shows the relation between q and μ. It indicates that inequality is valid whenever $q \to 1-$. The quantum number represents to the history of the movement of the coefficients towards the normal case (maximum value). The quantum inequalities that display quantum behavior are best described and modeled using quantum calculus. Quantum calculus offers a superior mathematical structure in circumstances where classical calculus is unable to adequately convey the complexities of quantum processes. These non-commutative forms may be represented in a manner that is more natural thanks to quantum calculus. There are restrictions on how accurately some combinations of attributes, like as location and momentum, may be known at the same time, according to the principle of quantum calculus. One may recognize a property less closely dependent on how precisely one is measured.

THEOREM 4.3. *If $f(z) \in \mathcal{US}^*(q;\eta,\lambda)$, where $f(z)$ in (1) then*

$$\mathcal{H}_2(1) = |a_3 - a_2^2| \leq \frac{\lambda(1+\eta)}{q(1+q)\chi_3},$$

where χ_3 have been included in (10) with $j = 3$.

Proof. In view of the subordination condition (13), we obtain

$$\frac{z\mathfrak{D}_q(\mathcal{K}^{\mu}_{q,\nu,k}f(z))}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)} \prec \Omega\left(\upsilon(z)\right). \tag{22}$$

After making a simplification of $\Omega(z)$, we obtain

$$\Omega(z) = 1 + \lambda(1+\eta)z + \eta\lambda^2(1+\eta)z^2 + \eta^2\lambda^3(1+\eta)z^3 + \cdots. \tag{23}$$

We proceed to illustrate the function $\omega(z)$, as follows:

$$\omega\left(z\right) = \frac{1+\upsilon\left(z\right)}{1-\upsilon\left(z\right)} = 1 + \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \omega_4 z^4 + \ldots.$$

Clearly, $\omega \in \mathcal{P}$ then

$$\begin{aligned}\upsilon\left(z\right) &= \frac{\omega\left(z\right)-1}{\omega\left(z\right)+1}\\ &= \frac{1}{2}\omega_1 z + \frac{1}{2}\left(\omega_2 - \frac{1}{2}\omega_1^2\right)z^2 + \frac{1}{2}\left(\omega_3 - \omega_1\omega_2 + \frac{1}{4}\omega_1^3\right)z^3 + \cdots.\end{aligned}$$

In view of $\Omega(z)$ and $\upsilon(z)$, we deduce that

$$\begin{aligned}\Omega(\upsilon(z)) = 1 &+ \frac{\lambda(1+\eta)}{2}\omega_1 z + \left\{\left(\frac{\lambda(1+\eta)}{2}\right)\left(\omega_2 - \frac{1}{2}\omega_1^2\right) + \frac{\eta\lambda^2(1+\eta)}{4}\omega_1^2\right\}z^2\\ &+ \frac{\lambda(1+\eta)}{2}\left\{\omega_3 + \eta\lambda\omega_1\omega_2 - \omega_1\omega_2 + \frac{1}{4}\omega_1^3 - \frac{\eta\lambda}{2}\omega_1^3 + \frac{\eta^2\lambda^2}{4}\omega_1^3\right\}z^3.\end{aligned} \tag{24}$$

Similarly, from (16), we have

$$\begin{aligned}&\frac{z\mathfrak{D}_q\left(\mathcal{K}^{\mu}_{q,\nu,k}f(z)\right)}{\mathcal{K}^{\mu}_{q,\nu,k}f(z)}\\ &= 1 + q\chi_2 a_2 z + \left\{q\left(1+q\right)\chi_3 a_3 - q\chi_2^2 a_2^2\right\}z^2\\ &+ \left\{q\left(1+q+q^2\right)\chi_4 a_4 - q\left(2+q\right)\chi_2\chi_3 a_2 a_3 + q\chi_2^3 a_2^3\right\}z^3\\ &+ \left\{q\left(1+q+q^2+q^3\right)\chi_5 a_5 - q\left(2+q+q^2\right)\chi_2\chi_4 a_2 a_4 - q\left(1+q\right)\chi_3^2 a_3^2\right.\\ &\left. + q\left(3+q\right)\chi_2^2\chi_3 a_2^2 a_3 - q\chi_2^4 a_2^4\right\}z^4 + \cdots.\end{aligned}$$

