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On some class of Bernstein – type operators which represent a generalization of standard Bernstein operators

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Abstract: The purpose of this paper is to introduce and study a new sequence of linear positive operators $\overline{\mathcal{B}}_{n,r}$ which represent a generalization of Bernstein – type operators to approximate a function belong to the space $C[0,1]$. Our goal is to study approximation properties of this sequence and study the uniformly convergence of it on some continous functions on a compact set $[0,1]$. We find a rate of this covnvergence using a modulus of continuity. Finaly we establish a formula of a Voronoviskaja- type asymptotic formula on $\overline{\mathcal{B}}_{n,r}$.

MSC: Primary41A25; 41A35; 41A36.

Key words: Bernstein operators, Korovkin theorem, Modified Bernstein basis polynomial, Linear positive operators, p-th order moment, Modulus of continuity, Voronovikaja theorem.

1. INTRODUCTION

In 1885[14], Weierstrassintroduced his famous theorem (the fandemantal theorem in approximation). There are some proofs of it one of them which simple and important is Bernstein proof, in 1912 [2] the Russian mathematician S. N. Bernstein gave his proof on a compact set when he defined a sequence of polynomials that approximate all the function $f \in C[0,1]$. The standard Bernstein operators are defined as follow [1],[5]:

$$\mathcal{B}_n: C[0,1] \to C[0,1],$$

$$\mathcal{B}_n(f(x);x) = \sum_{k=0}^{n} \mathfrak{b}_{n,k}(x)\, f(x)$$

$$\mathfrak{b}_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}\, f(x)$$

where $x \in [0,1]$ and n some integer. There are some known facts on this operators we need them in our work we review in the next lemmas and theorems:

Lemma 1.1[4], [8],[6], [10]: Suppose$f \in C[0,1]$for every $x \in [0,1]$**;** for Bernstein operators

$$\mathcal{B}_n(f(x);x) = \sum_{k=0}^{n} \mathfrak{b}_{n,k}(x)\, f(x)$$

$$\mathfrak{b}_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} f(x)$$

Then Korovkin's conditions and the$p$-th order momentcan be obtained as bellow

1- $\mathcal{B}_n(1;x) = 1$
2- $\mathcal{B}_n(t;x) = x$;
3- $\mathcal{B}_n(t^2;x) = \left(1 - \frac{1}{n}\right) x^2 + \frac{x}{n}$
4- $\mathcal{B}_n(t^3;x) = \left(\frac{n^2-3n+2}{n^2}\right) x^3 + \left(\frac{n+1}{n^2}\right) x^2 + \frac{x}{n^2}$
5- $\mathcal{T}_{n,0}(x) = 1, \mathcal{T}_{n,1}(x) = 0$
6- $n\mathcal{T}_{n,p+1}(x) = x(1-x)\left[\mathcal{T}'_{n,p}(x) + p\mathcal{T}_{n,p-1}(x)\right]$, where $\mathcal{T}_{n,p}(x) \equiv \mathcal{B}_n((t-x)^p;x)$ represent the p-th order moment for the operators $\mathcal{B}_n(f;x)$.

We interest always to discussing a Voronoviskaja theorem for approximation [12], so we must to offer the next lemma.

Lemma 1.2:[12] Let $\mathfrak{b}_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$ the weight of Bernstein Polynomials[2],then we have:

for$\delta > 0$ and $x \in [0,1]$. If $\left|\frac{k}{n} - x\right| \geq \delta$then $\sum_{\left|\frac{k}{n}-x\right| \geq \delta} \mathfrak{b}_{n,k}(x) \leq \frac{1}{4n\delta^2}$.

Voronoviskaja Theorem 1.1[12], [10]: Let $f \in C[0,1]$ and suppose that $f''(x)$ exists and continuous at a point $x \in (0,1)$ then $\lim_{n\to\infty} n\{\mathcal{B}_n(f;x) - f(x)\} = \frac{x(1-x)}{2} f''(x)$.

