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Some Approximation Properties of New Family of Baskakov- Type Operators

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Abstract
The aim of this paper is to establish a new family of Baskakov – type operators represented by summation type generalized Baskakov operators. Primarily, we study the convergence of this sequences of linear positive operators. Further we view some approximation properties which lead us to establish a Voronovskaja-type asymptotic formula for this operators. Finally, we study the rate of convergence when we show this new family preserve properties of modulus of continuity on a continuous function.

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Keywords:

Baskakov operators, Voronovskaja-type asymptotic formula, Korovkin's e theorem, Modulus of continuity, Convergence theorems, Weighted space.

1. Introduction

In (1957), Baskakov [1] introduced a sequence of linear and positive operators $\{L_n\}$. $L_n: C(\mathbb{R}^+) \to C[0, A]$ for $x \in \mathbb{R}^+ = [0, \infty)$ defined by

$$L_n(f; x) = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi_n^{(k)}(x)}{k!} (x)^k f\left(\frac{k}{n}\right), \text{ for } n = 1,2, \dots \quad (1)$$

In the paper itself, Baskakov could define these operators as the following formula when he defined $\varphi_n^{(k)}(x)$,

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$$L_n(f;x) = \frac{1}{(1+x)^n} \sum_{k=0}^{\infty} \frac{n(n+1)\ldots(n+k-1)}{k!} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right) \qquad (2)$$

Which leads us to the following formula

$$L_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right). \qquad (3)$$

2. Preliminaries

In (2006), Vijay Gupta and others [2] defined the following summation - integral Baskakov operators for $f \in C_\alpha(\mathbb{R}^+)$, $\mathbb{R}^+ = [0,\infty)$.

$$V_n(f;x) = (n-1) \sum_{k=0}^{\infty} v_{n,k}(x) \int_0^{\infty} v_{n,k}(t)\, f(t) dt \qquad (4)$$

Where $v_{n,k}(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k}$. (5)

In this paper, we defined a new family of Baskakov - type operators when we benefited by the formulas which used by Z. Walczak [3], [4] and [5] also Ali J. and Haneen J. used these formulas in [6],[7] and others as following:

For $\alpha > 0$, $C_\rho[0,\infty) = \{f \in C[0,\infty);\ f$ is real- valued function and $S_\rho(x)$ is continuous and bounded on $[0,\infty)\}$,

where $S_\rho(x) = \begin{cases} 1 & if\ \rho = 0 \\ (1+x^p)^{-1} & if\ \rho \in N \end{cases}$ (6)

The norm on C_ρ is defined by the formula

$$\|f(x)\|_\rho \coloneqq sup_{x\in[0,\infty)} S_\rho(x) |f(x)|. \qquad (7)$$

We define $\mathcal{H}_n(f(t);x) := \frac{1}{g(x,r)} \sum_{k=0}^{\infty} \hbar_{n,k,r}(x)$, (8)

where, $\hbar_{n,k,r}(x) = \begin{pmatrix} n+k+r-1 \\ k+r \end{pmatrix} x^{k+r}(1+x)^{-n-k-r}$, (9)

$g(x,r) \coloneqq \sum_{k=0}^{\infty} \hbar_{n,k,r}(x)$, $0 < g(x,r) < 1$, $x \in [0,\infty)$.

3. Auxiliary Results

In this part, we shall give some properties of $\mathcal{H}_n(.\,;x)$ which we shall use them to proof main theorems and an approximation theorem which called Korovkin theorem [8].

