Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta
2022. Volume 59. Pp. 3–14

MSC2020: 53C25, 53D10, 53D15

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ON WEYL TENSOR OF ACR-MANIFOLDS OF CLASS C_{12} WITH APPLICATIONS

In this paper, we determine the components of the Weyl tensor of almost contact metric (ACR-) manifold of class C_{12} on associated G-structure (AG-structure) space. As an application, we prove that the conformally flat ACR-manifold of class C_{12} with $n > 2$ is an η-Einstein manifold and conclude that it is an Einstein manifold such that the scalar curvature r has provided. Also, the case when $n = 2$ is discussed explicitly. Moreover, the relationships among conformally flat, conformally symmetric, ξ-conformally flat and Φ-invariant Ricci tensor have been widely considered here and consequently we determine the value of scalar curvature r explicitly with other applications. Finally, we define new classes with identities analogously to Gray identities and discuss their connections with class C_{12} of ACR-manifold.

Keywords: almost contact metric manifold of class C_{12}, η-Einstein manifold, Weyl tensor.

DOI: 10.35634/2226-3594-2022-59-01

Introduction

Throughout this paper, we consider a Riemannian manifold with an odd dimension $2n+1$ and furnished by an almost contact structure (Φ, ξ, η). Chinea and Gonzalez [9] obtained a complete classification for ACR-manifold through the study of the covariant derivative of fundamental 2-forms on the manifolds in question and consequently, these classifications imply a new class which is called class C_{12} with the following condition:

$$\nabla_X(\Omega)(Y,Z) = \eta(X)\{\eta(Z)\nabla_\xi(\eta)\Phi Y - \eta(Y)\nabla_\xi(\eta)\Phi Z\} \quad \forall\ X,Y,Z \in X(M),$$

where ∇ is the Levi–Civita connection on M and $\Omega(X,Y) = g(X,\Phi Y)$. On the other hand, Bouzir et al. [6] studied the properties of the manifolds of class C_{12} when the dimension is 3, but the first author [2] determined the structure equations of Cartan for ACR-manifold of class C_{12} on the AG-structure space and determined the components of the Riemannian curvature tensor and Ricci tensor during this study; some of these results are given in the next section. Whereas, Candia and Falcitelli [7, 8], generalized the class C_{12} into the class $C_5 \oplus C_{12}$.

Moreover, as a quotation from the citation [19], we found that Weyl introduced a generalized curvature tensor which vanishes whenever the metric is conformally flat and this is why some times it is called conformal curvature tensor. The Weyl conformal curvature tensor field W is a tensor of type (3, 1) on ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ and is defined to be (see [11])

$$\begin{aligned} W(Z,U)Y &= R(Z,U)Y - \frac{1}{2n-1}[S(U,Y)Z - S(Z,Y)U + g(U,Y)LZ - g(Z,Y)LU] \\ &+ \frac{r}{2n(2n-1)}[g(U,Y)Z - g(Z,Y)U], \end{aligned} \tag{0.1}$$

for all $U,Y,Z \in X(M)$, where R is the Riemannian curvature tensor, $S(U,Z) = g(LU,Z)$ is the Ricci tensor, L denotes the Ricci operator and r is a scalar curvature. Moreover, the Weyl tensor W of type (4, 0) is defined by $W(X,Y,Z,U) = g(W(Z,U)Y,X)\ \forall\ X,Y,Z,U \in X(M)$. It is straightforward to show that the Weyl tensor possesses the same symmetries as the Riemannian tensor. However, Weyl tensor possesses very interesting property which is traceless tensor, in other words, it vanishes for any pair of contracted indices.

Geometrically, the Weyl tensor conveys information about the tidal force and shows how a body feels when moving along a geodesic. In fact, the major difference between the Weyl tensor and the Riemannian curvature tensor is that Weyl tensor does not decode information on how the volume of the body changes, but rather only how the shape (topology) of the body is distorted by the tidal force. More concretely, one could decompose the Riemannian curvature into trace and traceless parts which allows an easy proof that the Weyl curvature tensor is the conformally invariant part of the Riemannian curvature. So, without the slightest doubt, the Weyl tensor is no less important than the Riemannian curvature tensor from geometrical point of view.

In this light and as the trace part (Riemannian curvature) has been studied in [2], it would be interesting and reasonable to study the Weyl curvature tensor (conformally invariant part) of ACR-manifold of class C_{12} to complete the geometrical (trace and traceless parts) picture of ACR-manifold of class C_{12}. On the other hand, Hwang and Yun [13] studied the Weyl curvature tensor that is weakly harmonic with some conditions. Whereas, Blair and Yıldırım [5] discussed the conformally flat for another class.