By comparison, we have

$$a_2 = \frac{\lambda(1+\eta)\omega_1}{2q\chi_2}, \tag{25}$$

$$a_3 = \frac{\lambda(1+\eta)}{2q(1+q)\chi_3}\left\{\omega_2 + \left(\frac{\eta\lambda - 1}{2} + \frac{\lambda(1+\eta)}{2q}\right)\omega_1^2\right\} \tag{26}$$

and

$$a_4 = \frac{\lambda(1+\eta)}{2(1+q+q^2+q^3)\chi_4}\left\{\omega_3 + \Lambda_1\omega_1\omega_2 + \Lambda_2\omega_1^3\right\}, \tag{27}$$

where

$$\Lambda_1 := \eta\lambda - 1 + \frac{\lambda(1+\eta)}{2}\frac{(2+q)}{q(1+q)},$$

and

$$\Lambda_2 := \left\{ \begin{array}{l} \frac{1}{4} - \frac{\lambda\eta}{2} + \frac{\eta^2\lambda^2}{4} + \frac{\lambda(1+\eta)}{2}\frac{(2+q)}{q(1+q)} \\ \times\left(\frac{\eta\lambda-1}{2} + \frac{\lambda(1+\eta)}{2q}\right) - \frac{\lambda^2(1+\eta)^2}{4q^2} \end{array} \right\}.$$

By the similar way of Theorem 4.1, we obtain the required outcome. □

If $k = 0$ and $\mu = 1$, we obtain the following corollary

EXAMPLE 4.4. Consider the normalized analytic function $f(z)$. Then in view of Theorem 4.3, we have

$$\begin{aligned} |a_3 - a_2^2| \le \frac{\lambda(1+\eta)}{q(1+q)\chi_3} &= \frac{\lambda(1+\eta)}{q(1+q)\left(\left(\frac{[1+\nu]_q}{[j+\nu]_q}\right)^k \frac{[j]_q!}{[\mu+1;j]_q}\right)} \\ &= \frac{\lambda(1+\eta)}{q(1+q)\left(\left(\frac{1-q^{1+\nu}}{1-q^{3+\nu}}\right)^k\left(\frac{(1-q^3)(1-q^2)(1-q)}{(1-q^{3+\mu})(1-q^{2+\mu})(1-q^{1+\mu})}\right)\right)} \\ &= \frac{\lambda(1+\eta)}{q(1+q)}\left(\frac{1-q^{3+\nu}}{1-q^{1+\nu}}\right)^k\left(\frac{(1-q^{3+\mu})(1-q^{2+\mu})(1-q^{1+\mu})}{(1-q^3)(1-q^2)(1-q)}\right). \end{aligned}$$

Then

$$\frac{\lambda(1+\eta)}{q(1+q)}\left(\frac{1-q^{3+\nu}}{1-q^{1+\nu}}\right)^k\left(\frac{(1-q^{3+\mu})(1-q^{2+\mu})(1-q^{1+\mu})}{(1-q^3)(1-q^2)(1-q)}\right) \le 1,$$

whenever

$$\left(\frac{1-q^{3+\nu}}{1-q^{1+\nu}}\right)^k\left(\frac{(1-q^{3+\mu})(1-q^{2+\mu})(1-q^{1+\mu})}{(1-q^3)(1-q^2)(1-q)}\right) \le \frac{q(1+q)}{\lambda(1+\eta)}. \tag{28}$$