Theorem (3.1) (Korovkin Theorem):

Let $e_i(t) := t^i$, $i = 0,1,2,3$. Then the Baskakov operators satisfies

1) $\mathcal{H}_n(e_0; x) = 1,$ (10)

2) $\mathcal{H}_n(e_1; x) = x + \frac{(n+r-1)!x^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)},$ (11)

3) $\mathcal{H}_n(e_2; x) =$

$$x^2\left(1+\frac{1}{n}\right) + \frac{x}{n}\left\{(n+1)\frac{(n+r-1)!x^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)} + 1\right\} + \frac{(n+r-1)!x^r(1+x)^{-n-r+1}}{n!n(r-1)!g(x,r)}, \tag{12}$$

$\mathcal{H}_n(e_3; x) =$

$$\frac{x^3(n+1)(n+2)}{n^2} + \frac{3x^2(n+1)}{n^2} + \frac{x}{n^2} + \frac{r(1+x)\mathscr{k}_{n,r}(x)}{ng(x,r)}\left[\frac{x(2n+1)+x(n+2)[x(n+1)+r]+r^2}{n^2}\right]. \tag{13}$$

Proof:

1) $\mathcal{H}_n(e_0; x) = \frac{1}{g(x,r)}\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)$

$$= \frac{1}{g(x,r)}\sum_{k=0}^{\infty}\binom{n+k+r-1}{k+r}x^{k+r}(1+x)^{-n-k-r}=1$$

2) $\mathcal{H}_n(e_1; x) = \frac{1}{g(x,r)}\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)\left(\frac{k+r}{n}\right)$

$$= \frac{1}{ng(x,r)}\sum_{k=0}^{\infty}\binom{n+k+r-1}{k+r}x^{k+r}(1+x)^{-n-k-r}(k+r)$$

$$= \frac{1}{ng(x,r)}\sum_{k=0}^{\infty}\frac{(n+k+r-1)!}{(k+r)!\,(n-1)!}x^{k+r}(1+x)^{n-k-r}(k+r)$$

$$= \frac{1}{ng(x,r)}\frac{(n+r-1)!}{(r-1)!\,(n-1)!}x^r(1+x)^{-n-r}$$

$$+\frac{1}{ng(x,r)}\sum_{k=1}^{\infty}\frac{(n+k+r-1)!}{(k+r-1)!\,(n-1)!}x^{k+r}(1+x)^{-n-k-r}$$

$$= \frac{r}{ng(x,r)}\binom{n+r-1}{r}x^{r}(1+x)^{-n-r}$$

$$+\frac{1}{ng(x,r)}\sum_{k=0}^{\infty}\frac{(n+k+r)!}{(k+r)!\,(n-1)!}x^{k+r+1}(1+x)^{-n-k-r-1}$$

$$:= I_1 + I_2.$$

We obtain that $I_1 = r\mathcal{k}_{n,r}(x)$

$$I_2 = \frac{1}{ng(x,r)}\frac{x}{1+x}\sum_{k=0}^{\infty}(n+k+r)\binom{n+k+r-1}{k+r}x^{k+r}(1+x)^{-n-k-r}$$

$$= \frac{1}{ng(x,r)}\frac{x}{1+x}\times\left\{n\sum_{k=0}^{\infty}\mathcal{k}_{n,k,r}(x)+\sum_{k=0}^{\infty}\mathcal{k}_{n,k,r}(x)(k+r)\right\}$$

$$= \frac{1}{ng(x,r)}\frac{x}{1+x}\times\left\{ng(x,r)+\sum_{k=0}^{\infty}\mathcal{k}_{n,k,r}(x)(k+r)\right\}$$

$$= \frac{x}{1+x}+\frac{x}{(1+x)}\frac{1}{g(x,r)}\sum_{k=0}^{\infty}\mathcal{k}_{n,k,r}(x)(\frac{k+r}{n})$$

That means $= \frac{x}{1+x}\times\{1+K\}$ where $K = \frac{1}{g(x,r)}\sum_{k=0}^{\infty}\mathcal{k}_{n,k,r}(x)(\frac{k+r}{n})$

Hence, $\frac{1}{1+x}K = \frac{r\mathcal{k}_{n,r}(x)}{ng(x,r)}+\frac{x}{1+x}$;

Therefore, $K = x + \frac{r(1+x)\mathcal{k}_{n,r}(x)}{ng(x,r)}$.

