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Some Direct Theorems on A Certain Class of Szasz- Beta Operators

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Abstract. Our purpose of this paper is to define a class of positive linear operators based on some operators defined by Z. Walczak depend on Szãsz type operators and the famous Beta function. We established a Korovkin theorem and Voronovikaja theorem. Finally, we gave the rate of convergence of a new Szãsz- Beta type operators by using the modulus of continuity.

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INTRODUCTION AND PRELIMINARIES

In (2003), Zbigniew Walczak [1] defined a new family of Szãsz – Mirakyan operators and investigate approximation theorems and the direct results for these operators. The well known Szãsz – Mirakyan operators which defined as following

$$S_{\mathfrak{n}}(f;x) := e^{-\mathfrak{n}x}\sum_{\ell=0}^{\infty}\frac{(\mathfrak{n}x)^{\ell}}{\ell!}\, f\left(\frac{\ell}{\mathfrak{n}}\right),\ (x\in\mathbb{R}_0=[0,\infty),\mathfrak{n}\in\mathbb{N}). \tag{1}$$

Then for $\upsilon\in\mathbb{N}$, Z. Walczak defined a new class of operators as follows

$$A_{\mathfrak{n}}^{\upsilon}(f;\upsilon;x) := \frac{1}{g(\mathfrak{n}x;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(\mathfrak{n}x)^{\upsilon}}{(\ell+\upsilon)!}\, f\left(\frac{\ell+\upsilon}{\mathfrak{n}}\right),\ \text{where } x\in\mathbb{R}_0,\mathfrak{n}\in\mathbb{R}. \tag{2}$$

$$\text{where } (\mathrm{t};\upsilon) := \sum_{\ell=0}^{\infty}\frac{\mathrm{t}^{\ell}}{(\ell+\upsilon)!},\ (\mathrm{t}\in\mathbb{R}_0) \tag{3}$$

with the space C_p, $p\in\mathbb{N}_0=\{0,1,2,\ldots\}$ associated with the weight function

$$\begin{Bmatrix} \mathcal{L}_p(x)=\dfrac{1}{1+x^p} & if\ p\geq 1, \\ \mathcal{L}_p(x)=1 & if\ p=0. \end{Bmatrix}$$

Where $C_p=\{f: f$ is a real − valued function continuous on $\mathbb{R}_0$ and $\mathcal{L}_p f$ is bounded and uniformly continuous on $\mathbb{R}_0$ $\}$.

In the next lemmas (1.1) and (1.2) the author Z. Walczak [1] applied the Korovkin's conditions and he found the central moment for the operators $A_{\mathrm{n}}^{\upsilon}(f;\upsilon;x)$ to ensure the convergence for the test functions $a_i(\mathrm{t}) := \mathrm{t}^i$, $i = 0,1,2$.

Lemma 1.1. (Bohman Korovkin's Theorem). Suppose that $a_i(\mathrm{t}) := \mathrm{t}^i$, for $i = 0,1,2$. Then the operators defined in Eq. (2) satisfy:

1- $A_{\mathrm{n}}^{\upsilon}(a_0;\upsilon;x) = 1,$ (4)

2- $A_{\mathrm{n}}^{\upsilon}(a_1;\upsilon;x) = x + \frac{1}{(\upsilon-1)!\mathrm{n}g(\mathrm{n}x;\upsilon)},$ (5)

3- $A_{\mathrm{n}}^{\upsilon}(a_2;\upsilon;x) = x^2 + \frac{x}{n}\left(1 + \frac{1}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) + \frac{\upsilon}{(\upsilon-1)!\mathrm{n}^2 g(\mathrm{n}x;\upsilon)},$ (6)

4- $A_{\mathrm{n}}^{\upsilon}(a_3;\upsilon;x) = x^3 + \frac{x^2}{\mathrm{n}}\left(3 + \frac{1}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) + \frac{x}{\mathrm{n}^2}\left(1 + \frac{\upsilon+2}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) + \frac{\upsilon^2}{(\upsilon-1)!n^3 g(\mathrm{n}x;\upsilon)},$ (7)