2. APPROXIMATION OF NEW GENERALIZATION OF BERNSTEIN POLYNOMIAL: MAIN RESULTS

In the previous part of our work we discuss the operators;

$$\mathcal{B}_n(f(t);x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

In this paper, we introduce a new sequence of linear positive operators $\overline{\mathcal{B}}_{n,r}(f;x)$ of Bernstein – type operators to approximate a function f belong to the space $C[0,1]$, as follows for x belong to a compact set $[0,1]$ and $r \in \mathbb{N}^0 := \{0,1,2,\dots\}$

$$\overline{\mathcal{B}}_{n,r}(f;x) = (n+1)_r \sum_{k=0}^{n+r} \beta_{n,k,r}(x) f(t) \ ,$$

where $\beta_{n,k,r}(x) = \frac{(1-x)^r}{(n-k+1)_r} \mathfrak{b}_{n,k}(x)$,

$\mathfrak{b}_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$,

and $(x)_r = x(x+1)(x+2)\dots(x+r-1)$,

clearly when $r = 0$, then $\overline{\mathcal{B}}_{n,r}(f;x) = \mathcal{B}_n(f;x)$**.**

Definition 2.1[9](Modified Bernstein Basis Polynomial):

The modified Bernstein basis polynomials of degree $(n+r)$ on the interval $[0,1]$ are defined by;

$$\overline{\mathcal{B}}_{n,r,k}(f;x) \equiv \overline{\mathcal{B}}_{n+r,k}(f;x)$$

Take $(n+r) = 6$, for example

$$\overline{\mathcal{B}}_{6,0}(f;x) = (1-x)^6$$

$$\overline{\mathcal{B}}_{6,1}(f;x) = 6x(1-x)^5$$

$$\overline{\mathcal{B}}_{6,2}(f;x) = 15x^2(1-x)^4$$

$$\overline{\mathcal{B}}_{6,3}(f;x) = 40x^3(1-x)^3$$

$$\overline{\mathcal{B}}_{6,4}(f;x) = 15x^4(1-x)^2$$

$$\overline{\mathcal{B}}_{6,5}(f;x) = 6x^5(1-x)$$

$$\overline{\mathcal{B}}_{6,6}(f;x) = x^6$$

In the next lemma we study Korovkin's conditions for the operators $\overline{\mathcal{B}}_{n,r}(f;x)$;

Lemma2.1:

For the operators $\overline{\mathcal{B}}_{n,r}(f;x)$, we have;

1) $\overline{\mathcal{B}}_{n,r}(1;x) = 1$.
2) $\overline{\mathcal{B}}_{n,r}(t;x) = x$.
3) $\overline{\mathcal{B}}_{n,r}(t^2;x) = x^2\left\{1 - \frac{1}{(n+r)}\right\} + x\frac{1}{(n+r)}$.

4) $\overline{\mathcal{B}}_{n,r}(t^3;x) = x^3\left\{\frac{(n+r)^2-3(n+r)+2}{(n+r)^2}\right\} + 3x^2\left\{\frac{(n+r)-1}{(n+r)^2}\right\} + \frac{x}{(n+r)^2}.$

Proof:

1) $\overline{\mathcal{B}}_{n,r}(1;x) = (n+1)_r \sum_{k=0}^{n+r} \beta_{n,k,r}(x)$

$$= (n+1)_r \sum_{k=0}^{n+r} \frac{(1-x)^r}{(n-k+1)_r}\binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n+r} \frac{(n+1)\ldots(n+r)}{(n-k+1)\ldots(n-k+r)} \frac{n!}{k!\,(n-k)!} x^k (1-x)^{n+r-k}$$

$$= \sum_{k=0}^{n+r} \frac{(n+r)!}{k!\,(n-k+r)!} x^k (1-x)^{n+r-k}$$

$$= \sum_{k=0}^{n+r} \binom{n+r}{k} x^k (1-x)^{n+r-k} = 1$$

2) $\overline{\mathcal{B}}_{n,r}(t;x) = (n+1)_r \sum_{k=0}^{n+r} \beta_{n,k,r}(x)\, t$

$$= (n+1)_r \sum_{k=0}^{n+r} \frac{(1-x)^r}{(n-k+1)_r}\binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n+r}$$

$$= \sum_{k=1}^{n+r} \frac{(n+r)!}{(n+r)(k-1)!\,(n-k+r)!} x^k (1-x)^{n+r-k}$$

$$= x \sum_{k=0}^{n+r-1} \frac{(n+r-1)!}{k!\,(n+r-k-1)!} x^k (1-x)^{n+r-k-1}$$

$$\overline{\mathcal{B}}_{n,r}(t;x) = x \sum_{k=0}^{n+r-1} \binom{n+r-1}{k} x^k (1-x)^{n+r-k-1} = x$$