3) $\mathcal{H}_n(e_2;x) = \frac{1}{g(x,r)}\sum_{k=0}^{\infty}\mathcal{k}_{n,k,r}(x)\left(\frac{k+r}{n}\right)^2$

$$= \frac{1}{n^2g(x,r)}\sum_{k=0}^{\infty}\binom{n+k+r-1}{k+r}x^{k+r}(1+x)^{-n-k-r}(k+r)^2$$

$$=\frac{1}{n^2 g(x,r)}\sum_{k=0}^{\infty}\frac{(n+k+r-1)!}{(k+r-1)!\,(n-1)!}x^{k+r}(1+x)^{-n-k-r}(k+r)$$

$$=\frac{1}{n^2 g(x,r)}\frac{(n+r-1)!}{(r-1)!\,(n-1)!}x^{r}(1+x)^{-n-r}r$$

$$+\frac{1}{n^2 g(x,r)}\sum_{k=1}^{\infty}\frac{(n+k+r-1)!}{(k+r-1)!\,(n-1)!}x^{k+r}(1+x)^{-n-k-r}(k+r)$$

$$=\frac{r^2 \mathscr{k}_{n,r}(x)}{n^2 g(x,r)}+\frac{1}{n^2 g(x,r)}\sum_{k=0}^{\infty}\frac{(n+k+r)!}{(k+r)!\,(n-1)!}x^{k+r+1}(1+x)^{-n-k-r-1}(k+r+1)$$

$\coloneqq I_3 + I_4$, where $I_3 = \frac{r^2 \mathscr{k}_{n,r}(x)}{n^2 g(x,r)}$ and

$$I_4=\frac{1}{n^2 g(x,r)}\frac{x}{1+x}\sum_{k=0}^{\infty}\frac{(n+k+r)(n+k+r-1)!}{(k+r)!\,(n-1)!}x^{k+r}(1+x)^{-n-k-r}(k+r+1)$$

$$=\frac{1}{n^2 g(x,r)}\frac{x}{1+x}\sum_{k=0}^{\infty}(n+k+r)(k+r+1)\binom{n+k+r-1}{k+r}x^{k+r}(1+x)^{-n-k-r}$$

$$=\frac{1}{n^2 g(x,r)}\frac{x}{(1+x)}\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)\left\{(k+r)(n+1)+n+(k+r)^2\right\}$$

$$=\frac{1}{n^2 g(x,r)}\frac{x}{(1+x)}(n+1)\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)(k+r)$$

$$+\frac{1}{n^2 g(x,r)}\frac{x}{(1+x)}\left\{n\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)+\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)\,(k+r)^2\right\}$$

$$I_4=\frac{x}{n(1+x)}(n+1)K+\frac{x}{n(1+x)}+\frac{x}{(1+x)}L$$

Where $L=\frac{1}{g(x,r)}\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)(\frac{k+r}{n})^2$; $K=\frac{1}{g(x,r)}\sum_{k=0}^{\infty}\mathscr{k}_{n,k,r}(x)\,(\frac{k+r}{n})$

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Since $L := I_3 + I_4$

$$L = \frac{r^2 \mathcal{k}_{n,r}(x)}{n^2 g(x,r)} + \frac{x}{n(1+x)}(n+1)K + \frac{x}{n(1+x)} + \frac{x}{(1+x)}L$$

$$L = \frac{r^2(1+x)\mathcal{k}_{n,r}(x)}{n^2 g(x,r)} + \frac{x(n+1)}{n}\left\{x + \frac{r(1+x)\mathcal{k}_{n,r}(x)}{ng(x,r)}\right\} + \frac{x}{n}$$

$$L = x^2 + \frac{x^2}{n} + \frac{x}{n} + \frac{r(1+x)\mathcal{k}_{n,r}(x)}{n^2 g(x,r)}\{r + x(n+1)\}.$$