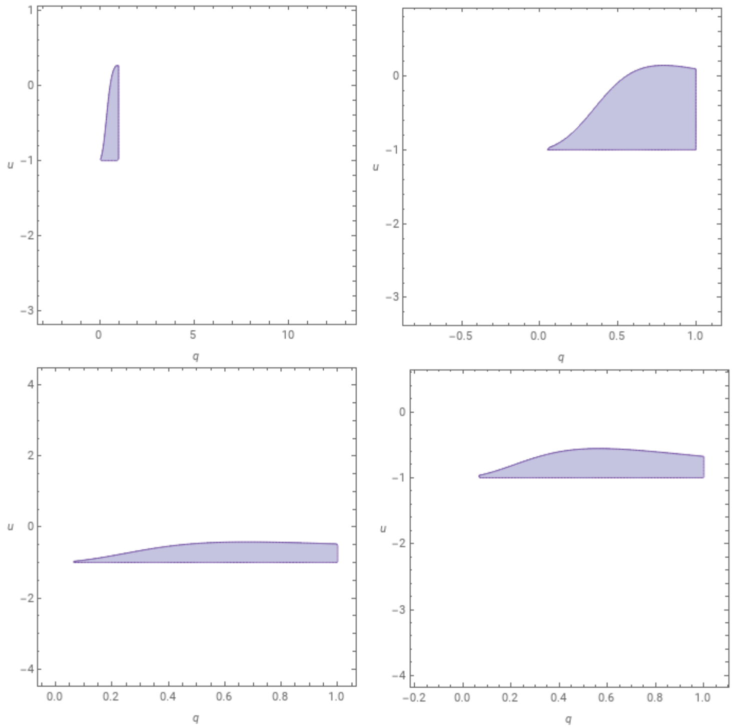

Figure 2: Inequality plot of (28) shows the relation between $q \in (0,1)$ and μ some $(k,\nu) = (1,2),(2,3),(3,1)$ and $(4,1)$ respectively for $\lambda = \eta = 1/2$.

Figure 2 shows the relation between q and μ when $\lambda = \eta = 1/2$. It indicates that inequality is valid whenever $q \to 1-$. Again, the quantum number shows the history of the movement of the coefficients towards the maximum case. Moreover, when $\eta = \lambda$ and $\mu = 0, k = 2$, we obtain $q = 0.7446$.

Corollary 4.5. *If $f(z) \in \mathcal{US}^*(q;\eta,\lambda)$, where $f(z)$ in (1) then*

$$\mathcal{H}_2(1) = |a_3 - a_2^2| \leq \frac{\lambda(1+\eta)}{q(1+q)}.$$

Corollary 4.6. *[24] If $f(z) \in \mathcal{S}^*$, where $f(z)$ in (1) then*

$$|a_3 - a_2^2| \leq 1. \tag{29}$$

Theorem 4.7. *If $f(z) \in \mathcal{KS}^*(q;k,\mu)$, where $f(z)$ in (1) then*

$$\mathcal{H}_2(2) = |a_2 a_4 - a_3^2| \leq \frac{1}{q^2 \chi_3^2},$$

where χ_3 have been included in (10) with $j = 3$.

Proof. From the values a_2, a_3, and a_4 of Theorem 4.1, the result is

$$a_2a_4 - a_3^2 = \left(\frac{(1+2q+q^2)}{4q^2(1+q+q^2)\chi_2\chi_4}\right)\omega_1\omega_3$$
$$-\left(\frac{(1+2q+q^2)(q-2)(1+2q)}{8q^3(1+q+q^2)\chi_2\chi_4}+\frac{(1+q^2)}{4q^3\chi_3^2}\right)\omega_1^2\omega_2-\left(\frac{1}{4q^2\chi_3^2}\right)\omega_2^2$$
$$+\left(\frac{(1+2q+q^2)(1+q^2)(1-q+q^2)}{16q^3(1+q+q^2)\chi_2\chi_4}-\frac{(1+q^2)^2}{16q^4\chi_3^2}\right)\omega_1^4.$$

