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A Survey on Riemannian Curvature Tensor for Certain Classes of Almost Contact Metric Manifolds

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ABSTRACT

This paper surveyed the components of Riemannian curvature tensor over the associated space of G-structure for certain classes of almost contact metric manifolds. These classes under consideration are only twelve and known as cosymplectic manifolds, Sasakian manifolds, Kenmotsu manifolds, C_9-manifolds, C_{12}-manifolds, normal manifolds of Killing type (CNK-manifold), nearly Kenmotsu manifolds, locally conformal almost cosymplectic manifolds (LCAC-manifolds), quasi-Sasakian manifolds, almost $C(\lambda)$-manifolds, nearly cosymplectic manifolds, and Kenmotsu type manifolds.

Introduction

The Riemannian curvature tensor (RC-tensor) is one of the interesting fields in the studying differential geometry. The Riemannian manifold of flat RC-tensor is locally isometric to the Euclidean space. Also, RC-tensor win its importance in the gravity theory and general relativity theory because its contraction is the Ricci tensor that a central mathematical tool in Einstein's theory. Based on the above, many authors studied RC-tensor of the manifolds and specially the almost contact metric manifolds that classified by D. Chinea and C. Gonzalez [1]. Especially among them, E. S. Volkova [2] determined the components of RC-tensor of CNK-manifolds. S. V. Umnova [3] established the components of RC-tensor of Kenmotsu manifolds and generalized Kenmotsu manifolds (nearly Kenmotsu manifolds). V. F. Kirichenko and A. R. Rustanov [4] deduced the components of RC-tensor of quasi-Sasakian manifolds. N. N. Dondukova [5].

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found the components of RC-tensor of cosymplectic manifolds and Sasakian manifolds. S. V. Kharitonova [6].

Concluded the components of RC-tensor of LCAC-manifolds. V. F. Kirichenko and E. V. Kusova [7] studied the components of RC-tensor of weakly cosymplectic manifolds (nearly cosymplectic manifolds).

So, according to the previous, we summarize these results in this paper and more than ones to have a survey about the RC-tensor of almost contact metric manifolds.

Preliminaries

Definition 1. [8] A topological space M is said to be a smooth manifold of dimension n and denoted by M^n, if M is T_2 −space, second countable, locally homeomorphic to $\mathbb{R}^n$, and has a smooth structure.

The symbol $\mathcal{X}(M)$ denotes to the module of whole vector fields on M^n.

Definition 2. [8] A bilinear map $g : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathbb{R}$ is said to be a metric tensor on M^n, if g is symmetric and positive definite.

Definition 3. [1] If a Riemannian manifold (M^{2n+1}, g) is provided by triple of a structure tensor (Φ, η, ξ), where η, ξ, Φ are tensors over M of types (1, 0), (0, 1), and (1, 1) respectively, such that $\forall\, Z_1, Z_2 \in \mathcal{X}(M)$, the following achieved:

$\eta(\xi) = 1;\quad \eta \circ \Phi = 0;\quad \Phi(\xi) = 0;\quad id + \Phi^2 = \eta \otimes \xi;$

$g(\Phi Z_1, \Phi Z_2) + \eta(Z_1)\eta(Z_2) = g(Z_1, Z_2)$, then it is known an almost contact metric (ACM-) manifold and denoted by $(M^{2n+1}, \xi, \eta, \Phi, g)$.

Definition 4. [8] A connection on a smooth manifold M is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by $\nabla(Z_1, Z_2) = \nabla_{Z_1} Z_2$ and it attains the subsequent properties:

(1) $\boldsymbol{\nabla_{f_1 Z_1 + f_2 Z_2} Z_3 = f_1 \nabla_{Z_1} Z_3 + f_2 \nabla_{Z_2} Z_3}$;
(2) $\boldsymbol{\nabla_{Z_3}(f_1 Z_1 + f_2 Z_2) = f_1 \nabla_{Z_3} Z_1 + f_2 \nabla_{Z_3} Z_2 + Z_3(f_1) Z_1 + Z_3(f_2) Z_2}$,

for all $f_1, f_2 \in C^\infty(M)$ and $Z_1, Z_2, Z_3 \in \mathcal{X}(M)$.