4- $\mathcal{H}_n(e_3; x) = \frac{1}{g(x,r)} \sum_{k=0}^{\infty} \mathcal{k}_{n,k,r}(x) \left(\frac{k+r}{n}\right)^3$

$$= \frac{1}{n^3 g(x,r)} \sum_{k=0}^{\infty} \binom{n+k+r-1}{k+r} x^{k+r}(1+x)^{-n-k-r}(k+r)^3$$

$$= \frac{1}{n^3 g(x,r)} \sum_{k=0}^{\infty} \frac{(n+k+r-1)!}{(k+r-1)!\,(n-1)!} x^{k+r}(1+x)^{-n-k-r}(k+r)^2$$

$$= \frac{1}{n^3 g(x,r)} \frac{r^2(n+r-1)!}{(r-1)!\,(n-1)!} x^r (1+x)^{-n-r} + \frac{1}{n^3 g(x,r)} \sum_{k=1}^{\infty} \frac{(n+k+r-1)!}{(k+r-1)!\,(n-1)!} x^{k+r}(1+x)^{-n-k-r}(k+r)^2$$

$$= \frac{r^3 \mathcal{k}_{n,r}(x)}{n^3 g(x,r)} + \frac{1}{n^3 g(x,r)} \sum_{k=0}^{\infty} \frac{(n+k+r)!}{(k+r)!\,(n-1)!} x^{k+r+1}(1+x)^{-n-k-r-1}(k+r+1)^2$$

$\coloneqq I_5 + I_6$, where $I_5 = \frac{r^3 \mathcal{k}_{n,r}(x)}{n^3 g(x,r)}$ and

$$I_6 = \frac{1}{n^3 g(x,r)} \frac{x}{1+x} \sum_{k=0}^{\infty} \frac{(n+k+r)(n+k+r-1)!}{(k+r)!\,(n-1)!} x^{k+r}(1+x)^{-n-k-r}(k+r+1)^2$$

$$= \frac{1}{n^3 g(x,r)} \frac{x}{1+x} \sum_{k=0}^{\infty} (n+k+r) \binom{n+k+r-1}{k+r} x^{k+r} (1+x)^{-n-k-r} \{(k+r)^2 + 2(k+r) + 1\}$$

$$= \frac{1}{n^3 g(x,r)} \frac{x}{1+x} \sum_{k=0}^{\infty} (n+k+r) \{\mathcal{k}_{n,k,r}(x)(k+r)^2 + 2\mathcal{k}_{n,k,r}(x)(k+r) + \mathcal{k}_{n,k,r}(x)\}$$

$I_6 = \frac{1}{g(x,r)} \frac{x}{1+x} \sum_{k=0}^{\infty} \mathcal{k}_{n,k,r}(x) \left(\frac{k+r}{n}\right)^3 + \frac{(n+2)}{ng(x,r)} \frac{x}{1+x} \sum_{k=0}^{\infty} \mathcal{k}_{n,k,r}(x) \left(\frac{k+r}{n}\right)^2 + \frac{(2n+1)}{n^2 g(x,r)} \frac{x}{1+x} \sum_{k=0}^{\infty} \mathcal{k}_{n,k,r}(x) \left(\frac{k+r}{n}\right) + \frac{1}{n^2 g(x,r)} \frac{x}{1+x} \sum_{k=0}^{\infty} \mathcal{k}_{n,k,r}(x)$.