5- $A_{\mathrm{n}}^{\upsilon}(a_4;\upsilon;x) = x^4 + \frac{x^3}{\mathrm{n}}\left(6 + \frac{1}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) + \frac{x^2}{\mathrm{n}^2}\left(7 + \frac{\upsilon+5}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) + \frac{x}{\mathrm{n}^3}\left(1 + \frac{\upsilon^2+3\upsilon+3}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) +$
$\frac{\upsilon^3}{(\upsilon-1)!\mathrm{n}^4 g(\mathrm{n}x;\upsilon)}.$ (8)

In the next lemma, Z. Walczak [1] introduced the central moment with order $\eth \in \mathbb{N}_0$, $\psi_x^{\eth}(x) := A_{\mathrm{n}}^{\upsilon}\left((\mathrm{t}-x)^{\eth};\upsilon;x\right)$ for the L. P. O. $A_{\mathrm{n}}^{\upsilon}(.;\upsilon;x)$ defined in Eq. (2) for each $x \in \mathbb{R}_0$ and $f \in C_p$,as following:

Lemma 1.2. Let $\psi_x^{\eth}(x) := (\mathrm{t}-x)^{\eth}$, $\eth \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$, be the central moment of the operators $A_{\mathrm{n}}^{\upsilon}(f;r;x)$. Then we have

1- $A_{\mathrm{n}}^{\upsilon}(\psi_x^0;\upsilon;x) = 1,$ (9)

2- $A_{\mathrm{n}}^{\upsilon}(\psi_x^1;\upsilon;x) = \frac{1}{(\upsilon-1)!\mathrm{n}g(\mathrm{n}x;\upsilon)},$ (10)

3- $A_{\mathrm{n}}^{\upsilon}(\psi_x^2;\upsilon;x) = \frac{x}{\mathrm{n}}\left(1 - \frac{1}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) + \frac{\upsilon}{(\upsilon-1)!\mathrm{n}^2 g(\mathrm{n}x;\upsilon)},$ (11)

4- $A_{\mathrm{n}}^{\upsilon}(\psi_x^3;\upsilon;x) = \left(\frac{\upsilon}{\mathrm{n}} - x\right)^2 \frac{1}{(\upsilon-1)!\mathrm{n}g(\mathrm{n}x;\upsilon)} + \frac{x}{\mathrm{n}^2}\left(1 + \frac{2}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right),$ (12)

5- $A_{\mathrm{n}}^{\upsilon}(\psi_x^4;\upsilon;x) = \left(\frac{\upsilon}{\mathrm{n}} - x\right)^3 \frac{1}{(\upsilon-1)!\mathrm{n}g(\mathrm{n}x;\upsilon)} + \frac{x^2}{\mathrm{n}^2}\left(3 - \frac{3}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right) + \frac{x}{\mathrm{n}^3}\left(1 + \frac{3\upsilon+3}{(\upsilon-1)!g(\mathrm{n}x;\upsilon)}\right).$ (13)