Lemma 1. [8] Suppose that ∇ is a connection over M and $U, V \in \mathcal{X}(M)$. If $U = 0$, or $V = 0$ then $\nabla_U V = 0$.

Definition 5. [8] A Riemannian connection over the Riemannian manifold (M, g) is a connection ∇ on M that possess the following properties:

(i) $\boldsymbol{\nabla_{Z_1} Z_2 - \nabla_{Z_2} Z_1 = [Z_1, Z_2]}$, where $\boldsymbol{[Z_1, Z_2] = Z_1 \circ Z_2 - Z_2 \circ Z_1}$;
(ii) $\boldsymbol{Z_1(g(Z_2, Z_3)) = g(\nabla_{Z_1} Z_2, Z_3) + g(Z_2, \nabla_{Z_1} Z_3)}$,

for all $Z_1, Z_2, Z_3 \in \mathcal{X}(M)$.

There are several classes of ACM-manifolds $(M^{2n+1}, \xi, \eta, \Phi, g)$. We define some of these classes according to their Riemannian connection as the following:

Table 1. Some defining classes

Classes	Defining conditions
Cosymplectic [9]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 = 0}$
Nearly cosymplectic [10]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \nabla_{Z_2}(\Phi) Z_1 = 0}$
Kenmotsu [11]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + g(Z_1, \Phi Z_2)\xi = -\eta(Z_2)\Phi Z_1}$
Sasakian [12]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \eta(Z_2) Z_1 = g(Z_1, Z_2)\xi}$
$\mathbf{C_9}$ [13]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 = \eta(Z_2)\nabla_{\Phi Z_1}\xi - g(\Phi Z_1, \nabla_{Z_2}\xi)\xi}$
$\mathbf{C_{12}}$ [14]	$\boldsymbol{-\eta(Z_1)\{\eta(Z_2)\Phi(\nabla_\xi \xi) + g(\nabla_\xi \xi, \Phi Z_2)\xi\} = \nabla_{Z_1}(\Phi) Z_2}$
CNK [2]	**Normal and** $\boldsymbol{\nabla_{Z_1}(\eta) Z_2 + \nabla_{Z_2}(\eta) Z_1 = 0}$
Nearly Kenmotsu [15]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \nabla_{Z_2}(\Phi) Z_1 = -\eta(Z_2)\Phi Z_1 - \eta(Z_1)\Phi Z_2}$
Kenmotsu type [16]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \eta(Z_2)\Phi Z_1 = \nabla_{\Phi Z_1}(\Phi)\Phi Z_2}$

for all $Z_1, Z_2 \in \mathcal{X}(M)$, where ∇ refer to Riemannian connection. Moreover, an ACM-manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is called normal if $2N + \xi \otimes d\eta = 0$, where for all $\mathcal{U}, \mathcal{V} \in \mathcal{X}(M)$:

$$N(\mathcal{U}, \mathcal{V}) = \frac{1}{4}([\Phi\mathcal{U}, \Phi\mathcal{V}] + \Phi^2[\mathcal{U}, \mathcal{V}] - \Phi[\Phi\mathcal{U}, \mathcal{V}] - \Phi[\mathcal{U}, \Phi\mathcal{V}]),$$

is the Nijenhuis tensor of the structure tensor Φ (see [2]).

Definition 6. [6] An ACM-manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is bearing an almost cosymplectic manifold if $d\Omega = 0$ and $d\eta = 0$, where

$$\Omega(Z_1, Z_2) = g(Z_1, \Phi Z_2) \quad \text{and}$$
$$2d\eta(Z_1, Z_2) = \nabla_{Z_1}(\eta) Z_2 - \nabla_{Z_2}(\eta) Z_1;$$
$$3d\Omega(Z_1, Z_2, Z_3) = \nabla_{Z_1}(\Omega)(Z_2, Z_3) + \nabla_{Z_2}(\Omega)(Z_3, Z_1) + \nabla_{Z_3}(\Omega)(Z_1, Z_2),$$
for all $Z_1, Z_2, Z_3 \in \mathcal{X}(M)$.