Where $Z \coloneqq \frac{1}{g(x,r)} \sum_{k=0}^{\infty} \mathcal{k}_{n,k,r}(x) \left(\frac{k+r}{n}\right)^3$,

Then $Z = \frac{r^3 \mathcal{k}_{n,r}(x)}{n^3 g(x,r)} + \frac{x}{1+x} \left[Z + \frac{n+2}{n} L + \frac{2n+1}{n^2} K + \frac{1}{n^2}\right]$

From (11) and (12), we get:

$Z = \frac{r^3(1+x)\mathcal{k}_{n,r}(x)}{n^3 g(x,r)} + x\left(\frac{n+2}{n}\right)\left[x^2\left(1+\frac{1}{n}\right) + \frac{x}{n}\left((n+1)\frac{r(1+x)\mathcal{k}_{n,r}(x)}{ng(x,r)} + 1\right) + \frac{r(1+x)\mathcal{k}_{n,r}(x)}{ng(x,r)}\right] + x\left(\frac{2n+1}{n^2}\right)\left[x + \frac{r(1+x)\mathcal{k}_{n,r}(x)^2}{ng(x,r)}\right] + \frac{x}{n^2}$,

$$Z = \frac{x}{n^2}\{x(n+2)[1+x(n+1)] + x(2n+1) + 1\} + \frac{r(1+x)\mathcal{k}_{n,r}(x)}{ng(x,r)}\left[\frac{x(2n+1) + x(n+2)[x(n+1)+r] + r^2}{n^2}\right]$$

$$= \frac{x^3(n+1)(n+2)}{n^2} + \frac{3x^2(n+1)}{n^2} + \frac{x}{n^2} + \frac{r(1+x)\mathcal{k}_{n,r}(x)}{ng(x,r)}\left[\frac{x(2n+1) + x(n+2)[x(n+1)+r] + r^2}{n^2}\right].$$

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So, the proof is complete. ■

Here, we establish the τ −th order moment $\mathcal{Y}_{n,r,\tau}(x) := \mathcal{H}_n((t-x)^\tau; x)$ for the operators $\mathcal{H}_n(.;x)$ defined in (8) above for $\tau = 0,1,2,3 \ldots$ for every $x \in [0,\infty)$ and $f \in C_\rho$,as following:

$$\mathcal{Y}_{n,r,\tau}(x) := \mathcal{H}_n((t-x)^\tau; x) := \frac{1}{g(x,r)} \sum_{k=0}^{\infty} \hbar_{n,k,r}(x) (\frac{k+r}{n} - x)^\tau.$$

Lemma(3.1):

Let $\mathcal{Y}_{n,r,\tau}(x) := \mathcal{H}_n((t-x)^\tau; x)$, $\tau = 0,1,2,3 \ldots$. Then we have

1- $\mathcal{Y}_{n,r,0}(x) = 1,$ (14)

2- $\mathcal{Y}_{n,r,1}(x) = \frac{(n+r-1)! x^r (1+x)^{-n-r+1}}{n!(r-1)! g(x,r)},$ (15)

3- $\mathcal{Y}_{n,r,2}(x) = \frac{x^2}{n} + \frac{x}{n}\left\{1 + \frac{(n+r-1)! x^r (1+x)^{-n-r+1}}{n!(r-1)! g(x,r)}\right\}$

$+(x-1)\frac{(n+r-1)! r x^r (1+x)^{-n-r+1}}{n!(r-1)! g(x,r)},$ (16)

4- $\mathcal{Y}_{n,r,3}(x) =$

$\frac{2x^3}{n^2} + \frac{x^2}{n}\left\{\frac{2(n+r-1)! x^r (1+x)^{-n-r+1}}{n!(r-1)! g(x,r)}\left(\frac{n^2-n+1}{n}\right) + \frac{3}{n}\right\} +$

$\frac{x}{n}\left\{\frac{(n+r-1)! x^r (1+x)^{-n-r+1}}{n!(r-1)! g(x,r)}\left(\frac{n(4-3r)+1}{n}\right) + \frac{1}{n}\right\} + \frac{r^2}{n}\left\{\frac{(n+r-1)! x^r (1+x)^{-n-r+1}}{n!(r-1)! g(x,r)}\right\} +$

$\frac{(n+r-1)! x^r (1+x)^{-n-r+2}}{n!(r-1)! g(x,r)}.$ (17)

Definition3.1 (Weighted space)[9]: The classes of functions which satisfying the condition $S_p(x) = (1+x^p)^{-1}$ if $p \in N$, and $S_p(x) = 1$ if $p = 0$, with the norm $\|f(x)\|_p := sup_{x\in[0,\infty)} S_p(x)|f(x)|$, where $S_p(x)$ is continuous and bounded on $[0,\infty)$ are said to be weighted spaces, where f is continuous bounded function on $[0,\infty)$.