CONSTRUCTION OF OPERATORS

Definition 2.1. (Weighted Space)([2]and [3]) Let $\sigma(x) = (1 + x^{\omega})$; where $\omega \geq 1$, be the weight function and γ_f be a positive constant, then we have
$C_{\sigma,\mathrm{K}}(\mathbb{R}_0) = \left\{f \in C[0,\infty) : \left|f(x) \leq \gamma_f \sigma(x)\right|, and \lim_{\mathrm{n}\to\infty} \frac{f(x)}{\sigma(x)} = \mathrm{K}_f < \infty\right\}$. The space $C_{\sigma,\mathrm{K}}$ consists of real-valued functions f, with the norm $\|f(x)\|_{\sigma} := sup_{x\in\mathbb{R}_0} \frac{|f(x)|}{\sigma(x)}$. We can say weighted space on all the functions belonging to $C_{\sigma,\mathrm{K}}(\mathbb{R}_0)$.
we used the class of operators which introduced by Z. Walczak [1] when he developed the famous Szãsz – Mirakyan operators ([4] and [5]) which defined in Eq. (1), Walczak introduced his operators for fix $\upsilon \in \mathbb{N}$, as follows
$A_{\mathrm{n}}^{\upsilon}(f;\upsilon;x) := \frac{1}{g(\mathrm{n}x;\upsilon)} \sum_{\ell=0}^{\infty} \frac{(\mathrm{n}x)^{\upsilon}}{(\ell+\upsilon)!} f\left(\frac{\ell+\upsilon}{\mathrm{n}}\right)$, $(x \in \mathbb{R}_0, \mathrm{n} \in \mathbb{R})$,
where $(\mathrm{t};\upsilon) := \sum_{\ell=0}^{\infty} \frac{\mathrm{t}^{\ell}}{(\ell+\upsilon)!}$, $(\mathrm{t} \in \mathbb{R}_0)$.
And then, used the definition of Beta function to establish a new Beta – type operators ([6], [7] and [8])as following:

$B_{\mathfrak{n},\ell+v}(f;x) = \sum_{\ell=0}^{\infty} b_{\mathfrak{n},\ell+v}(x)\, f(x),$ (14)

$b_{\mathfrak{n},\ell+v}(x) = \sum_{\ell=0}^{\infty} \frac{(\ell+v)!(\mathfrak{n}-1)!}{(\mathfrak{n}+\ell+v)\,!} x^{\ell+v}(1+x)^{-(\mathfrak{n}+\ell+v+1)},$ (15)

Now, we construct a new sequence of L. P. O., as below

For fix $v \in \mathbb{N}$, and $(x \in \mathbb{R}_0, \mathfrak{n} \in \mathbb{R})$

$\widetilde{A_{\mathfrak{n}}^{v}}(f;v;x) := \frac{1}{g(\mathfrak{n}x;v)} \sum_{\ell=0}^{\infty} \frac{(\mathfrak{n}x)^v}{(\ell+v)!} \int_0^{\infty} b_{\mathfrak{n},\ell+v}(\mathrm{t})\, f(\mathrm{t})\, dt,$ (16)

Where $\mathrm{t} = \frac{\ell+v}{\mathfrak{n}}$, $0 < g(\mathfrak{n}x;v) < 1$, and for a function f belong to the weighted space $C_{\sigma,\mathrm{K}}(\mathbb{R}_0)$, which we can define it as:

$C_{\sigma,\mathrm{K}}(\mathbb{R}_0) = \{f \in CB(\mathbb{R}_0)\colon f \text{ is bounded and continuous, such that } f(x) = O(1+x^{\rho}) \text{ for some } \rho \in \mathbb{N}\}$. Also the norm on $C_{\sigma,\mathrm{K}}(\mathbb{R}_0)$ is defined as

$\|f\|_{\rho} \equiv \|f(.)\|_{\rho} := sup_{x\in\mathbb{R}_0} \frac{|f(x)|}{1+x^{\rho}}.$ (17)

In our next computations need the following scientefic fact

$\int_0^{\infty} b_{\mathfrak{n},\ell}(\mathrm{t})\mathrm{t}^m\, dt = \frac{(m+\ell)!\,(\mathfrak{n}-m-1)!}{\ell!\,(\mathfrak{n}-1)!}$, for $m = 0,1,2,\ldots.$ (18)

AUXILIARY RESULTS

In this part of this paper we give a direct theorem including Korovkin's Theorem and then find the central moment on the operators $\widetilde{A_{\mathfrak{n}}^{v}}(.\,;v;x)$ defined in Eq. (16).