By Lemma 3.2, it follows that

$$a_2a_4 - a_3^2 = \left(\frac{(1+2q+q^2)(1+q^2)(1-q+q^2)}{16q^3(1+q+q^2)\chi_2\chi_4}-\frac{(q^2+1)^2}{16q^4\chi_3^2}\right)\omega_1^4$$
$$+\left(\frac{(1+2q+q^2)}{16q^2(1+q+q^2)\chi_2\chi_4}\right)\omega_1\{\omega_1^3+2\omega_1(4-\omega_1^2)\xi$$
$$-\omega_1(4-\omega_1^2)\xi^2+2(4-\omega_1^2)(1-|\xi|^2)z\}$$
$$-\left(\frac{(1+2q+q^2)(q-2)(1+2q)}{16q^3(1+q+q^2)\chi_2\chi_4}+\frac{(1+q^2)}{8q^3\chi_3^2}\right)\omega_1^2\{(\omega_1^2+\xi(4-\omega_1^2))\}$$
$$-\left(\frac{1}{16q^2\chi_3^2}\right)\left\{\omega_1^4+2\xi(4-\omega_1^2)\omega_1^2+(4-\omega_1^2)^2\xi^2\right\}.$$

If we take $\omega_1=\omega$ with $|\xi|=\delta$ then

$$|a_2a_4-a_3^2| \le \frac{1}{\aleph_1(q)}\left[g(q)\omega^4+2q(1+2q+q^2)\chi_3^2\omega(4-\omega^2)\right.$$
$$+\aleph_2(q)(4-\omega^2)\omega^2\delta+\left(q(q+1)^2\chi_3^2\omega^2+q(4-\omega^2)\right.$$
$$\left.\left.\cdot(1+q+q^2)\chi_2\chi_4-2q(1+2q+q^2)\chi_3^2\omega\right)(4-\omega^2)\delta^2\right]=F_2(\omega,\delta),$$

where

$$\aleph_1(q) := 16q^3(1+q+q^2)\chi_2\chi_3^2\chi_4,$$

$$g(q) :=\left|\,(q^4-q^3+3q+3)(1+2q+q^2)\chi_3^2-(1+3q+2q^2+2q^3+q^4)\right.$$
$$\left.\cdot(1+q+q^2)\chi_2\chi_4\,\right|$$

and

$$\aleph_2(q) := |(1+q)^2(2q^2-5q-2)\chi_3^2+2q(2+q^2)(1+q+q^2)\chi_2\chi_4|.$$

Now, by the partially differentiating of $F_2(\omega,\delta)$ with respect to δ, we have

$$\frac{\partial F_2(\omega,\delta)}{\partial\delta} = \left(\frac{1}{\aleph_1(q)}\right)\left[\aleph_2(q)(4-\omega^2)\omega^2+2\left(q(1+2q+q^2)\chi_3^2\omega^2+q(4-\omega^2)\right.\right.$$
$$\left.\left.(1+q+q^2)\chi_2\chi_4-2q(1+2q+q^2)\chi_3^2\omega\right)(4-\omega^2)\delta\right]>0.$$

This leads to the fact that the function $F_2(\omega,\delta)$ is an increasing function of δ, $(\delta \in [0,1])$, then we get

$$\max\{F_2(\omega,\delta)\} = F_2(\omega,1) = G_2(\omega),$$

where

$$\begin{aligned} G_2(\omega) := (\frac{1}{\aleph_1(q)})[&(g(q) - \aleph_2(q) - q(1+2q+q^2)\chi_3^2 + (q+q^2+q^3)\chi_2\chi_4)\omega^4 \\ &+ (4\aleph_2(q) + 4q(1+2q+q^2)\chi_3^2 - 8(q+q^2+q^3)\chi_2\chi_4)\omega^2 \\ &+ 16q(1+q+q^2)\chi_2\chi_4], \end{aligned} \tag{30}$$

and

$$\begin{aligned} G_2'(\omega) = (\frac{1}{\aleph_1(q)})[&4(g(q) - \aleph_2(q) - q(1+2q+q^2)\chi_3^2 + (q+q^2+q^3)\chi_2\chi_4)\omega^3 \\ &+ 2(4\aleph_2(q) + 4q(1+2q+q^2)\chi_3^2 - 8(q+q^2+q^3)\chi_2\chi_4)\omega]. \end{aligned}$$