Here, we shall study the asymptotic behavior of given operators by applying a main theorem (Voronoviskaja- type theorem) [10].

Theorem (3.2) Voronoviskaja Theorem:

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Let $f \in C_\rho[0,\infty)$, be twice differentiable and continuous at $x \in (0,\infty)$, then;

$$\lim_{n\to\infty} n\{\mathcal{H}_n(f(t);x) - f(x)\} = \frac{x(x+1)}{2} f''(x). \tag{18}$$

Proof:

By Taylor's expansion of f about x;

$$f(t) = \begin{cases} f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!} f''(x) + (t-x)^2\varepsilon(t;x) & \text{if } t \neq x \\ 0 & \text{if } t = x \end{cases},$$

Where $\varepsilon(t;x)$ is a function belonging to $C_\rho[0,\infty)$ and $\varepsilon(t;x) \to 0$ as $t \to x$, applying the operators $\mathcal{H}_n(.;x)$ defined in (8) we have

$$\mathcal{H}_n(f(t);x) = f(x) + \mathcal{Y}_{n,r,1}(x)f'(x) + \frac{1}{2}\mathcal{Y}_{n,r,2}(x)f''(x) + \mathcal{H}_n((t-x)^2\varepsilon(t);x).$$

$$\begin{aligned}\mathcal{H}_n(f;x) - f(x) = &\left\{\frac{(n+r-1)!\,x^r(1+x)^{-n-r+1}}{n!\,(r-1)!\,g(x,r)}\right\} f'(x) \\ &+ \frac{1}{2}\left\{\frac{x^2}{n} + \frac{x}{n}\left\{1 + \frac{(n+r-1)!\,x^r(1+x)^{-n-r+1}}{n!\,(r-1)!\,g(x,r)}\right\} + (x \right. \\ &\left. - 1)\frac{(n+r-1)!\,rx^r(1+x)^{-n-r+1}}{n!\,(r-1)!\,g(x,r)}\right\} f''(x) \\ &+ \lim_{n\to\infty} n\mathcal{H}_n((t-x)^2\varepsilon(t);x)\end{aligned}$$

Using Lemma (3.1) and since ε is arbitrary, then $n\mathcal{Y}_{n,r,1}(x) \to 0$ as $n \to \infty$,

and $n\mathcal{Y}_{n,r,2}(x) = x(1+x)$ as $n \to \infty$, we get:

$$\lim_{n\to\infty} n\{\mathcal{H}_n(f(t);x) - f(x)\} = \frac{x(1+x)}{2} f''(x) + \lim_{n\to\infty} n\mathcal{H}_n(\varepsilon(t,x)(t-x)^2;x)$$

By using Cauchy- Schwarz inequality and (16), we obtain:

$$|n\mathcal{H}_n(\varepsilon(t,x)(t-x)^2;x)| \leq \left(\mathcal{H}_n(\varepsilon^2(t,x);x)\right)^{\frac{1}{2}}(n^2\mathcal{H}_n(t-x)^4;x)^{\frac{1}{2}}$$

From the linearity of $\mathcal{H}_n(.;x)$, theorem (3.1) and the properties of ε, we get

$$\lim_{n\to\infty} \mathcal{H}_n(\varepsilon^2(t,x);x) \equiv \lim_{n\to\infty} \varepsilon^2(t,x)\mathcal{H}_n(1;x) = \varepsilon^2(t,x) = 0 \text{ as } n \to \infty.$$

From Lemma (3.1) we have $\mathcal{H}_n((t-x)^2\varepsilon(t);x) \to 0$ as $n \to \infty$.