Definition 7. [6] An ACM-manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is bearing a LCAC-manifold if the ACM-manifold $(M^{2n+1}, \tilde{\xi}, \tilde{\eta}, \Phi, \tilde{g})$ is an almost cosymplectic manifold, where $\tilde{\xi} = exp(\alpha)\xi$; $\tilde{\eta} = exp(-\alpha)\eta$; $\tilde{g} = exp(-2\alpha) g$, and α is a smooth function.

Definition 8. [17] An ACM-manifold M^{2n+1} is known as quasi-Sasakian manifold if $d\Omega = 0$ and M is normal.

Definition 9. [8] An RC-tensor of type (3, 1) on a Riemannian manifold (N, g) is a tensor $R : \mathcal{X}(N) \times \mathcal{X}(N) \times \mathcal{X}(N) \to \mathcal{X}(N)$ that defined by $R(Z_1, Z_2)Z_3 = ([\nabla_{Z_1}, \nabla_{Z_2}] - \nabla_{[Z_1, Z_2]})Z_3$, for all $Z_1, Z_2, Z_3 \in \mathcal{X}(N)$, where ∇ is Riemannian connection over N. Furthermore, the RC-tensor R of type (4, 0) is given by the formula $R(Z_1,Z_2,Z_3,Z_4)=g(R(Z_3,Z_4)Z_2,Z_1)$, with $Z_4\in\mathcal{X}(N)$.

Definition 10. [18] The associated space of G-structure for an ACM-manifold $(M^{2n+1},\xi,\eta,\Phi,g)$ is a set of all A-frame $(x;\Upsilon_0=\xi,\Upsilon_1,\dots,\Upsilon_n,\Upsilon_{\hat{1}},\dots,\Upsilon_{\hat{n}})$, where $x\in M$, $\Upsilon_a=\frac{1}{\sqrt{2}}\left(\chi_a-\sqrt{-1}\Phi(\chi_a)\right)$, $\Upsilon_{\hat{a}}=\frac{1}{\sqrt{2}}\left(\chi_a+\sqrt{-1}\Phi(\chi_a)\right)$, $a=1,2,\dots,n$, $\hat{a}=a+n$ and $\{\chi_0=\xi,\chi_1,\dots,\chi_n,\chi_{\hat{1}},\dots,\chi_{\hat{n}}\}$ is a basis of $\mathcal{X}(M)$ which satisfies $g(\chi_p,\chi_q)=\delta_{pq}$, for all $p,q=0,1,\dots,2n$.

Lemma 2. [19] Suppose that $(M^{2n+1},\xi,\eta,\Phi,g)$ is an ACM-manifold and R its RC-tensor of kind (4, 0) with components R_{pqrs} on the associated space of G-structure. Then the subsequent relations are satisfied:

(1) $\boldsymbol{R_{pqrs}=-R_{qprs}}$;
(2) $\boldsymbol{R_{pqrs}=-R_{pqsr}}$;
(3) $\boldsymbol{R_{pqrs}=R_{rspq}}$;
(4) $\boldsymbol{R_{pqrs}+R_{psqr}+R_{prsq}=0}$,

where $p,q,r,s=0,1,\dots,2n$.

Definition 11. [20] An ACM-manifold $(M^{2n+1},\xi,\eta,\Phi,g)$ is said to be an almost $C(\lambda)$-manifold if its RC-tensor R fulfill the following identity:

$$\begin{aligned}&g(R(Z_3,Z_4)Z_2,Z_1)\\&=g(R(\Phi Z_3,\Phi Z_4)Z_2,Z_1)\\&-\lambda\{g(Z_1,Z_4)g(Z_2,Z_3)\\&-g(Z_1,Z_3)g(Z_2,Z_4)\\&-g(Z_1,\Phi Z_4)g(Z_2,\Phi Z_3)\\&+g(Z_1,\Phi Z_3)g(Z_2,\Phi Z_4)\},\end{aligned}$$

where $Z_1,Z_2,Z_3,Z_4\in\mathcal{X}(M)$, and $\lambda\in\mathbb{R}$. Moreover, a normal almost $C(\lambda)$-manifold is said to be $C(\lambda)$-manifold.