We can get the following result by differentiating the function $G_2'(\omega)$ with respect to ω

$$\begin{aligned} G_2''(\omega) = (\frac{1}{\aleph_1(q)})[&12(g(q) - \aleph_2(q) - q(1+2q+q^2)\chi_3^2 + (q+q^2+q^3)\chi_2\chi_4)\omega^2 \\ &+ 2(4\aleph_2(q) + 4q(1+2q+q^2)\chi_3^2 - 8(q+q^2+q^3)\chi_2\chi_4)]. \end{aligned}$$

Hence, we have the following inequality:

$$|a_2a_4 - a_3^2| \le \frac{1}{q^2\chi_3^2}.$$

□

Theorem 4.8. *If $f(z) \in \mathcal{US}^*(q;\eta,\lambda)$, where $f(z)$ in (1) then*

$$\mathcal{H}_2(2) = |a_2a_4 - a_3^2| \le \frac{\lambda^2(1+\eta)^2}{q^2(1+q)^2\chi_3^2},$$

where χ_3 have been included in (10) with $j=3$.

Proof. With the aid of (25), (26) and (27), we have

$$a_2a_4 - a_3^2 = \frac{\lambda^2(1+\eta)^2}{4}\left\{ \begin{array}{c} \Upsilon_1\omega_1\omega_3 + (\Upsilon_1\Lambda_1 - 2\Upsilon_2\Lambda_3)\,\omega_1^2\omega_2 \\ -\Upsilon_2\omega_2^2 + \left(\Upsilon_1\Lambda_2 - \Upsilon_2\Lambda_3^2\right)\omega_1^4 \end{array} \right\}, \tag{31}$$

where

$$\Upsilon_1 = \frac{1}{q(1+q+q^2+q^3)\chi_2\chi_4}, \Upsilon_2 = \frac{1}{q^2(1+q)^2\chi_3^2}$$

and

$$\Lambda_3 = \frac{\eta\lambda - 1}{2} + \frac{\lambda(1+\eta)}{2q\chi_2}.$$

Also, by the similar way of Theorem 4.7, we obtain the required outcome. □

If $k=0$ and $\mu=1$, the following corollary is obtained.

COROLLARY 4.9. ([2]) *If $f(z) \in \mathcal{US}^*(q;\eta,\lambda)$, where $f(z)$ in* (1) *then*

$$\mathcal{H}_2(2) = |a_2a_4 - a_3^2| \leq \frac{\lambda^2(1+\eta)^2}{q^2(1+q)^2}.$$

According to the values of $q, k, \mu; \lambda,$ and η of the classes $\mathcal{KS}^*(q;k,\mu)$ and $\mathcal{US}^*(q;\eta,\lambda)$, we get the obtained estimates of [22]-Theorem 3.1, as follows:

COROLLARY 4.10. ([22]) *If $f(z) \in \mathcal{KS}^*(1;0,1) = \mathcal{S}^*$ or $f(z) \in \mathcal{US}^*(q \to 1-;1,1) = \mathcal{S}^*$, where $f(z)$ in* (1) *then*

$$\mathcal{H}_2(2) = |a_2a_4 - a_3^2| \leq 1.$$

THEOREM 4.11. *If $f(z) \in \mathcal{KS}^*(q;k,\mu)$, where $f(z)$ in* (1) *then*

$$|a_2a_3 - a_4| \leq \frac{(1+q)\,\Upsilon_q}{q\,(q+q^2+q^3)\,\chi_2\chi_3\chi_4},$$

where

$$\Upsilon_q := |\left(1+q+q^2\right)^2\chi_4 - \left(q^4 - 3q + 6q^2 + q + 1\right)\chi_2\chi_3| \tag{32}$$

and χ_2, χ_3, χ_4 have been included in (10) *with $j = 2, 3, 4$.*

Proof. From (17) - (19) of Theorem 4.1 and by simplification, we get

$$\begin{aligned} a_2a_3 - a_4 = &\left(\frac{(1+q)\left(1+q^2\right)}{8q^3\chi_2\chi_3} - \frac{(1+q)\left(1+q^2\right)\left(1-q+q^2\right)}{8q^3\left(1+q+q^2\right)\chi_4}\right)\omega_1^3 \\ &- \frac{(1+q)}{2q\left(1+q+q^2\right)\chi_4}\omega_3 + \left(\frac{(1+q)}{4q^2\chi_2\chi_3} - \frac{(1+q)\,(q-2)\,(2q+1)}{4q^2\left(1+q+q^2\right)\chi_4}\right)\omega_1\omega_2. \end{aligned} \tag{33}$$