3) $\overline{\mathcal{B}}_{n,r}(t^2;x) = (n+1)_r \sum_{k=0}^{n+r} \beta_{n,k,r}(x)\, t^2$

$$= (n+1)_r \sum_{k=0}^{n+r} \frac{(1-x)^r}{(n-k+1)_r}\binom{n}{k} x^k (1-x)^{n-k} \left(\frac{k}{n+r}\right)^2$$

$$= \frac{1}{(n+r)} \sum_{k=1}^{n+r} \frac{(n+r-1)!}{(k-1)!\,(n+r-k)!} x^k (1-x)^{n+r-k} k$$

$$= \frac{x}{(n+r)} \sum_{k=0}^{n+r-1} \frac{(n+r-1)!}{k!\,(n+r-1-k)!} x^k (1-x)^{n+r-1-k} (k+1)$$

$$= \frac{x}{(n+r)} \sum_{k=1}^{n+r-1} \frac{(n+r-1)!}{(k-1)!\,(n+r-1-k)!} x^k (1-x)^{n+r-1-k}$$

$$+ \frac{x}{(n+r)} \sum_{k=0}^{n+r-1} \binom{n+r-1}{k} x^k (1-x)^{n+r-1-k}$$

$$= \frac{x^2(n+r-1)}{(n+r)} \sum_{k=0}^{n+r-2} \frac{(n+r-2)!}{(k-1)!\,(n+r-2-k)!} x^k (1-x)^{n+r-2-k} + \frac{x}{(n+r)}$$

$$\overline{\mathcal{B}}_{n,r}(t^2;x) = x^2 \left\{1 - \frac{1}{(n+r)}\right\} + x\frac{1}{(n+r)}$$

4) $\overline{\mathcal{B}}_{n,r}(t^3;x) = (n+1)_r \sum_{k=0}^{n+r} \beta_{n,k,r}(x)\, t^3$

$$= (n+1)_r \sum_{k=0}^{n+r} \frac{(1-x)^r}{(n-k+1)_r} \binom{n}{k} x^k (1-x)^{n-k} \left(\frac{k}{n+r}\right)^3$$

$$= \frac{1}{(n+r)^3} \sum_{k=1}^{n+r} \frac{(n+r)!}{(n+r-k)!\,(k-1)!} x^k (1-x)^{n+r-k-1} k^2$$

$$= \frac{x}{(n+r)^2} \sum_{k=0}^{n+r-1} \frac{(n+r-1)!}{(n+r-k-1)!\ k!} x^k (1-x)^{n+r-k-1} (k+1)^2$$

$$= \frac{x}{(n+r)^2} \sum_{k=1}^{n+r-1} \binom{n+r-1}{k} x^k (1-x)^{n+r-1-k} k^2$$

$$+ \frac{2x}{(n+r)^2} \sum_{k=1}^{n+r-1} \binom{n+r-1}{k} x^k (1-x)^{n+r-1-k}\, k$$

$$+ \frac{x}{(n+r)^2} \sum_{k=0}^{n+r-1} \binom{n+r-1}{k} x^k (1-x)^{n+r-1-k}$$

$$= \frac{x(n+r-1)}{(n+r)^2} \sum_{k=0}^{n+r-2} \frac{(n+r-2)!}{k!\,(n+r-2-k)!} x^{k+1} (1-x)^{n+r-2-k}\ (k+1)$$

$$+ \frac{2x(n+r-1)}{(n+r)^2} \sum_{k=0}^{n+r-2} \frac{(n+r-2)!}{k!\,(n+r-2-k)!} x^{k+1} (1-x)^{n+r-2-k} + \frac{x}{(n+r)^2}$$