The paper is structured as follows. In Section 1, we recall some definitions and theorems about ACR-manifold. In Section 2, we calculate the components of the Weyl tensor on AG-structure and its relation with η-Einstein manifold as an application. Also, we focus on ξ-conformally flat manifold of the class C_{12} and the result shows it is Φ-invariant Ricci tensor; then the scalar curvature is calculated explicitly. In Section 3, interesting theorems are obtained on the discussions of the contact analog of Gray identities on Riemannian curvature tensor of the class C_{12} and their generalization to Weyl tensor.

As a future work, the authors can develop this work in the direction of the citations [4, 10, 12, 18, 20].

§ 1. Preliminaries

We denote by M^{2n+1} and g, the smooth manifold M of dimension $2n+1$ and the Riemannian metric respectively.

D e f i n i t i o n 1.1 (see [3, 15]). A Riemannian manifold (M^{2n+1}, g) is called an *ACR-manifold* if it is supplied by a structure of triple (ξ, η, Φ), where Φ is a (1, 1)-tensor over M, ξ is a vector field on M and η is a 1-form of M, such that $\forall\ U, V \in X(M)$, the following hold:

$$\Phi(\xi) = 0; \quad \eta(\xi) = 1; \quad \eta \circ \Phi = 0; \quad \Phi^2 + \mathrm{id} = \eta \otimes \xi;$$

$$g(\Phi U, \Phi V) + \eta(U)\eta(V) = g(U, V).$$

Note that $X(M)$ is the module of all vector fields on M. On the other hand, for the background of AG-structure space, the researchers can refer to the citation [15, 17]. Moreover, on AG-structure space, the tensors g and Φ of ACR-manifold M^{2n+1} are given by the following [15]:

$$(g_{kl}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & O & I_n \\ 0 & I_n & O \end{pmatrix}; \qquad (\Phi^k_l) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & O \\ 0 & O & -\sqrt{-1}I_n \end{pmatrix}; \tag{1.1}$$

where $k, l = 0, 1, \ldots, 2n$ and I_n is $n \times n$ identity matrix.

T h e o r e m 1.1 (see [2]). *The components of Riemann curvature tensor R over AG-structure space of the class C_{12} with dimension $2n+1$ are given by:*

1) $R^a_{0b0} + C^a\ C_b = C^a_b$;

2) $R^a_{0\hat{b}0} + C^a\ C^b = C^{ab}$;

3) $R^{b}_{ad\hat{c}} = A^{bc}_{ad}$,

and the other components are 0 or can be obtained by the features of R *or the conjugates (i. e.,* $\overline{R^{i}_{jkl}} = R^{\hat{i}}_{\hat{j}\hat{k}\hat{l}}$*) to the above components, where* $a, b, c, d = 1, 2, \ldots, n$*,* $\hat{a} = a + n$*,* $A^{[ad]}_{bc} = A^{ad}_{[bc]} = C_{[bd]} = C^{[bd]} = 0$ *and* C^{a}, C_{a} *are the components of* 6^{th} *structure tensor* G *(see* [16]*).*

T h e o r e m 1.2 (see [2]). *The components of Ricci tensor* S *over AG-structure space that coming from the class* C_{12} *with dimension* $2n+1$ *are provided as follows:*

1) $S_{00} + 2C^{a}\, C_{a} = 2C^{a}_{a}$;

2) $S_{a0} = 0$;

3) $S_{ab} + C_{a}\, C_{b} = C_{ab}$;

4) $S_{\hat{a}b} + C^{a}\, C_{b} = C^{a}_{b} + A^{ac}_{cb}$,

and the remaining components are set by the symmetries or conjugates to the above components.

D e f i n i t i o n 1.2 (see [19]). An ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ is called

(i) *ξ-conformally* flat if $W(X, \xi, Y, Z) = 0$;

(ii) *conformally symmetric* if $W(X, Y, Z, \Phi U) = 0$;

(iii) *$\Phi-$conformally flat* if $W(\Phi U, \Phi X, \Phi Y, \Phi Z) = 0$,

for all $U, X, Y, Z \in X(M)$.