Theorem 3.1. (Bohman-Korovkin's Theorem)

Let $a_i(\mathrm{t}) := \mathrm{t}^i$, for $i = 0,1,2$ be the test functions. Then the operators defined in Eq. (16) satisfy:

1-$\widetilde{A_{\mathfrak{n}}^{v}}(a_0;v;x) = 1,$ (19)

2-$\widetilde{A_{\mathfrak{n}}^{v}}(a_1;v;x\ ) = \frac{\mathfrak{n}}{\mathfrak{n}-2}\ x + \frac{1}{(r-1\,)!(\,\mathfrak{n}-2)g(\mathfrak{n}\,x;v)} + \frac{1}{\mathfrak{n}-2},$ (20)

$$3\text{-}\widetilde{A_{\mathfrak{n}}^{v}}(a_2;v;x) = \frac{\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)}x^2 + \frac{\mathfrak{n}x}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\left(4 + \frac{1}{(v-1)!g(\mathfrak{n}x;v)}\right) + \frac{v+3}{(v-1\,)!(\mathfrak{n}-1)(\mathfrak{n}-2)g(\mathfrak{n}x;v)} + \frac{2}{(\mathfrak{n}-1)(\mathfrak{n}-2)}, \quad (21)$$

$$4\text{-}\widetilde{A_{\mathfrak{n}}^{v}}(a_3;v;x) = \frac{\mathfrak{n}^3}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}x^3 + \frac{\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}\left(9 + \frac{1}{(v-1)!g(\mathfrak{n}x;v)}\right)x^2 + \frac{\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}\left(18 + \frac{v+3}{(v-1)!g(\mathfrak{n}x;v)}\right)x + \frac{v^2+11}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(v-1)!g(\mathfrak{n}x;v)} + \frac{6}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}, \quad (22)$$

$$5\text{-}\widetilde{A_{\mathfrak{n}}^{v}}(a_4;v;x) = \frac{\mathfrak{n}^4}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)}x^4 + \frac{\mathfrak{n}^3}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)}\left(16 + \frac{1}{(v-1)!g(\mathfrak{n}x;v)}\right)x^3 + \frac{\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-3)(\mathfrak{n}-4)}\left(72 + \frac{v+6}{(v-1)!g(\mathfrak{n}x;v)}\right)x^2 + \frac{\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)}\left(46 + \frac{v^2+13v+108}{(v-1)!g(\mathfrak{n}x;v)}\right)x + \frac{v^3+10\,v^2+35\,v+50}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)(v-1)!g(\,\mathfrak{n}x;v)} + \frac{24}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\,\mathfrak{n}-4)}. \quad (23)$$

Proof. It is easy to prove Eq. (19) above; now to prove Eq. (20) we have

$$\widetilde{A_{\mathfrak{n}}^{v}}(a_1;v;x) = \frac{1}{g(\mathfrak{n}x;v)} \sum_{\ell=0}^{\infty} \frac{(\mathfrak{n}x)^v}{(\ell+v)!} \int_0^{\infty} b_{\mathfrak{n},\ell+v}(\mathrm{t}) \frac{\ell+v}{\mathfrak{n}}\, dt,$$

Using the fact in Eq. (18) we get

$$\widetilde{A_n^{\upsilon}}(a_1;\upsilon;x)=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\left\{\frac{\ell+\upsilon}{n-2}+\frac{1}{n-2}\right\}$$

Using some simplifications and Lemma (1.1) we get

$\widetilde{A_n^{\upsilon}}(a_1;\upsilon;x\ )=\frac{n}{n-2}x+\frac{1}{(\upsilon-1)!(n-2)g(nx;\upsilon)}+\frac{1}{n-2};$

To prove Eq. (21)

$$\widetilde{A_n^{\upsilon}}(a_2;\upsilon;x)=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\int_0^{\infty} b_{n,\ell+\upsilon}(\mathrm{t})\,(\frac{\ell+\upsilon}{n})^2\,dt,$$