$$\overline{\mathcal{B}}_{n,r}(t^3;x)=\frac{x^2(n+r-1)}{(n+r)^2}\sum_{k=1}^{n+r-2}\binom{n+r-2}{k}x^k(1-x)^{n+r-2-k}\,k$$
$$+\frac{x^2\,(n+r-1)}{(n+r)^2}\sum_{k=0}^{n+r-2}\binom{n+r-2}{k}x^k(1-x)^{n+r-2-k}$$
$$+\frac{2x^2\,(n+r-1)}{(n+r)^2}\sum_{k=0}^{n+r-2}\binom{n+r-2}{k}x^k(1-x)^{n+r-2-k}+\frac{x}{(n+r)^2}$$

$$=\frac{x^2(n+r-1)(n+r-2)}{(n+r)^2}\sum_{k=0}^{n+r-3}\frac{(n+r-3)!}{k!\,(n+r-3-k)!}x^k(1-x)^{n+r-3-k}$$
$$+\frac{3x^2(n+r-1)}{(n+r)^2}+\frac{x}{(n+r)^2}$$

$$=\frac{x^3(n+r-1)(n+r-2)}{(n+r)^2}+\frac{3x^2(n+r-1)}{(n+r)^2}+\frac{x}{(n+r)^2}$$

$$=x^3\left\{\frac{(n+r)^2-3(n+r)+2}{(n+r)^2}\right\}+3x^2\left\{\frac{(n+r)-1}{(n+r)^2}\right\}+\frac{x}{(n+r)^2}. \qquad \blacksquare$$

Theorem:[3] 2.1: For $f\in C[0,1]$ then the sequence of modified Bernstein operators $\overline{\mathcal{B}}_{n,r}$ converge uniformly to $f\in C[0,1]$.

Proof:

From Bohman-Korovkin theorem [8], [4] we can see that $\lim_{n\to\infty}\left\|\overline{\mathcal{B}}_{n,r}(t^j-x^j)\right\|=0$, for $j=0,1,2$.We can be obtain that by lemma (2.1). $\blacksquare$

The next Lemma2.2 explains some properties of the p -th order moment $\overline{\mathcal{T}}_{n,r,p}(x)$ to the operators $\overline{\mathcal{B}}_{n,r}(f(t);x)$where $\overline{\mathcal{T}}_{n,r,p}(x)\equiv\overline{\mathcal{B}}_{n,r}((t-x)^p;x)$.

Lemma 2.2: Let $r\in\mathbb{R}^+$, then for all $x\in[0,1]$, we have

1) $\overline{\mathcal{T}}_{n,r,0}(x)=1$,
2) $\overline{\mathcal{T}}_{n,r,1}(x)$=0,
3) $\overline{\mathcal{T}}_{n,r,2}(x)=\frac{x(1-x)}{n+r}$,
4) $\overline{\mathcal{T}}_{n,r,3}(x)=x^3\left[\frac{(n+r)-3(n+r)+2}{(n+r)^2}-\frac{3(n+r)+1}{(n+r)}+4\right]+x^2\left[\frac{3(n+r)-1}{(n+r)^2}-\frac{3}{n+r}\right]+\frac{x}{(n+r)^2}$.

also, $(n+r)\overline{\mathcal{T}}_{n,r,p+1}(x)=x(1-x)\overline{\mathcal{T}}'_{n,r,p}(x)+px(1-x)\overline{\mathcal{T}}_{n,r,p-1}(x)$

Proof:

1) $\overline{\mathcal{T}}_{n,r,0}(x)=\overline{\mathcal{B}}_{n,r}((t-x)^0;x)=1$,

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2) $\bar{\mathcal{T}}_{n,r,1}(x) = \bar{\mathcal{B}}_{n,r}((t-x)^1; x) = \bar{\mathcal{B}}_{n,r}(t; x) - x\bar{\mathcal{B}}_{n,r}(1; x) = x - x = 0.$