D e f i n i t i o n 1.3 (see [17]). An ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ is called

(i) *of class* CR_1 if $g(R(\Phi U, \Phi X)\Phi Y, \Phi Z) = g(R(\Phi^2 U, \Phi^2 X)\Phi Y, \Phi Z)$;

(ii) *of class* CR_2 if

$$g(R(\Phi X, \Phi Y)\Phi Z, \Phi U) = g(R(\Phi^2 X, \Phi^2 Y)\Phi Z, \Phi U) + g(R(\Phi^2 X, \Phi Y)\Phi^2 Z, \Phi U) + g(R(\Phi^2 X, \Phi Y)\Phi Z, \Phi^2 U);$$

(iii) *of class* CR_3 if $g(R(\Phi U, \Phi X)\Phi Y, \Phi Z) = g(R(\Phi^2 U, \Phi^2 X)\Phi^2 Y, \Phi^2 Z)$,

for all $X, U, Y, Z \in X(M)$. Moreover, over AG-structure space, the aforementioned classes are equivalent to the following:

$$CR_1 \Longleftrightarrow R_{\hat{a}bcd} = R_{abcd} = R_{\hat{a}\hat{b}cd} = 0;$$
$$CR_2 \Longleftrightarrow R_{\hat{a}bcd} = R_{abcd} = 0;$$
$$CR_3 \Longleftrightarrow R_{\hat{a}bcd} = 0.$$

D e f i n i t i o n 1.4 (see [14]). An ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ is said to be *η-Einstein manifold* if the Ricci tensor S of M attains the following equation:

$$S(U, V) = \alpha\, g(U, V) + \beta\, \eta(U)\, \eta(V) \quad \forall\, U, V \in X(M),$$

where $\alpha, \beta \in C^{\infty}(M)$, (the set of all smooth functions on M). In particular, if $\beta = 0$ then M becomes Einstein manifold.

D e f i n i t i o n 1.5 (see [15]). An ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ has *Φ-invariant Ricci tensor property* if it satisfies the condition:

$$S(\Phi U, V) + S(U, \Phi V) = 0 \quad \forall\, U, V \in X(M).$$

L e m m a 1.1 (see [1]). *An ACR-manifold* $(M^{2n+1}, g, \Phi, \xi, \eta)$ *possesses Φ-invariant Ricci tensor, if and only if, the components* S_{a0}, S_{ab} *and their conjugates vanish, where* $a, b = 1, 2, \ldots, n$.

§ 2. The geometry of Weyl tensor on class C_{12}

In this section, we discuss the geometry of Weyl tensor on class C_{12} as below.

T h e o r e m 2.1. *The components of Weyl tensor on AG-structure space of the class C_{12} are given by:*

1) $W_{\hat{a}0c0} = C^a_c - C^a C_c - \frac{1}{2n-1}\{\delta^a_c\ S_{00} + S_{\hat{a}c}\} + \frac{r}{2n(2n-1)}\delta^a_c;$

2) $W_{\hat{a}0\hat{c}0} = C^{ac} - C^a C^c - \frac{1}{2n-1} S_{\hat{a}\hat{c}};$

3) $W_{\hat{a}bcd} = \frac{1}{2n-1}\{S_{bc}\ \delta^a_d - S_{bd}\ \delta^a_c\};$

4) $W_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - \frac{1}{2n-1}\{S_{b\hat{d}}\ \delta^a_c + S_{\hat{a}c}\ \delta^d_b\} + \frac{r}{2n(2n-1)}\delta^a_c\ \delta^d_b;$

5) $W_{\hat{a}b\hat{c}\hat{d}} = \frac{1}{2n-1}\{S_{\hat{a}\hat{d}}\ \delta^c_b - S_{\hat{a}\hat{c}}\ \delta^d_b\};$

6) $W_{\hat{a}\hat{b}cd} = \frac{1}{2n-1}\{S_{\hat{b}c}\ \delta^a_d - S_{\hat{b}d}\ \delta^a_c - S_{\hat{a}c}\ \delta^b_d + S_{\hat{a}d}\ \delta^b_c\} + \frac{r}{2n(2n-1)}\{\delta^b_d\ \delta^a_c - \delta^b_c\ \delta^a_d\},$

and the other components are 0 or obtained by the features of W or conjugates to the above components.

P r o o f. Suppose that $(M^{2n+1}, g, \Phi, \xi, \eta)$ is an ACR-manifold of class C_{12}, then according to equation (0.1) and for each $X, Y, Z, U \in X(M)$, we get

$$\begin{aligned} W(X,Y,Z,U) &= g(W(Z,U)Y,X) = \\ &= R(X,Y,Z,U) - \frac{1}{2n-1}\{S(Y,U)g(X,Z) \\ &- S(Y,Z)g(X,U) + g(Y,U)S(X,Z) - g(Y,Z)S(X,U)\} \\ &+ \frac{r}{2n(2n-1)}\{g(Y,U)g(X,Z) - g(Y,Z)g(X,U)\}. \end{aligned}$$

So, the components of Weyl tensor over AG-structure space are given by:

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{2n-1}\{S_{jl}\ g_{ik} - S_{jk}\ g_{il} + g_{jl}\ S_{ik} - g_{jk}\ S_{il}\} \\ &+ \frac{r}{2n(2n-1)}\{g_{jl}\ g_{ik} - g_{jk}\ g_{il}\}, \end{aligned} \tag{2.1}$$

where $i, j, k, l = 0, 1, \ldots, 2n$. Now, we choose $i = 0, a, \hat{a}$, $j = 0, b, \hat{b}$, $k = 0, c, \hat{c}$ and $l = 0, d, \hat{d}$, where $a, b, c, d = 1, 2, \ldots, n$ and $\hat{a}, \hat{b}, \hat{c}, \hat{d} = n+1, n+2, \ldots, 2n$. If we take all possible cases of indexes i, j, k, l, then we have only six cases and their conjugates or symmetries in which W_{ijkl} does not vanish and these six cases are

$$(i,j,k,l) \in \{(\hat{a},0,c,0),\ (\hat{a},0,\hat{c},0),\ (\hat{a},b,c,d),\ (\hat{a},b,c,\hat{d}),\ (\hat{a},b,\hat{c},\hat{d}), (\hat{a},\hat{b},c,d)\}.$$

So, using the components of R in Theorem 1.1, components of S in Theorem 1.2 and components of g in equation (1.1) and substituting them in equation (2.1), we arrive to the results by taking into account $R^i_{jkl} = R_{\hat{i}jkl}$.

C o r o l l a r y 2.1. *If the ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ is of class C_{12}, then for all X, Y, Z, $U \in X(M)$, we have*

$$W(\Pi(X), \Pi(Y), \Pi(Z), \Pi(U)) = W(\overline{\Pi}(X), \overline{\Pi}(Y), \overline{\Pi}(Z), \overline{\Pi}(U)) = 0,$$

where $\Pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$, and $\overline{\Pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$.

P r o o f. Since on AG-structure space the following are equivalent:

$$W(\Pi(X), \Pi(Y), \Pi(Z), \Pi(U)) \Longleftrightarrow W_{abcd}; \quad a, b, c, d = 1, 2, \ldots, n,$$
$$W(\overline{\Pi}(X), \overline{\Pi}(Y), \overline{\Pi}(Z), \overline{\Pi}(U)) \Longleftrightarrow W_{\hat{a}\hat{b}\hat{c}\hat{d}}; \quad \hat{a}, \hat{b}, \hat{c}, \hat{d} = n+1, n+2, \ldots, 2n.$$

Then Theorem 2.1 gives the results.

T h e o r e m 2.2. *If the ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ of class C_{12} having $n > 2$, is conformally flat then it is η-Einstein manifold with $\alpha = \frac{1}{2n-4}(S_{00} - \frac{r}{n})$ and $\beta = S_{00} - \alpha$.*

P r o o f. Suppose that M is conformally flat, then $W_{ijkl} = 0$ for all $i, j, k, l = 0, 1, \ldots, 2n$. Then we get from Theorem 2.1 that $W_{\hat{a}0c0} = 0$, and $W_{\hat{a}bc\hat{d}} = 0$. Then replacing c by b in the first and contracting (c, d) in the second we get respectively the following:

$$S_{\hat{a}b} = (2n-1)\{C^a_b - C^a C_b\} - \delta^a_b\ S_{00} + \frac{r}{2n}\delta^a_b. \tag{2.2}$$

$$2S_{\hat{a}b} = (2n-1)A^{ac}_{bc} + \frac{r}{2n}\delta^a_b. \tag{2.3}$$

Now, adding equations (2.2), (2.3), using the fact $A^{ac}_{bc} = A^{ac}_{cb}$ and then from Theorem 1.2, item 4, we obtain $S_{\hat{a}b} = \frac{1}{2n-4}(S_{00} - \frac{r}{n})\delta^a_b$, if $n > 2$. But, $W_{\hat{a}0\hat{c}0} = 0$ gives $S_{\hat{a}\hat{c}} = (2n-1)\{C^{ac} - C^a C^c\}$ and from Theorem 1.2, item 3, we have $S_{\hat{a}\hat{c}} = 0$. Also, we note that the remaining items of Theorem 2.1 satisfy the previous results. But, Definition 1.4 leads M to be an η-Einstein manifold having $S_{00} = \alpha + \beta$ and $\alpha = \frac{1}{2n-4}(S_{00} - \frac{r}{n})$.

C o r o l l a r y 2.2. *If the ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ of the class C_{12} with $n > 2$ is conformally flat then it is Einstein manifold with scalar curvature $r = -n(2n-5)S_{00}$.*

P r o o f. If M is conformally flat, then M is η-Einstein manifold according to Theorem 2.2. But if $\beta = 0$, then M is Einstein manifold with $S_{00} = \alpha = \frac{1}{2n-4}(S_{00} - \frac{r}{n})$, and this implies that the value of r is given.