Also, using the fact in Eq. (18) we get

$\widetilde{A_n^{\upsilon}}(a_2;\upsilon;x)=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\left\{\frac{(\ell+\upsilon)^2}{(n-1)(n-2)}+\frac{3(\ell+\upsilon)}{(n-1)(n-2)}+\frac{2}{(n-1)(n-2)}\right\}$

$=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\frac{(\ell+\upsilon)^2}{(n-1)(n-2)}+\frac{3}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\frac{(\ell+\upsilon)}{(n-1)(n-2)}+\frac{2}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\frac{1}{(n-1)(n-2)}$

Using some simplifications and lemma (1.1) we get

$\widetilde{A_n^{\upsilon}}(a_2;\upsilon;x)=\frac{n^2}{(n-1)(n-2)}x^2+\frac{nx}{(n-1)(n-2)}\left(4+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)+\frac{\upsilon+3}{(\upsilon-1)!(n-1)(n-2)g(nx;\upsilon)}+\frac{2}{(n-1)(n-2)}.$

To prove Eq. (22)

$$\widetilde{A_n^{\upsilon}}(a_3;\upsilon;x)=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\int_0^{\infty} b_{n,\ell+\upsilon}(\mathrm{t})\,(\frac{\ell+\upsilon}{n})^3\,dt,$$

Also, using the fact in Eq.(18) we get

$\widetilde{A_n^{\upsilon}}(a_3;\upsilon;x)=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\left\{\frac{(\ell+\upsilon)^3}{(n-1)(n-2)(n-3)}+\frac{6(\ell+\upsilon)^2}{(n-1)(n-2)(n-3)}+\frac{11(\ell+\upsilon)}{(n-1)(n-2)(n-3)}+\frac{6}{(n-1)(n-2)(n-3)}\right\}$

$=\frac{n^3}{(n-1)(n-2)(n-3)}\left[x^3+\frac{x^2}{n}\left(3+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)+\frac{x}{n^2}\left(1+\frac{\upsilon+2}{(\upsilon-1)!g(nx;\upsilon)}\right)+\frac{\upsilon^2}{(\upsilon-1)!n^3g(nx;\upsilon)}\right]+\frac{6n^2}{(n-1)(n-2)(n-3)}\left[x^2+\frac{x}{n}\left(1+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)+\frac{\upsilon}{(\upsilon-1)!n^2g(nx;\upsilon)}\right]+\frac{11n}{(n-1)(n-2)(n-3)}\left[x+\frac{1}{(\upsilon-1)!ng(nx;\upsilon)}\right]+\frac{6}{(n-1)(n-2)(n-3)}\,,$

$\widetilde{A_n^{\upsilon}}(a_3;\upsilon;x)=\frac{n^3}{(n-1)(n-2)(n-3)}x^3+\frac{n^2}{(n-1)(n-2)(n-3)}\left(9+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)x^2+\frac{n}{(n-1)(n-2)(n-3)}\left(18+\frac{\upsilon+3}{(\upsilon-1)!g(nx;\upsilon)}\right)x+\frac{\upsilon^2+11}{(\upsilon-1)!(n-1)(n-2)(n-3)g(nx;\upsilon)}+\frac{6}{(n-1)(n-2)(n-3)}.$

Finally, we must to prove Eq. (23)

$\widetilde{A_n^{\upsilon}}(a_4;\upsilon;x)=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\int_0^{\infty} b_{n,\ell+\upsilon}(\mathrm{t})\,(\frac{\ell+\upsilon}{n})^4\,dt,$