So we prove (1) and (2). By direct calculating we can prove the others,

3) $\bar{\mathcal{T}}_{n,r,2}(x) = \bar{\mathcal{B}}_{n,r}((t-x)^2; x) = \bar{\mathcal{B}}_{n,r}(t^2; x) - 2x\bar{\mathcal{B}}_{n,r}(t; x) + x^2\bar{\mathcal{B}}_{n,r}(1; x)$

$$= x^2 \frac{(n+r)-1}{(n+r)} + \frac{x}{(n+r)} - x^2$$

$\bar{\mathcal{T}}_{n,r,2}(x) = \frac{x(1-x)}{n+r},$

4) $\bar{\mathcal{T}}_{n,r,3}(x) = \bar{\mathcal{B}}_{n,r}((t-x)^3; x) =$
$\bar{\mathcal{B}}_{n,r}(t^3; x) - 3x\bar{\mathcal{B}}_{n,r}(t^2; x) + 3x^2\bar{\mathcal{B}}_{n,r}(t; x) + x^3\bar{\mathcal{B}}_{n,r}(1; x)$

$$= x^3\left[\frac{(n+r)^2 - 3(n+r) + 2}{(n+r)^2}\right] + 3x^2\left[\frac{(n+r)-1}{(n+r)^2}\right] + \frac{x}{(n+r)^2} - 3x\left[x^2\left(\frac{(n+r)-1}{n+r}\right) + \frac{x}{n+r}\right] + 3x^2x + x^3$$

$$\bar{\mathcal{T}}_{n,r,3}(x) = x^3\left[\frac{(n+r) - 3(n+r) + 2}{(n+r)^2} - \frac{3\big((n+r)+1\big)}{(n+r)} + 4\right] + x^2\left[\frac{3(n+r)-1}{(n+r)^2} - \frac{3}{n+r}\right] + \frac{x}{(n+r)^2}.$$

It remain to prove the repetitive formula

$$\bar{\mathcal{T}}_{n,r,p}(x) = \bar{\mathcal{B}}_{n,r}((t-x)^p; x)$$

$$\bar{\mathcal{T}}_{n,r,p}(x) = (n+1)_r \sum_{k=0}^{n+r} \beta_{n,k,r}(x)\,(t-x)^p$$

$$\bar{\mathcal{T}}'_{n,r,p}(x) = -(n+1)_r\, p \sum_{k=0}^{n+r} \beta_{n,k,r}(x)\,(\frac{k}{n+r} - x)^{p-1} + (n+1)_r \sum_{k=0}^{n+r} \beta'_{n,k,r}(x)\,(\frac{k}{n+r} - x)^p$$

Since $\beta_{n,k,r}(x) = \frac{(1-x)^r}{(n-k+1)_r}\binom{n}{k} x^k (1-x)^{n-k} = \binom{n}{k}\frac{1}{(n-k+1)_r} x^k (1-x)^{n+r-k}$

$$\beta'_{n,k,r}(x) = \binom{n}{k}\frac{1}{(n-k+1)_r}\{(n+r-k)x^k(1-x)^{n+r-k-1} + kx^{k-1}(1-x)^{n+r-k}\}$$
$$= \frac{-(n+r-k)}{1-x}\binom{n}{k}\frac{1}{(n-k+1)_r}x^k(1-x)^{n+r-k} + \frac{k}{x}\binom{n}{k}\frac{1}{(n-k+1)_r}x^k(1-x)^{n+r-k}$$

$$\beta'_{n,k,r}(x) = -\frac{(n+r-k)}{1-x}\beta_{n,k,r}(x) + \frac{k}{x}\beta_{n,k,r}(x) = \beta_{n,k,r}(x)\left[\frac{k}{x} - \frac{(n+r-k)}{1-x}\right]$$

$$\beta'_{n,k,r}(x) = \beta_{n,k,r}(x)\left(\frac{k-kx-(n+r)x+kx}{x(1-x)}\right)$$

$$x(1-x)\beta'_{n,k,r}(x) = [k-(n+r)x]\beta_{n,k,r}(x)$$

$$x(1-x)\bar{\mathcal{T}}'_{n,r,p}(x) = -x(1-x)(n+1)_r p\sum_{k=0}^{n+r}\beta_{n,k,r}(x)\left(\frac{k}{n+r}-x\right)^{p-1} + (n+1)_r\sum_{k=0}^{n+r}\beta_{n,k,r}(x)\left(k-(n+r)x\right)\left(\frac{k}{n+r}-x\right)^{p}$$

$$x(1-x)\bar{\mathcal{T}}'_{n,r,p}(x) = -x(1-x)p\bar{\mathcal{T}}_{n,r,p-1}(x) + (n+r)\bar{\mathcal{T}}_{n,r,p+1}(x)$$

$$(n+r)\bar{\mathcal{T}}_{n,r,p+1}(x) = x(1-x)\bar{\mathcal{T}}'_{n,r,p}(x) + px(1-x)\bar{\mathcal{T}}_{n,r,p-1}(x). \qquad ■$$