C o r o l l a r y 2.3. *If the ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ of the class C_{12} is conformally flat then it has Φ-invariant Ricci tensor.*

P r o o f. If M is conformally flat then Theorems 1.2 and 2.2 yields $S_{a0} = S_{ab} = 0$. But, Lemma 1.1 attains the requirement.

C o r o l l a r y 2.4. *If the ACR-manifold $(M^5, g, \Phi, \xi, \eta)$ of class C_{12} is conformally flat then $r = 2S_{00}$ and $A^{ac}_{ca} = 0$.*

P r o o f. Suppose M^5 is conformally flat, then regarding the proof of Theorem 2.2, we have $r = 2S_{00}$. But $r = 2S_{\hat{a}a} + S_{00} \Longrightarrow S_{\hat{a}a} = \frac{1}{2}S_{00}$. Thus, contracting item 4, in Theorem 1.2, we get $A^{ac}_{ca} = 0$.

T h e o r e m 2.3. *The ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ of class C_{12} is conformally symmetric, if and only if, it is conformally flat.*

P r o o f. If M is conformally flat, then $W(X, Y, Z, U) = 0$ for all $X, Y, U, Z \in X(M)$. So, if we replace U by ΦU, then also we have $W(X, Y, Z, \Phi U) = 0$ and this implies that M is conformally symmetric according to Definition 1.2.

Conversely, suppose that M is conformally symmetric. Then according to Definition 1.2, we have

$$W(X, Y, Z, \Phi U) = 0 \quad \forall\ X, Y, Z, U \in X(M),$$
$$W_{ijkt}\ X^i\ Y^j\ Z^k\ \Phi^t_l\ U^l = 0 \quad i, j, k, l, t = 0, 1, \ldots, 2n,$$
$$W_{ijkt}\ \Phi^t_l = 0.$$

Considering the components of Φ in equation (1.1), we get:

$$W_{ijkd}=W_{ijk\hat{d}}=0;\quad d=1,2,\dots,n,\quad \hat{d}=d+n.$$

Now, from the symmetries of W, we obtain that all the components of W in Theorem 2.1 must be vanishing. Thus M is conformally flat.

R e m a r k 2.1. Regarding Theorem 2.3, we discover the previous theorems and corollaries are stay valid if we replace the statement "conformally flat" by "conformally symmetric".

T h e o r e m 2.4. *If the ACR-manifold $(M^{2n+1},g,\Phi,\xi,\eta)$ of the class C_{12} is ξ-conformally flat with $A^{ab}_{bc}=\gamma\ \delta^a_c$, and $\gamma\in C^\infty(M)$ then it is η-Einstein manifold, with*

$$\alpha=\frac{1}{2n-2}\{S_{00}+(2n-1)\gamma-\frac{r}{2n}\},\quad \beta=S_{00}-\alpha.$$

P r o o f. Suppose that M is ξ-conformally flat with $A^{ab}_{bc}=\gamma\ \delta^a_c$, then we pay attention to Definition 1.2 and obtain $W(X,\xi,Y,Z)=0$. Thus, on AG-structure space, we have $W_{i0jk}=0$, where $i,j,k=0,1,\dots,2n$. Taking into account Theorem 2.1, we get $W_{\hat{a}0c0}=W_{\hat{a}0\hat{c}0}=0$, and this implies that $S_{\hat{a}\hat{c}}=0$ and $S_{\hat{a}c}=(2n-1)\{C^a_c-C^aC_c\}-\delta^a_c\ S_{00}+\frac{r}{2n}\delta^a_c$. Moreover, from Theorem 1.2, we have $S_{\hat{a}c}=(2n-1)\{S_{\hat{a}c}-A^{ab}_{bc}\}-\delta^a_c\ S_{00}+\frac{r}{2n}\delta^a_c$. Therefore,

$$(2n-2)S_{\hat{a}c}=(2n-1)A^{ab}_{bc}+\{S_{00}-\frac{r}{2n}\}\delta^a_c.$$

Since $A^{ab}_{bc}=\gamma\ \delta^a_c$, then M is η-Einstein manifold, with

$$\alpha=\frac{1}{2n-2}\{S_{00}+(2n-1)\gamma-\frac{r}{2n}\},\quad \beta=S_{00}-\alpha.$$