The Components of Riemannian Curvature Tensor on the Associated Space of G-Structure

In this section, we review the ingredients of RC-tensor on the associated space of G-structure for certain classes of ACM-manifolds.

Theorem 1. [5] The components of RC-tensor of cosymplectic manifolds are given by: $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}$, and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc}=A^{ad}_{[bc]}=0$.

Theorem 2. [5] The components of RC-tensor of Sasakian manifolds are given by:

1. $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}-2\delta^a_b\delta^d_c-\delta^a_c\delta^d_b$;
2. $R_{\hat{a}\hat{b}cd}=\delta^{ab}_{cd}=\delta^a_c\delta^b_d-\delta^a_d\delta^b_c$;
3. $R_{\hat{a}0b0}=\delta^a_b$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc}=A^{ad}_{[bc]}=0$.

Theorem 3. [5] The components of RC-tensor of Kenmotsu manifolds are given by:

1. $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}-\delta^a_c\delta^d_b$;
2. $R_{\hat{a}\hat{b}cd}=\delta^{ab}_{dc}=\delta^a_d\delta^b_c-\delta^a_c\delta^b_d$;
3. $R_{\hat{a}0b0}=-\delta^a_b$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc}=A^{ad}_{[bc]}=0$.

Theorem 4. [20] The components of RC-tensor of almost $C(\lambda)$-manifolds are given by:

1. $R_{\hat{a}\hat{b}cd}=\lambda\delta^{ab}_{cd}$;
2. $R_{\hat{a}0b0}=\lambda\delta^a_b$;
3. $R_{\hat{a}bc\hat{d}}-R_{\hat{a}cb\hat{d}}=-\lambda\delta^{ad}_{bc}$,

and the other components are vanish or given by Lemma 2, or their conjugates.

Theorem 5. [13] The components of RC-tensor of C_9-manifolds are given by:

1. $R_{0a\hat{b}0}=F_{ac}F^{cb}$;
2. $R_{0ab0}=-F_{ab0}$;
3. $R_{0ab\hat{c}}=-F_{ab}{}^{c}$;
4. $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}+F^{ad}F_{bc}$;
5. $R_{abcd}=-2F_{a[c}F_{|b|d]}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc}=A^{ad}_{[bc]}=0$, and F^{ab}, F_{ab}, F_{ab0}, $F_{ab}{}^{c}$ are components of Kirichenko's fifth structure tensor F (see [16]) and their covariant derivatives respectively.

Theorem 6. [14] The components of RC-tensor of C_{12}-manifolds are given by:

1. $C^a_b-C^a\,C_b=R_{\hat{a}0b0}$;

2. $C^{a\,b} - C^{a}\,C^{b} = R_{\hat{a}0\hat{b}0}$;
3. $A^{ad}_{bc} = R_{\hat{a}bc\hat{d}}$,

and the disappeared components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$, and C^{a}, C_{a}, $C^{a\,b}$, C^{a}_{b} are components of Kirichenko's sixth structure tensor G (see [16]) and their covariant derivatives respectively.

Theorem 7. [16] The components of RC-tensor over the manifolds of Kenmotsu type are seemed as follow:

1. $-\delta^{a}_{c} = R_{\hat{a}0c0}$;
2. $2A^{a}_{bcd} = R_{\hat{a}bcd}$;
3. $A^{ad}_{bc} - \delta^{a}_{c}\,\delta^{d}_{b} - B^{ah}{}_{c}\,B_{bh}{}^{d} = R_{\hat{a}bc\hat{d}}$;
4. $2\left(-\delta^{a}_{[c}\,\delta^{b}_{d]} + B^{ab}{}_{[cd]}\right) = R_{\hat{a}\hat{b}cd}$;
5. $-B^{ab}{}_{h}\,B^{hd}{}_{c} + B^{ab}{}_{c}{}^{d} = R_{\hat{a}\hat{b}c\hat{d}}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} and A^{a}_{bcd} are suitable smooth functions and $B^{ab}{}_{c}$, $B_{ab}{}^{c}$, $B^{ab}{}_{cd}$, $B^{ab}{}_{c}{}^{d}$ are components of Kirichenko's first structure tensor B (see [16]) and their covariant derivatives respectively.