Also, using the fact in Eq. (18) we get

$\widetilde{A_n^{\upsilon}}(a_4;\upsilon;x)=\frac{1}{g(nx;\upsilon)}\sum_{\ell=0}^{\infty}\frac{(nx)^{\upsilon}}{(\ell+\upsilon)!}\left\{\frac{(\ell+\upsilon)^4}{(n-1)(n-2)(n-3)(n-4)}+\frac{10(\ell+\upsilon)^3}{(n-1)(n-2)(n-3)(n-4)}+\frac{35(\ell+\upsilon)^2}{(n-1)(n-2)(n-3)(n-4)}+\frac{50(\ell+\upsilon)}{(n-1)(n-2)(n-3)(n-4)}+\frac{24}{(n-1)(n-2)(n-3)(n-4)}\right\}$

$=\frac{n^4}{(n-1)(n-2)(n-3)(n-4)}\left[x^4+\left(6+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)\frac{x^3}{n}+\left(7+\frac{\upsilon+5}{(\upsilon-1)!g(nx;\upsilon)}\right)\frac{x^2}{n^2}+\left(1+\frac{\upsilon^2+3\upsilon+3}{(\upsilon-1)!g(nx;\upsilon)}\right)\frac{x}{n^3}+\frac{\upsilon^3}{(\upsilon-1)!n^4g(nx;\upsilon)}\right]+\frac{10n^3}{(n-1)(n-2)(n-3)(n-4)}\left[x^3+\frac{x^2}{n}\left(3+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)+\frac{x}{n^2}\left(1+\frac{\upsilon+2}{(\upsilon-1)!g(nx;\upsilon)}\right)+\frac{\upsilon^2}{(\upsilon-1)!n^3g(nx;\upsilon)}\right]+\frac{35n^2}{(n-1)(n-2)(n-3)(n-4)}\left[x^2+\frac{x}{n}\left(1+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)+\frac{\upsilon}{(\upsilon-1)!n^2g(nx;\upsilon)}\right]+\frac{50n}{(n-1)(n-2)(n-3)(n-4)}\left[x+\frac{1}{n(\upsilon-1)!g(nx;\upsilon)}\right]+\frac{24}{(n-1)(n-2)(n-3)(n-4)}\,,$

$\widetilde{A_n^{\upsilon}}(a_4;\upsilon;x\ )=\frac{n^4}{(n-1)(n-2)(n-3)(n-4)}x^4+\frac{n^3}{(n-1)(n-2)(n-3)(n-4)}\left(16+\frac{1}{(\upsilon-1)!g(nx;\upsilon)}\right)x^3+\frac{n^2}{(n-1)(n-2)(n-3)(n-4)}\left(72+\frac{\upsilon+6}{(\upsilon-1)!g(nx;\upsilon)}\right)x^2+\frac{n}{(n-1)(n-2)(n-3)(n-4)}\left(46+\frac{\upsilon^2+13\upsilon+108}{(\upsilon-1)!g(nx;\upsilon)}\right)x+\frac{\upsilon^3+10\upsilon^2+35\upsilon+50}{(\upsilon-1)!(n-1)(n-2)(n-3)(n-4)g(nx;\upsilon)}+\frac{24}{(n-1)(n-2)(n-3)(n-4)}.$

So, we get the required.

■

Next, from theorem (3.1) we institute the moment for the operators $\widetilde{A_n^{\upsilon}}(f;\upsilon;x)$ as following:

Lemma 3.1. Let us obtain the results for the moments of the operators $\widetilde{A_\mathfrak{n}^\upsilon}(.\,;\upsilon;x)$ defined in Eq. (18) above for $ð \in \mathbb{N}^0$ and each $x \in [0,\infty)$, where $\widetilde{\psi_\mathfrak{n}^ð}(x) := (\mathrm{t}-x)^ð$ we find the central moments $\widetilde{A_\mathfrak{n}^\upsilon}(\widetilde{\psi_\mathfrak{n}^ð};\upsilon;x)$ with order ð , we get

1-$\widetilde{A_\mathfrak{n}^\upsilon}(\widetilde{\psi_\mathfrak{n}^0};\upsilon;x) = 1,$ (24)