C o r o l l a r y 2.5. *If the ξ-conformally flat ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ of class C_{12} with $A^{ab}_{bc}=\gamma\ \delta^a_c$, is Einstein manifold, where $\gamma\in C^\infty(M)$, then the scalar curvature*

$$r=-2n\{(2n-3)S_{00}-(2n-1)\gamma\}.$$

P r o o f. Suppose M is Einstein manifold, then from Theorem 2.4, we have $\beta=0$ and this implies that $S_{00}=\alpha=\frac{1}{2n-2}\{S_{00}+(2n-1)\gamma-\frac{r}{2n}\}$. So, the last equation gives:

$$r=-2n\{(2n-3)S_{00}-(2n-1)\gamma\}.$$

C o r o l l a r y 2.6. *Every ξ-conformally flat ACR-manifold $(M^{2n+1},g,\Phi,\xi,\eta)$ of class C_{12} has Φ-invariant Ricci tensor.*

P r o o f. Suppose that M is ξ-conformally flat, then $W_{\hat{a}0c0}=W_{\hat{a}0\hat{c}0}=0$. Thus, from Theorems 1.2 and 2.1, we deduce that $S_{a0}=S_{ab}=0$. Then we establish the desired.

T h e o r e m 2.5. *If the ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ of class C_{12} is $\Phi-$conformally flat with $A^{ac}_{cb}=\gamma\ \delta^a_b$, then it is η-Einstein manifold with $\alpha=\frac{r}{4n}+(2n-1)\frac{\gamma}{2}$, and $\beta=S_{00}-\alpha$.*

P r o o f. Suppose that M is Φ-conformally flat, then from Definition 1.2, we have

$$W(\Phi X,\Phi Y,\Phi Z,\Phi U)=0\quad \forall\ X,Y,Z,U\in X(M),$$
$$W_{ijkl}\ (\Phi X)^i\ (\Phi Y)^j\ (\Phi Z)^k\ (\Phi U)^l=0,\quad i,j,k,l=0,1,\dots,2n,$$
$$W_{ijkl}\ \Phi^i_{t_1}\ \Phi^j_{t_2}\ \Phi^k_{t_3}\ \Phi^l_{t_4}=0,\quad t_1,t_2,t_3,t_4=0,1,\dots,2n.$$

According to the above equations and the components of Φ in equation (1.1), we establish $W_{ijkl}=0$ for $i,j,k,l=1,2,\dots,2n$. Regarding Theorem 2.1, we acquire

$$W_{\hat{a}bcd}=W_{\hat{a}bc\hat{d}}=W_{\hat{a}b\hat{c}\hat{d}}=W_{\hat{a}\hat{b}cd}=0.$$

Regarding the proof of Theorem 2.2, we attain $S_{ab}=0$, and $2S_{\hat{a}b}=(2n-1)A^{ac}_{cb}+\frac{r}{2n}\delta^a_b$. Since $A^{ac}_{cb}=\gamma\ \delta^a_b$, then M is η-Einstein manifold having $\alpha=\frac{r}{4n}+(2n-1)\frac{\gamma}{2}$, and $\beta=S_{00}-\alpha$.

C o r o l l a r y 2.7. *If the Φ-conformally flat ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ of class C_{12} with $A^{ac}_{cb} = \gamma\ \delta^a_b$, then it is Einstein manifold with the scalar curvature $r = 4nS_{00} - 2n(2n-1)\gamma$.*

P r o o f. Suppose that M is Φ-conformally flat. The consideration of Theorem 2.5 gives M to be Einstein manifold if $\beta = 0$, and then $S_{00} = \alpha = \frac{r}{4n} + (2n-1)\frac{\gamma}{2}$. Thus, $r = 4nS_{00} - 2n(2n-1)\gamma$.

C o r o l l a r y 2.8. *If the ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ of class C_{12} is Φ-conformally flat, then it possesses Φ-invariant Ricci tensor.*

P r o o f. Suppose that M is Φ-conformally flat. Then the proof of Theorem 2.5 gives $S_{ab} = 0$, and from Theorem 1.2 we have $S_{a0} = 0$. Then Lemma 1.1 produces the claim of this corollary.

C o r o l l a r y 2.9. *The ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ of class C_{12} is conformally flat, if and only if, it is ξ-conformally flat and Φ-conformally flat.*

P r o o f. The assertion of this corollary is achieved from Theorems 2.1, 2.2, 2.4, and 2.5.

§ 3. The contact analogs of Gray identities on class C_{12}

In this section, we discuss the contact analogs of Gray identities on the Riemannian curvature tensor of the class C_{12} and their generalization to Weyl tensor.

T h e o r e m 3.1. *The classes CR_1, CR_2, and CR_3 are equivalent on the ACR-manifold M of class C_{12}.*

P r o o f. Suppose that M is ACR-manifold of class C_{12}. Then under Theorem 1.1, and Definition 1.3, we have

$$\begin{aligned} R_{\hat{a}bcd} = R_{abcd} = R_{\hat{a}\hat{b}cd} = 0. &\Longrightarrow M \in CR_1; \\ R_{\hat{a}bcd} = R_{abcd} = 0. &\Longrightarrow M \in CR_2; \\ R_{\hat{a}bcd} = 0. &\Longrightarrow M \in CR_3. \end{aligned}$$

Then the classes CR_1, CR_2, and CR_3 are equivalent on M.