Making use of Lemma 3.2 with $\omega_1 \leq 2$. Also, by Lemma 3.1, if we take $\omega_1 = \omega$ when $0 \leq \omega \leq 2$. Then, by applying the trigonometric inequality on (33) and $|\xi| = \delta$, we have

$$|a_2a_3 - a_4| \leq \mathcal{A}(\omega,\delta),$$

where

$$\begin{aligned} \mathcal{A}(\omega,\delta) := &\left(\frac{(1+q)}{8q^3\left(1+q+q^2\right)\chi_2\chi_3\chi_4}\right)\left[\Upsilon_q\omega^3 + \Lambda_q\omega\left(4-\omega^2\right)\delta\right. \\ &\left.+2q^2\chi_2\chi_3\left(4-\omega^2\right) + q^2\chi_2\chi_3\,(\omega-2)\left(4-\omega^2\right)\delta^2\right], \end{aligned}$$

with $\Lambda_q := |q(1+q+q^2)\chi_4 + q(2q^2-q-2)\chi_2\chi_3|$ and Υ_q well-known in (32). Now, if we differentiate the function $\mathcal{A}(\omega,\delta)$ with respect to δ, we get

$$\mathcal{A}'(\omega,\delta) = \left(\frac{(1+q)}{8q^3(1+q+q^2)\chi_2\chi_3\chi_4}\right)[\Lambda_q\omega(4-\omega^2) + 2q^2\chi_2\chi_3(\omega-2)(4-\omega^2)\delta] > 0.$$

This leads to the fact that the function $\mathcal{A}(\omega,\delta)$ is an increasing function of δ when $\delta \in [0,1]$. That means the maximum value is achieved at $\delta = 1$, then

$$\mathcal{A}(\omega,\delta) \leq \mathcal{A}(\omega,1).$$

Therefore,

$$\max\{\mathcal{A}(\omega,\delta)\} = \mathcal{A}(\omega,1) \leq \mathcal{B}(\omega),$$

where

$$\mathcal{B}(\omega) := \left(\frac{(1+q)}{8q^3(1+q+q^2)\chi_2\chi_3\chi_4}\right)[(\Upsilon_q - \Lambda_q - q^2\chi_2\chi_3)\omega^3 + (4\Lambda_q + 4q^2\chi_2\chi_3)\omega].$$

Since $0 \leq \omega \leq 2$, that means the maximum point is $\omega = 2$, hence

$$\mathcal{B}(\omega) \leq \frac{(1+q)\,\Upsilon_q}{q^3\,(1+q+q^2)\,\chi_2\chi_3\chi_4},$$

this corresponds to $\omega = 2$ and $\delta = 1$ and the desired limit. □

Similarly, we obtain the upper bound of the inequality $|a_2a_3 - a_4|$ of the class $\mathcal{US}^*(q;\eta,\lambda)$ as follows:

THEOREM 4.12. *If $f(z) \in \mathcal{US}^*(q;\eta,\lambda)$, where $f(z)$ in* (1) *then*

$$|a_2a_3 - a_4| \leq \frac{\lambda(1+\eta)\Phi_q}{q^3(1+q)^2\,(1+q^2)\,(1+q+q^2+q^3)\,\chi_2\chi_3\chi_4},$$

where

$$\Phi_q = \left\{ \begin{array}{l} \lambda\left(1+q+q^2+q^3\right)\left((1+q)\left(1+q^2\right)(1+\alpha)(\lambda+\eta\lambda+q\eta\lambda)\chi_4\right. \\ \left.-q\left(q(2+q)(1+\eta)\eta\lambda+\lambda+\eta\left(2+\left(1+q^2+q^3\right)\eta\right)\lambda\right)\chi_2\chi_3\right) \end{array} \right\} \tag{34}$$

and χ_2, χ_3, χ_4 have been included in (10) *with $j = 2, 3, 4$.*

According to the values of $q, k, \mu; \lambda$ and η of the classes $\mathcal{KS}^*(q;k,\mu)$ and $\mathcal{US}^*(q;\eta,\lambda)$, we get the obtained estimates of [[5], Theorem 3.1].

COROLLARY 4.13. ([5]) *If $f(z) \in \mathcal{KS}^*(1;0,1) = \mathcal{S}^*$ or $f(z) \in \mathcal{US}^*(q \to 1-;1,1) = \mathcal{S}^*$, where $f(z)$ in* (1) *then*

$$|a_2a_3 - a_4| \leq 2.$$

THEOREM 4.14. *If $f(z) \in \mathcal{KS}^*(q;k,\mu)$, where $f(z)$ in* (1) *then*

$$\mathcal{H}_3(1) \leq \left[\frac{\left(1+q+q^2\right)}{q^4\chi_2^3} + \frac{\Upsilon_q\Xi_q}{q^5(1+q+q^2)^2\chi_2\chi_3\chi_4^2} + \frac{\phi_q}{q^5\,(1+q+q^2+q^3)\,(1+q+q^2)\,\chi_3\chi_5}\right],$$

where Υ_q is defined by (32) *and*

$$\Xi_q = (1+q)^2\left(q^4 - 3q^3 + 6q^2 + q + 1\right), \tag{35}$$

$$\phi_q = (1+q)\left(4q^7 + 2q^6 + 6q^5 + 7q^4 + 13q^3 - q - 1\right). \tag{36}$$

Proof. From the TOHD

$$\mathcal{H}_3(1) \leq |a_3|\mathcal{H}_2(2) + |a_4||a_2a_3 - a_4| + |a_5|\mathcal{H}_2(1).$$

Using Lemma 3.1, we show that

$$|a_4| \leq \frac{(1+q)\left(1+q+6q^2-3q^3+q^4\right)}{q^3\left(1+q+q^2\right)\chi_4}.$$

Also, we have

$$|a_5| \leq \frac{\phi_q}{q^4\left(1+q+q^2+q^3\right)\left(1+q+q^2\right)\chi_5},$$

where ϕ_q is defined in (36).

To obtain the TOHD, we must apply all the results obtained in the Theorems 4.1-4.11. Thus Theorem 4.14 is complete. □

5. Fekete-Szegö type inequality.

In the following section , we will investigate the Fekete-Szegö type inequality $|a_3 - \gamma a_2^2|$ for the classes $\mathcal{KS}^*(q;k,\mu)$ and $\mathcal{US}^*(q;\eta,\lambda)$ of q-starlike functions.

THEOREM 5.1. *If $f(z) \in \mathcal{KS}^*(q;k,\mu)$, where $f(z)$ in (1) then*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{\left(1+q+q^2\right)\chi_2^2 - \gamma(1+q)^2\chi_3}{q^2\chi_3\chi_2^2} & \left(\gamma < \frac{\left(q^2+1\right)\chi_2^2}{(1+q)^2\chi_3}\right) \\ \frac{1}{q\chi_3} & \left(\frac{\left(q^2+1\right)\chi_2^2}{(1+q)^2\chi_3} \leq \gamma \leq \frac{\chi_2^2}{\chi_3}\right) \\ \frac{\gamma(1+q)^2\chi_3 - \left(1+q+q^2\right)\chi_2^2}{q^2\chi_3\chi_2^2} & \left(\gamma > \frac{\chi_2^2}{\chi_3}\right), \end{cases}$$

where χ_j defined in (10).