Definition[3], [11], [13]2.2(modulus of continuity):

For $\delta > 0$and $f \in C[0,1]$, the modulus of continuity $\omega(f;\delta)$, of the function f is defined by

$\omega(f;\delta) = Sup_{|x-y|\leq\delta}|f(x)-f(y)|$, for every$x, y \in [0,1]$.

Theorem2.2[3]: Let$f \in C[0,1]$, then $\left|\bar{\mathcal{B}}_{n,r}(f;x) - f(x)\right| \leq 2\omega(f;\delta)$, where $\delta = \sqrt{\frac{x(1-x)}{n+r}}$.

Proof:

By offered the following property of modulus of continuity

$|f(t)-f(x)| \leq \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2}+1\right)$,

We using the above property, we have

$$\left|\bar{\mathcal{B}}_{n,r}(f;x) - f(x)\right| \leq \bar{\mathcal{B}}_{n,r}(|f(t)-f(x)|;x) \leq \omega(f;\delta)\left(\frac{1}{\delta^2}\bar{\mathcal{B}}_{n,r}((t-x)^2;x)+1\right),$$

$$= \omega(f;\delta)\left(\frac{1}{\delta^2}\frac{x(1-x)}{n+r}+1\right)$$

Choose $\delta = \sqrt{\frac{x(1-x)}{n+r}}$,

$\left|\overline{\mathcal{B}}_{n,r}(f;x) - f(x)\right| = 2\omega(f;\delta)$.

So, the proof is completed. ■

Lemma2.3:

Let$\beta_{n,k,r}(x) = \frac{(1-x)^r}{(n-k+1)_r}\binom{n}{k} x^k(1-x)^{n-k}, x \in [0,1]$ be the weight of modified Bernstein Polynomials. Then we have for $\delta > 0$ and $x \in [0,1]$. If $\left|\frac{k}{n+r} - x\right| \geq \delta$,

Then$(n+1)_r \sum_{\left|\frac{k}{n+r}-x\right|\geq\delta} \beta_{n,k,r}(x) \leq \frac{1}{4(n+r)\delta^2}$.

Proof: since $\left|\frac{k}{n+r} - x\right| \geq \delta$

$$\left(\frac{k}{n+r} - x\right)^2 \geq \delta^2 ,$$

$$\frac{\left(\frac{k}{n+r} - x\right)^2}{\delta^2} \geq 1$$

Hence $\sum_{\left|\frac{k}{n+r}-x\right|\geq\delta} \beta_{n,k,r}(x) \leq \frac{1}{\delta^2}\sum_{\left|\frac{k}{n+r}-x\right|\geq\delta}\left(\frac{k}{n+r} - x\right)^2 \beta_{n,k,r}(x)$,

$$(n+1)_r \sum_{\left|\frac{k}{n+r}-x\right|\geq\delta} \beta_{n,k,r}(x) \leq \frac{1}{\delta^2}(n+1)_r \sum_{\left|\frac{k}{n+r}-x\right|\geq\delta} \left(\frac{k}{n+r} - x\right)^2 \beta_{n,k,r}(x)$$

$$\leq \frac{1}{\delta^2}\bar{\mathcal{T}}_{n,s,2}(x)$$

$$\leq \frac{1}{\delta^2}\frac{x(1-x)}{n+r}$$

$$\leq \frac{1}{4\delta^2(n+r)}$$

Since the function $x(1-x)$ has a local maximum value at a point $x = \frac{1}{2}$ then $x(1-x) \leq \frac{1}{4}$. Hence $(n+1)_r \sum_{\left|\frac{k}{n+r}-x\right|\geq\delta} \beta_{n,k,r}(x) \leq \frac{1}{4(n+r)\delta^2}$. ■