D e f i n i t i o n 3.1. An ACR-manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$ is called

(i) *of class CW_1* if $g(W(\Phi U, \Phi X)\Phi Y, \Phi Z) = g(W(\Phi^2 U, \Phi^2 X)\Phi Y, \Phi Z)$;

(ii) *of class CW_2* if

$$\begin{aligned} g(W(\Phi X, \Phi Y)\Phi Z, \Phi U) = g(W(\Phi^2 X, \Phi^2 Y)\Phi Z, \Phi U) + g(W(\Phi^2 X, \Phi Y)\Phi^2 Z, \Phi U) \\ + g(W(\Phi^2 X, \Phi Y)\Phi Z, \Phi^2 U); \end{aligned}$$

(iii) *of class CW_3* if $g(W(\Phi X, \Phi Y)\Phi Z, \Phi U) = g(W(\Phi^2 X, \Phi^2 Y)\Phi^2 Z, \Phi^2 U)$,

for all $X, Y, Z, U \in X(M)$.

Now, since the Weyl tensor has the same properties as the Riemann curvature tensor, then from Definition 1.3, we get the following lemma.

L e m m a 3.1. *On AG-structure space, the above classes are equivalent to the following:*

$$\begin{aligned} &CW_1 \Longleftrightarrow W_{\hat{a}bcd} = W_{abcd} = W_{\hat{a}\hat{b}cd} = 0; \\ &CW_2 \Longleftrightarrow W_{\hat{a}bcd} = W_{abcd} = 0; \\ &CW_3 \Longleftrightarrow W_{\hat{a}bcd} = 0. \end{aligned}$$

Interesting relations with η-Einstein manifolds and Φ-invariant Ricci tensor are given in the following theorems.

T h e o r e m 3.2. *If the ACR-manifold* $(M^{2n+1}, g, \Phi, \xi, \eta)$ *of class* C_{12} *belongs to the class* CW_1*, then it is* η*-Einstein manifold with* $\alpha = \frac{1}{2n-4}\{S_{00} - \frac{r}{n}\}$ *and* $\beta = S_{00} - \alpha$*, provided that* $n > 2$*.*

P r o o f. Suppose that $M \in C_{12}$ and $M \in CW_1$, then from Lemma 3.1, we have $W_{\hat{a}bcd} = W_{abcd} = W_{\hat{a}\hat{b}cd} = 0$. According to Theorem 2.1, we get:

$$0 = \frac{1}{2n-1}\{S_{bc}\,\delta^a_d - S_{bd}\,\delta^a_c\},$$
$$0 = \frac{1}{2n-1}\{S_{\hat{b}c}\,\delta^a_d - S_{\hat{b}d}\,\delta^a_c - S_{\hat{a}c}\,\delta^b_d + S_{\hat{a}d}\,\delta^b_c\} + \frac{r}{2n(2n-1)}\{\delta^b_d\,\delta^a_c - \delta^b_c\,\delta^a_d\}.$$

Contracting the above equations with respect to the indexes (a, d), we obtain:

$$S_{bc} = 0,$$
$$S_{\hat{b}c} = \frac{1}{n-2}\{\frac{r(n-1)}{2n} - S_{\hat{a}a}\}\delta^b_c.$$

Since $r = 2S_{\hat{a}a} + S_{00}$, then M is η-Einstein manifold having $\alpha = \frac{1}{2n-4}\{S_{00} - \frac{r}{n}\}$ and $\beta = S_{00} - \alpha$.

C o r o l l a r y 3.1. *If the ACR-manifold* $(M^{2n+1}, g, \Phi, \xi, \eta)$*, having* $n > 2$ *belongs to the classes* C_{12} *and* CW_1 *then it is Einstein manifold with* $r = -n(2n-5)S_{00}$*.*

P r o o f. Using Theorem 3.2, we conclude that M is Einstein manifold if $\beta = 0$, and then $S_{00} = \alpha = \frac{1}{2n-4}\{S_{00} - \frac{r}{n}\}$. Thus, we obtain the result.

C o r o l l a r y 3.2. *If the ACR-manifold* $(M^{2n+1}, g, \Phi, \xi, \eta)$*, having* $n > 2$ *belongs to the classes* C_{12} *and* CW_1 *then it possesses* Φ*-invariant Ricci tensor.*

P r o o f. According to the proof of Theorem 3.2, we attain the claim of this corollary.