Theorem 8. [6] The components of RC-tensor of LCAC-manifolds are appeared as follow:

1. $2\left(A^{a}_{bcd} - \alpha_0 B_{b[d}\delta^{a}_{c]} + 4\alpha^{[a}\delta^{h]}_{[c}B_{d]hb}\right) = R_{\hat{a}bcd}$;
2. $2\left(2\delta^{[b}_{[c}\,\alpha^{a]}_{d]} - \delta^{a}_{[c}\delta^{b}_{d]}\alpha_0^2 + 2B^{hab}B_{hdc}\right) = R_{\hat{a}\hat{b}cd}$;
3. $A^{ad}_{bc} - 4B^{dah}B_{chb} + 4\alpha^{[a}\delta^{h]}_{c}\alpha_{[h}\delta^{d}_{b]} - \delta^{a}_{c}\delta^{d}_{b}\alpha_0^2 + B^{ad}B_{bc} = R_{\hat{a}bc\hat{d}}$;
4. $2\left(2B_{[c|ab|d]} + B_{a[c}B_{d]b} - 2\alpha_{[a}B_{b]cd}\right) = R_{abcd}$;
5. $2\left(\alpha_{0[c}\delta^{a}_{d]} - 2\alpha^{[a}\delta^{h]}_{[c}B_{d]h} + B^{ab}B_{bcd}\right) = R_{\hat{a}0cd}$;
6. $A^{ac0}_{b} - \delta^{c}_{b}\alpha_0\alpha^{a} + \alpha_b B^{ac} = R_{\hat{a}b\hat{c}0}$;
7. $2B_{cab}\alpha_0 + 2B_{cab0} = R_{abc0}$;
8. $-\delta^{a}_{b}\alpha_{00} - B_{cb}B^{ac} - \delta^{a}_{b}\alpha_0^2 - \alpha^{a}\alpha_b - \alpha^{a}_{b} + 2\alpha^{[a}\delta^{c]}_{b}\alpha_c = R_{\hat{a}0b0}$;
9. $2\alpha_0 B^{ab} + 2B^{bac}\alpha_c - D^{ab0} - \alpha^{ab} - \alpha^{a}\alpha^{b} = R_{\hat{a}0\hat{b}0}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc}, A^{ac0}_{b} and A^{a}_{bcd} are suitable smooth functions, B^{abc}, B_{abc}, B_{abcd}, B_{abc0} are components of Kirichenko's second structure tensor C (see [16]) and their covariant derivatives respectively, B^{ab}, B_{ab}, D^{ab0} are the components of Kirichenko's third structure tensor D (see [21]) and their covariant derivatives respectively, α^{a}, α_{a}, α_0 are the components of $d\alpha$, α^{a}_{b}, α^{ab} are the components of $d\alpha^{a}$, and α_{00}, α_{0a} are the components of $d\alpha_0$.

Theorem 9. [4] The components of RC-tensor of quasi-Sasakian manifolds are given by:

1. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - 2B^{a}_{b}B^{d}_{c} - B^{a}_{c}B^{d}_{b}$;
2. $R_{\hat{a}b0c} = B^{a}_{bc}$;
3. $R_{\hat{a}b0\hat{c}} = B^{ac}_{b}$;
4. $R_{\hat{a}0b0} = B^{a}_{c}B^{c}_{b}$;
5. $R_{\hat{a}\hat{b}cd} = 2B^{a}_{[c}B^{b}_{d]}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$, and B^{a}_{b}, B^{ac}_{b}, B^{a}_{bc}, are components of Kirichenko's fourth structure tensor E (see [16]) and their covariant derivatives respectively.