2-$\widetilde{A_\mathfrak{n}^\upsilon}(\widetilde{\psi_\mathfrak{n}^1};\upsilon;x) = \left\{\frac{\mathfrak{n}}{\mathfrak{n}-2}-1\right\} x + \frac{1+(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}{(\upsilon-1)!(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)},$ (25)

3-$\widetilde{A_\mathfrak{n}^\upsilon}(\widetilde{\psi_\mathfrak{n}^2};\upsilon;x) = \left\{\frac{\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)} - \frac{2\mathfrak{n}}{\mathfrak{n}-2}+1\right\}x^2 + \left\{\frac{\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\left(4+\frac{1}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) - \frac{2}{(\upsilon-1)!(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)} - \frac{2}{(\mathfrak{n}-2)}\right\}x +$
$\frac{\upsilon+3+2(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}{(\upsilon-1)!(\mathfrak{n}-1)(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)},$ (26)

4-$\widetilde{A_\mathfrak{n}^\upsilon}(\widetilde{\psi_\mathfrak{n}^3};\upsilon;x) = \left\{\frac{\mathfrak{n}^3}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)} - \frac{3\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)} + \frac{3\mathfrak{n}}{\mathfrak{n}-2} - 1\right\}x^3 + \left\{\frac{\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}\left(9+\frac{1}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) -\right.$
$3\frac{\mathfrak{n}x}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\left(4+\frac{1}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) + 3\frac{1}{(\upsilon-1)!(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)} + \left.\frac{3}{\mathfrak{n}-2}\right\}x^2 + \left\{\frac{\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}\left(18+\frac{\upsilon+3}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) -\right.$
$\left.\frac{3(\upsilon+3)}{(\upsilon-1)!(\mathfrak{n}-1)(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)} - \frac{6}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\right\}x + \frac{\upsilon^2+11+6(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\upsilon-1)!g(\mathfrak{n}x;\upsilon)},$ (27)

5-$\widetilde{A_\mathfrak{n}^\upsilon}(\widetilde{\psi_\mathfrak{n}^4};\upsilon;x) = \left\{\frac{\mathfrak{n}^4}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)} - 4\frac{\mathfrak{n}^3}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)} + 6\frac{\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)} - 4\frac{\mathfrak{n}}{\mathfrak{n}-2}+1\right\}x^4 +$

$\left\{\frac{\mathfrak{n}^3}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)}\left(16+\frac{1}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) - \frac{4\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}\left(9+\frac{1}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) +\right.$

$\left.\frac{6\mathfrak{n}x}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\left(4+\frac{1}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) - \frac{4+4(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}{(\upsilon-1)!(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)}\right\}x^3 + \left\{\frac{\mathfrak{n}^2}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-3)(\mathfrak{n}-4)}\left(72+\frac{\upsilon+6}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) -\right.$

$\left.\frac{4\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)}\left(18+\frac{\upsilon+3}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) + \frac{2(3\upsilon+9+6(\upsilon-1)!g(\mathfrak{n}x;\upsilon))}{(\upsilon-1)!(\mathfrak{n}-1)(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)}\right\}x^2 + \left\{\frac{\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)}\left(46+\frac{\upsilon^2+13\upsilon+108}{(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right) -\right.$

$\left.\frac{4(\upsilon^2+11-6(\upsilon-1)!g(\mathfrak{n}x;\upsilon))}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right\}x + \frac{\upsilon^3+10\upsilon^2+35\upsilon+50}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)(\upsilon-1)!g(\mathfrak{n}x;\upsilon)} + \frac{24}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\mathfrak{n}-3)(\mathfrak{n}-4)}.$ (28)

Theorem 3.2. (Voronoviskaja Theorem) Let $f \in C_{\sigma,\mathrm{K}}(\mathbb{R}_0)$. Then for any $x \in (0,\infty)$ at which $f''(x)$ exists, we have