Therefore, $\lim_{n\to\infty} n\{\mathcal{H}_n(f(t);x) - f(x)\} = \frac{x(x+1)}{2} f''(x)$. ■

4. Rate of Approximation of $\mathcal{H}_n(.;x)$ in Weighted Space

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Finally, we want to find the rate of approximation of the family $\{\mathcal{H}_n(.\,;x)\}$ for $f \in C_\rho[0,\infty)$. We firstly gave the definition of the first modulus of continuity.

Definition(4.1) (modulus of continuity):[11],[12] and [13]

Let $f \in C[a,b]$. For > 0 , then the modulus of continuity $\omega(f;\delta)$ is defined by

$\omega(f;\delta) = Sup\{|f(t)-f(x)|: t,x \in [a,b]; |t-x| \le \delta\}$,

Next, for the order of approximation we give the following theorem.

Theorem (4.1): Let $f \in C_\rho$, then

$$|\mathcal{H}_n(f;x) - f(x)| \le 2\omega(f;\delta) \tag{19}$$

where $\delta = \sqrt{\frac{x^2}{n} + \frac{x}{n}\left\{1 + \frac{(n+r-1)!x^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)}\right\} + (x-1)\frac{(n+r-1)!rx^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)}}$.

Proof:

By using the well-known property of modulus of continuity

$$|f(t)-f(x)| \le \omega(f;\delta)\left(\frac{|t-x|}{\delta}+1\right) \tag{20}$$

Since the polynomials $\mathcal{H}_n(f;x)$ are linear positive operators, then we get

$$|\mathcal{H}_n(f;x) - f(x)| \le \left(\frac{1}{\delta}\mathcal{H}_n(|t-x|;x)+1\right)\omega(f;\delta)$$

Applying Cauchy- Schwartz inequality, (10) and (16) we have

$$|\mathcal{H}_n(f;x) - f(x)| \le \omega(f;\delta)\left(\frac{1}{\delta}\sqrt{(\mathcal{H}_n(t-x)^2;x)}+1\right)$$

$$\le \omega(f;\delta)\left(\frac{1}{\delta}\left(\sqrt{\left|\mathcal{Y}_{n,r,2}(x)\right|}\right)+1\right)$$

$$\le \omega(f;\delta)\left(\frac{1}{\delta}\sqrt{\frac{x^2}{n} + \frac{x}{n}\left\{1 + \frac{(n+r-1)!x^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)}\right\} + (x-1)\frac{(n+r-1)!rx^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)}} + 1\right)$$

Choose

$$\delta = \sqrt{\frac{x^2}{n} + \frac{x}{n}\left\{1 + \frac{(n+r-1)!x^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)}\right\} + (x-1)\frac{(n+r-1)!rx^r(1+x)^{-n-r+1}}{n!(r-1)!g(x,r)}}$$

Hence, we get $|\mathcal{H}_n(f;x) - f(x)| \leq 2\omega(f;\delta)$ ■

Theorem (4.2) : Let $f \in C_\rho$ on the interval $[0,\infty)$. Then for a real number $\mathcal{S} > 0$, the limit relation

$$\lim_{n\to\infty} \mathcal{H}_n(f;x) = f(x), \tag{21}$$

holds uniformly on the interval $[0,\mathcal{S}]$.