Theorem 10. [2] The components of RC-tensor of CNK-manifolds are given by:

1. $R_{\hat{a}bcd} = 2A^{a}_{bcd}$;
2. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - 2B_{b}{}^{a}B_{c}{}^{d} - B_{c}{}^{a}B_{b}{}^{d} + B^{ah}{}_{c}B_{hb}{}^{d}$;
3. $R_{\hat{a}bc0} = -C^{a}_{bc} - B_{[b}{}^{h}B_{c]h}{}^{a}$;
4. $R_{\hat{a}0b0} = B_{b}{}^{h}B_{h}{}^{a}$;
5. $R_{\hat{a}\hat{b}cd} = 2\left(B^{ab}{}_{[dc]} + B_{[c}{}^{a}B_{d]}{}^{b}\right)$;
6. $R_{\hat{a}\hat{b}c0} = 2B_{h}{}^{[a}B^{b]h}{}_{c}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc}, A^{a}_{bcd} are suitable smooth functions, $B_{b}{}^{a}$, C^{a}_{bc} are the components of Kirichenko's fourth structure tensor E and their covariant derivatives respectively, and $B^{ab}{}_{c}$, $B_{ab}{}^{c}$, $B^{ab}{}_{cd}$ are the components of Kirichenko's first structure tensor B and their covariant derivatives respectively.

Theorem 11. [15] The components of RC-tensor of nearly Kenmotsu manifolds are given by:

1. $R_{\hat{a}bcd} = -\frac{2}{3}\delta^{a}_{b}F_{cd} + \frac{1}{3}\delta^{a}_{c}F_{db} + \frac{1}{3}\delta^{a}_{d}F_{bc}$;
2. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - C^{adh}C_{hbc} - \frac{1}{2}F^{ad}F_{bc} - \delta^{a}_{c}\delta^{d}_{b}$;
3. $R_{\hat{a}\hat{b}cd} = 2C^{abh}C_{hcd} + F^{ab}F_{cd} - 2\delta^{a}_{[c}\delta^{b}_{d]}$;
4. $R_{\hat{a}\hat{b}\hat{c}\hat{d}} = C^{acdb} - \frac{1}{2}\left(F^{ab}F^{cd} + F^{ac}F^{db} + F^{ad}F^{bc}\right)$;
5. $R_{\hat{a}00b} = F^{ac}F_{cb} + \delta^{a}_{b}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are suitable smooth functions, F^{ab}, F_{ab} are the components of Kirichenko's fifth structure tensor F, and C^{abc}, C_{abc},

C^{abcd} are the components of Kirichenko's second structure tensor C and their covariant derivatives respectively.

Theorem 12. [7] The components of RC-tensor of nearly cosymplectic manifolds are given by:

1. $R_{abcd} = -2B_{ab[cd]}$;
2. $R_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd}$;
3. $R_{\hat{a}0b0} = C^{ac}C_{bc}$;
4. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are suitable smooth functions, C^{ab}, C_{ab} are the components of Kirichenko's third structure tensor D, and B^{abc}, B_{abc}, B_{abcd} are the components of Kirichenko's second structure tensor C and their covariant derivatives respectively.

Conclusions

This paper collected the theories that determined the components of RC-tensors for 12 different classes of ACM-manifolds. So, the readers can be recognized the difference among these classes from the theorems in this paper. Then we concluded that the RC-tensor distinct according to its class.

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Conflict of Interest

The author declares no conflict of interest.

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مراجعة حول تنسر الانحناء الريماني لبعض فئات المنطويات المترية التلامسية تقريباً

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الخلاصة:

استعرض هذا البحث مركبات تنسر الانحناء الريماني على الفضاء المتعلق بالبنية – G لبعض فئات المنطويات المترية التلامسية تقريباً. الفئات التي تم دراستنا هي اثنتا عشرة فئة فقط والمعروفة بالاسماء منطويات كوسمبلكتك ومنطويات ساساكي ومنطويات كينموتسو ومنطويات C_9 – ومنطويات – C_{12} والمنطويات الطبيعية من النوع المعدوم ومنطويات كينموتسو التقريبي ومنطويات التحويل الكونفورمي المحلي للكوسمبلكتك تقريباً ومنطويات شبه ساساكي ومنطويات – $C(\lambda)$ تقريباً ومنطويات كوسمبلكتك التقريبي والمنطويات من نوع كينموتسو.

الكلمات المفتاحية : *منطويات كينموتسو، منطويات – $C(\lambda)$ ، منطويات ساساكي، منطويات كوسمبلكتك، منطويات اينشتاين.*