T h e o r e m 3.3. *If the ACR-manifold* $(M^5, g, \Phi, \xi, \eta)$ *belongs to the classes* C_{12} *and* CW_1*, then it possesses* Φ*-invariant Ricci tensor and* $r = 2S_{00}$*.*

P r o o f. Consider $M \in C_{12}$ and $M \in CW_1$, then with the proof of Theorem 3.2, we have $S_{ab} = 0$ and

$$r = 4S_{\hat{a}a}. \Longrightarrow r = 2r - 2S_{00}. \Longrightarrow r = 2S_{00}.$$

So, this completes the proof.

C o r o l l a r y 3.3. *If the ACR-manifold* $(M^5, g, \Phi, \xi, \eta)$ *belongs to the classes* C_{12} *and* CW_1*, then* $A^{ab}_{ba} = 0$*.*

P r o o f. Suppose that $M \in C_{12}$ and $M \in CW_1$. Note that $r = 2S_{\hat{a}a} + S_{00}$. Applying Theorem 3.3, we have $S_{\hat{a}a} = \frac{1}{2}S_{00}$. So, by using item 4 of Theorem 1.2, we obtain the result.

T h e o r e m 3.4. *The ACR-manifold* $(M^{2n+1}, g, \Phi, \xi, \eta)$ *of class* C_{12} *belongs to the class* CW_2*, if and only if, it has* Φ*-invariant Ricci tensor.*

P r o o f. Consider $M \in C_{12}$ and $M \in CW_2$, then $W_{\hat{a}bcd} = W_{abcd} = 0$ and this implies that $S_{ab} = 0$ according to Theorem 2.1. Thus M has Φ-invariant Ricci tensor according to the combination of Theorem 1.2, Lemma 1.1, and the consequence obtained. Conversely, if M has Φ-invariant Ricci tensor, then $S_{a0} = S_{ab} = 0$. Thus, combining the previous result with the Theorem 2.1, we have $W_{\hat{a}bcd} = W_{abcd} = 0$. So, M belongs to the class CW_2.

Т h e o r e m 3.5. *The ACR-manifold* $(M^{2n+1}, g, \Phi, \xi, \eta)$ *of class* C_{12} *belongs to the class* CW_3*, if and only if, it possesses* Φ*-invariant Ricci tensor.*

P r o o f. Consider $M \in C_{12}$ and $M \in CW_3$, then $W_{\hat{a}bcd} = 0$ and this implies that $S_{ab} = 0$ under Theorem 2.1. Thus M has Φ-invariant Ricci tensor according to the combination of Theorem 1.2, Lemma 1.1, and the resulting consequence. Conversely, if M has Φ-invariant Ricci tensor, then we apply Lemma 1.1 and get $S_{a0} = S_{ab} = 0$. So, Theorem 2.1, item 3, yields $W_{\hat{a}bcd} = 0$. Thus, we conclude the implication of this theorem.

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Received 11.01.2022
Accepted 25.04.2022

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Citation: M. Y. Abass, Q. S. A. Al-Zamil. On Weyl tensor of ACR-manifolds of class C_{12} with applications, *Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta*, 2022, vol. 59, pp. 3–14.

М. Ю. Абасс, К. С. А. Аль-Замиль
О тензоре Вейля ACR-многообразий класса C_{12} с приложениями

Ключевые слова: почти контактное метрическое многообразие класса C_{12}, η-эйнштейновское многообразие, тензор Вейля.

УДК: 514.77

DOI: 10.35634/2226-3594-2022-59-01

В данной работе мы определяем компоненты тензора Вейля почти контактного метрического (ACR-) многообразия класса C_{12} на ассоциированном пространстве G-структуры (AG-структуры). В качестве приложения мы доказываем, что конформно плоское ACR-многообразие класса C_{12} с $n > 2$ является η-эйнштейновским многообразием и заключаем, что это эйнштейновское многообразие такое, что скалярная кривизна r обеспечена. Также в явном виде обсуждается случай, когда $n = 2$. Более того, здесь широко рассмотрены отношения между конформно плоским, конформно симметричным, ξ-конформно плоским и Ф-инвариантным тензором Риччи, и поэтому мы определяем значение скалярной кривизны r в явном виде с другими приложениями. Наконец, мы определяем новые классы с тождествами, аналогичными тождествам Грея, и обсуждаем их связь с классом C_{12} ACR-многообразий.

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Поступила в редакцию 11.01.2022
Принята в печать 25.04.2022

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Цитирование: М. Ю. Абасс, К. С. А. Аль-Замиль. О тензоре Вейля ACR-многообразий класса C_{12} с приложениями // Известия Института математики и информатики Удмуртского государственного университета. 2022. Т. 59. С. 3–14.