Proof:

By using (10), (11) and (12) from theorem (3.1) we can see that:

$$\|\mathcal{H}_n(1;x) - 1\|_{C[0,\mathcal{S}]} = 0 \tag{22}$$

$$\|\mathcal{H}_n(t;x) - x\|_{C[0,\mathcal{S}]} = \max_{x\in[0,\mathcal{S}]} \frac{(n+r-1)!\, x^r(1+x)^{-n-r+1}}{n!\,(r-1)!\,g(x,r)}$$

$$\leq \frac{(n+r-1)!\mathcal{S}^r(1+\mathcal{S})^{-n-r+1}}{n!(r-1)!g(\mathcal{S},r)} \to 0 \text{ as } n \to \infty.$$

$$\|\mathcal{H}_n(t^2;x) - x^2\|_{C[0,\mathcal{S}]}$$
$$= \max_{x\in[0,\mathcal{S}]} \frac{x^2}{n} + \frac{x}{n}\left\{(n+1)\frac{(n+r-1)!\, x^r(1+x)^{-n-r+1}}{n!\,(r-1)!\,g(x,r)} + 1\right\} + \frac{(n+r-1)!\, x^r(1+x)^{-n-r+1}}{n!\,n(r-1)!\,g(x,r)}$$

$$\leq \frac{\mathcal{S}^2}{n} + \frac{\mathcal{S}}{n}\left\{(n+1)\frac{(n+r-1)!\mathcal{S}^r(1+\mathcal{S})^{-n-r+1}}{n!(r-1)!g(\mathcal{S},r)} + 1\right\} + \frac{(n+r-1)!\mathcal{S}^r(1+\mathcal{S})^{-n-r+1}}{n!n(r-1)!g(\mathcal{S},r)} \to 0$$ for sufficiently large n.

The proof of this theorem can be obtained by P. P. Korovkin [8]. ■

References

[1] Baskakov, V. A. : An instance of a sequence of linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR **113**(2) (1957), 249 – 251.

[2] Vijay Gupta, Muhammad Aslam Noor, Man Singh Beniwal, and M.K. Gupta: On Simultaneous Approximation for Certain Baskakov Durrmeyer type operators. Journal of Inequalities in Pure and Applied Mathematics **7**(4), Art, 125(2006), 1-15.

[3] Zbigniew Walczak : Approximation properties of certain linear positive operators in exponential weighted spaces of functions of two variables. Acta Univ. palacki. Olomuc., Fac. rer. nat., Mathematica. **42** (2003),131-138.

[4] Zbigniew Walczak: Approximation properties of certain linear positive operators in polynomial weighted spaces of functions of one and two variables.Univ.Beograd. Publ. Elektrotehn. Fak.

14(2003), 51- 64.

[5] Zbigniew Walczak : On the rate of convergence for some linear operators .Hiroshima Math. J. **35** (2005),115-124.

[6] Ali J. Mohammed and Haneen J. Sadiq: A Sequence Of Linear And Positive Operators For The Functions Of Growth 2^x, Journal of Basrah Research ((Sciences)), **40**, 87-93, (2014).

[7] Haneen J. Sadiq: Approximation of bounded functions by positive linear operators in C_ρ. Basrah Journal of Science, **37** (3) 412-429, (2019). Doi: 10.29072/basjs.20190306.

[8] P. P. Korovkin: Linear Operators and Approximation Theory, Hindustan Publ. Corp. Delhi, 1960 (Translated from Russian Edition) (1960).

[9] Aydin İzgi: Approximation By Composition Of Chlodowsky Operators And Százs-Durrmeyer Operators On Weighted Spaces, LMS J. Comput. Math., **16** (2013), 388-397. doi:10.1112/s1461157013000090.

[10] Voronovskaja E.: Détermination de la forme asymptotiqued' ápproximation des functions par les polynômes de S. N. Bernstein, C. R. Acad. Sci. USSR(1932),79-85.

[11] Bodur M., Manav N., Taşdelen F.: Bernstein -Kantorovich-Stancu Operators With Shape Parameter λ, (April, 12, 2019), 1-12.

[12] Swetits J. J.: On Summability and Positive Linear Operators, Journal of Approximation Theory, **25**, (1979), 186-188.

[13] Wafi A., Habib A., Khan H.H.: On Generalized Bernstein Polynomials, Indian Journal of Pure and Applied Mathematics, **9**, (1978), 867-870.