Theorem [7]2.3: For $f \in C[0,1]$, then

$\lim_{n\to\infty} \overline{\mathcal{B}}_{n,r}(f;x) = f(x)$, for every $x \in [0,1]$

Proof:

By using Korovkin′s conditions (lemma 2.1)we get;

$$\lim_{n\to\infty} max_{x\in[0,1]}\left|\overline{\mathcal{B}}_{n,r}(t^2;x) - x^2\right| =$$

$$\lim_{n\to\infty} max_{x\in[0,1]}\left|\frac{x}{n+r} - \frac{x^2}{n+r}\right| =$$

$$\lim_{n\to\infty} max_{x\in[0,1]}\left|\frac{x(1-x)}{n+r}\right| =$$

$\lim\limits_{n\to\infty} \frac{1}{4(n+r)} = 0$; as sufficiently large n.

$\lim_{n\to\infty} max_{x\in[0,1]}\left|\overline{\mathcal{B}}_{n,r}(t^j;x) - x^j\right| = 0,\, j = 0,1,2.$

Can be obtained by lemma(2.1) above. ■

Voronoviskaja Theorem 2.4: Suppose that $f \in C[0,1]$ and suppose that $f''(x)$ exists and continuous at a point $x \in (0,1)$ then,

$$\lim_{n\to\infty}(n+r)\left\{\overline{\mathcal{B}}_{n,r}(f;x) - f(x)\right\} = \frac{x(1-x)}{2}f''(x)$$

Proof: By Taylor's expansion of $f$about x;

$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + (t-x)^2\varepsilon(t;x)$, where $\varepsilon(t;x) \to 0\; as\; t \to x$

Put $t = \frac{k}{n+r}$, then

$f\left(\frac{k}{n+r}\right) = f(x) + \left(\frac{k}{n+r} - x\right)f'(x) + \frac{1}{2!}(\frac{k}{n+r} - x)^2 f''(x) + (\frac{k}{n+r} - x)^2\varepsilon(\frac{k}{n+r};x),$

where$\frac{k}{n+r} \to x\; as\; n \to \infty$.

Hence $\overline{\mathcal{B}}_{n,r}(f(t);x) = \overline{\mathcal{B}}_{n,r}(f(x);x) + \overline{\mathcal{B}}_{n,r}\left(f'(x)\left(\frac{k}{n+r} - x\right);x\right) + \overline{\mathcal{B}}_{n,r}\left(\frac{1}{2!}\left(\frac{k}{n+r} - x\right)^2 f''(x);x\right) + \overline{\mathcal{B}}_{n,r}\left(\left(\frac{k}{n+r} - x\right)^2 \varepsilon\left(\frac{k}{n+r};x\right);x\right)$

$$= f(x) + f'^{(x)}\mathcal{T}_{n,s,1}(x) + \frac{1}{2}f''(x)\bar{\mathcal{T}}_{n,s,2}(x)$$
$$+ (n+1)_r \sum_{k=0}^{n+r} \left(\frac{k}{n+r} - x\right)^2 \beta_{n,k,r}(x)\varepsilon\left(\frac{k}{n+r}; x\right).$$

$$= f(x) + \frac{1}{2}f''(x)\left(\frac{x(1-x)}{n+r}\right) + (n+1)_r \sum_{k=0}^{n+r} \left(\frac{k}{n+r} - x\right)^2 \beta_{n,k,r}(x)\varepsilon\left(\frac{k}{n+r}; x\right)$$

Since $\varepsilon\left(\frac{k}{n+r}; x\right) \to 0$ as $\frac{k}{n+r} \to x$(from continuity), then given $\varepsilon > 0$, there exist $\delta > 0$ such that

$\left|\frac{k}{n+r} - x\right| < \delta \implies \left|\varepsilon(\frac{k}{n+r}; x)\right| < \varepsilon$for $\left|\frac{k}{n+r} - x\right| \geq \delta, \exists\, M > 0$, such that

$\left|\varepsilon(\frac{k}{n+r}; x)\left(\frac{k}{n+r} - x\right)^2\right| \leq M$,