Proof. In view of a_2 and a_3 in (15) and (17) of Theorem 4.1, we obtain the required result. □

THEOREM 5.2. *If $f(z) \in \mathcal{US}^*(q;\eta,\lambda)$, where $f(z)$ in (1), then*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{\lambda(1+\eta)}{q(1+q)\chi_3}\left\{\begin{array}{c} \left(\frac{\lambda(1+\eta)}{q} + \lambda\eta\right) \\ -\frac{\lambda(1+\eta)(1+q)\chi_3}{\chi_2^2}\gamma \end{array}\right\}, & \text{if } \kappa_1 \leq \gamma \leq \frac{1}{2}, \\ \frac{\lambda(1+\eta)}{q(1+q)\chi_3}, & \text{if } \frac{1}{2} \leq \gamma \leq \kappa_2, \\ \frac{\lambda(1+\eta)}{q(1+q)\chi_3}\left\{\begin{array}{c} \frac{\lambda(1+\eta)(1+q)\chi_3}{\chi_2^2}\gamma \\ -\left(\eta\lambda + \frac{\lambda(1+\eta)}{q}\right) \end{array}\right\}, & \text{if } \gamma \geq \kappa_2, \end{cases} \tag{37}$$

where

$$\kappa_1 = \frac{q\eta\lambda\chi_2^2 + \lambda(1+\eta)}{q(1+q)\lambda(1+\eta)\chi_3},$$

$$\kappa_2 = \frac{q(\eta\lambda - 1)\chi_2^2 + \lambda(1+\eta)}{q(1+q)\chi_3}.$$

Proof. In view of a_2 and a_3 in (25) and (26) of Theorem 4.3, we have

$$|a_3 - \gamma a_2^2| = \frac{\lambda(1+\eta)}{2q(1+q)\chi_3} |(\Psi_1 - \gamma\Psi_2)\,\omega^2 + \omega_2|,$$

where

$$\Psi_1 = \frac{\eta\lambda - 1}{2} + \frac{\lambda(1+\eta)}{2q},$$
$$\Psi_2 = \frac{\lambda(1+\eta)(1+q)}{2\chi_2^2}.$$

Making use of Lemma 3.2 and after some of simplifications, we get the desired outcome. □

COROLLARY 5.3. ([16]) *If* $f(z) \in \mathcal{KS}^*(1;0,1) = \mathcal{S}^*$ *or* $f(z) \in \mathcal{US}^*(q \to 1-;1,1) = \mathcal{S}^*$, *where* $f(z)$ *in* (1) *then*

$$|a_3 - \gamma a_2^2| \le \begin{cases} 3-4\gamma, & \text{if } \mu \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le \gamma \le 1 \\ 4\gamma - 3, & \text{if } \gamma \ge 1. \end{cases}$$

6. Conclusion. Using the notion of quantum calculus (or q-analysis), we have presented the q-analogue integral operator $\mathcal{K}^{\mu}_{q,\nu,k}f(z)$ defined by the q-derivative of a modified q-BIO. In the unit disc $\mathbb{U}$, this operator has been applied to define the classes $\mathcal{KS}^*(q;k,\mu)$ and $\mathcal{US}^*(q;\eta,\lambda)$ of q-starlike functions. For these classes, we have attractively attained the upper bounds of SOHD $\mathcal{H}_2(1)$ and TOHD $\mathcal{H}_3(1)$. Theorems 4.1-4.14 state and prove our main findings, as well as the Fekete-Szegö type inequality. These main outcomes are enhanced basically by their several special cases, some of which have been discussed in Corollaries 4.5, 4.6, 4.9, 4.10, 4.13, and 5.3. All these findings and generalizations lead to the important cases and also new works, such as finding the Hankel and Toeplitz determinants for the q-convex and q-close-to-convex analytic functions and their inverses. This issue is still ongoing and relates to findings examined by Wang and Jiang [45].

Acknowledgments. The authors express many thanks to the Editor-in-Chief, handling editor, and the reviewers for their outstanding comments that improve our paper.

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Received 4 February, 2023 and in revised form 10 December, 2023.