Then $(n+1)_r \sum_{k=0}^{n+r}\left(\frac{k}{n+r} - x\right)^2 \left|\varepsilon\left(\frac{k}{n+r}; x\right)\right| \beta_{n,k,r}(x)$

$$\leq (n+1)_r\, \varepsilon \sum_{\left|\frac{k}{n+r}-x\right|<\delta} \left(\frac{k}{n+r} - x\right)^2 \beta_{n,k,r}(x) + (n+1)_r M \sum_{\left|\frac{k}{n+r}-x\right|\geq\delta} \beta_{n,k,r}(x)$$

$= \varepsilon\, \bar{\mathcal{T}}_{n,2}(x) + \frac{M}{4(n+r)\delta^2}$from lemma 2.3

$< \varepsilon\, \frac{x(1-x)}{n+r} + \frac{M}{4(n+r)\delta^2}$ from lemma 2.2

$$< \varepsilon\, \frac{1}{4(n+r)} + \frac{M}{4(n+r)\delta^2}$$

So, $(n+1)_r \sum_{k=0}^{n+r}\left(\frac{k}{n+r} - x\right)^2 \left|\varepsilon\left(\frac{k}{n+r}; x\right)\right| \beta_{n,k,r}(x) \to 0$ as $n \to \infty$.

Therefore $\lim_{n\to\infty}(n+r)\left\{\bar{\mathcal{B}}_{n,r}(f; x) - f(x)\right\} = \frac{x(1-x)}{2}f''(x)$. ■

References

[1] Arab H. K., Dehghan M., Eslahchi M. R.:*ANew Approach to Improve the Order of Approximation of the Bernstein Operators: Theory and Applications*. Numerical Algorithms (2017).

[2] Bernstein S. N.: *Démonstration du théoréme de Weierstrass fondée sur lecalcul de probabilités,* Comm. Soc. Math. Kharkow**13** (1912/13), 1-2.

[3] Bodur M., Manav N., Taşdelen F.: *Bernstein -Kantorovich-Stancu Operators With Shape Parameter λ*, (April, 12, 2019), 1-12.

[4] Bohman H.: *On Approximation of Continuous and Analytic Function,* Ark. Mat. **2** (1952), 43-56.

[5] Cai Q. B., Lian B. Y., Zhou G.: *Approximation Properties of λ-Bernstein Operators,* Journal of Inequalities and Applications **2018**, 2-11, (2018).

[6] Derriennic M. M.:*Sur I 'approximation de functions intégrabal sur [0,1] par des polynômes de Bernstein modliés,* J. Approx. Theory **31** (1981), 325-345.

[7] İzgi A.: *Approximation by a Class of New Type Bernstein Polynomials of One And Two Variables,* Global Journal of Pure and Applied Mathematics. 8 (2012)55-71, ISSN 0973-1768.

[8] Korovkin P. P.: *Linear Operators and Approximation Theory,* Hindustan Publ. Corp. Delhi, 1960 (Translated from Russian Edition) (1960).

[9] Qasim A. F., AL- Rawi E. S.: *A domain decomposition method with modified Bernstein polynomials for solving equations*, Hindawi Journal of Applied Mathematics, **2018** (2018), 1-9.

[10] Sikkema P. C.: *On Some Linear Positive Operators*, (communicated by prof. R. Timman at the meeting of April 25,1970).

[11] Swetits J. J.: *On Summability and Positive Linear Operators*, Journal of Approximation Theory, **25**, (1979), 186-188.

[12] Voronovskaja E.: *Détermination de la forme asymptotiqued' ápproximation des functions par les polynômes de S. N. Bernstein,* C. R. Acad. Sci. USSR(1932),79-85.

[13] Wafi A., Habib A., Khan H.H.: *On Generalized Bernstein Polynomials*, Indian Journal of Pure and Applied Mathematics, **9**, (1978), 867-870.

[14] Weierstrass K.: *Uber die analytischeDarstellbarkeitsogenannterwillkurlicher Funktioneneinerreellen Verdnderlichen (on the analytic representability of So-called arbitrary functions of a real variable)*. Minutes of the Academy in Berlin,

(1885).