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{n}\left\{\widetilde{A_\mathfrak{n}^\upsilon}(f(\mathrm{t});\upsilon;x) - f(x)\right\} = (2x+1)f'(x) + \frac{x(6-x)}{2}f''(x). \quad (29)$$

Proof: Let $x \in [0,\infty)$ be fixed. by Taylor's expansion, we get

$$f(\mathrm{t}) = f(x) + (\mathrm{t}-x)f'(x) + \frac{(\mathrm{t}-x)^2}{2!}f''(x) + (\mathrm{t}-x)^2\varepsilon(\mathrm{t};x), \quad (30)$$

Where $\varepsilon(\mathrm{t};x)$ is the Peano form of the remainder, $\varepsilon(\mathrm{t};x) \in C_{\sigma,\mathrm{K}}(\mathbb{R}_0)$, using L'Hopital's rule, we get

$$\lim_{\mathrm{t}\to x} \varepsilon(\mathrm{t};x) = \lim_{\mathrm{t}\to x} \frac{f(\mathrm{t}) - f(x) + (\mathrm{t}-x)f'(x) + \frac{(\mathrm{t}-x)^2}{2!}f''(x)}{(\mathrm{t}-x)^2} = \lim_{\mathrm{t}\to x}\frac{f'(\mathrm{t}) - f'(x) + (\mathrm{t}-x)f''(x)}{2(\mathrm{t}-x)}$$
$$= \lim_{\mathrm{t}\to x}\frac{f''(\mathrm{t}) - f''(x)}{2} = 0$$

Applying the operators $\widetilde{A_\mathfrak{n}^\upsilon}(.\,;\upsilon;x)$ to Eq. (30) we obtain

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{n}\left[\widetilde{A_\mathfrak{n}^\upsilon}(f;\upsilon;x) - f(x)\right] = f'(x)\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\widetilde{A_n^\upsilon}(\mathrm{t}-x;\upsilon;x) + \frac{f''(x)}{2}\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\widetilde{A_\mathfrak{n}^\upsilon}((\mathrm{t}-x)^2;\upsilon;x) +$$
$$\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\widetilde{A_\mathfrak{n}^\upsilon}(\varepsilon(\mathrm{t};x)(\mathrm{t}-x)^2;\upsilon;x). \quad (31)$$

Depending on Cauchy - Schwarz inequality, we have

$$\widetilde{A_{\mathfrak{n}}^{\upsilon}}(\varepsilon(\mathrm{t};x)(\mathrm{t}-x)^2;\upsilon;x) \leq \sqrt{\widetilde{A_{\mathfrak{n}}^{\upsilon}}(\varepsilon^2(\mathrm{t};x);\upsilon;x)}\sqrt{\widetilde{A_{\mathfrak{n}}^{\upsilon}}((\mathrm{t}-x)^4;\upsilon;x)}, \tag{32}$$

Since $\varepsilon^2(x;x)=0$, then we can obtain

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{n}\widetilde{A_{\mathfrak{n}}^{\upsilon}}(\varepsilon(\mathrm{t};x)(\mathrm{t}-x)^2;\upsilon;x)=0 \tag{33}$$

So, by lemma (3.1), theorem (3.1) and Eq. (31), (33), we get

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{n}\left\{\widetilde{A_{\mathfrak{n}}^{\upsilon}}(f(\mathrm{t});\upsilon;x)-f(x)\right\}=(2x+1)f'(x)+\frac{x(6-x)}{2}f''(x).$$

Therefore, the proof of theorem (3.2) is obtained.

SOME APPROXIMATION PROPERTIES IN WEIGHTED SPACE

At the last part of this paper, we would like to study the estimate of order for approximation of the function f by a certain L.P.O. $\left\{\widetilde{A_{\mathfrak{n}}^{\upsilon}}(.\,;\upsilon;x)\right\}$ using the modulus of continuity.

At the first, need to introduce the concept of the modulus of continuity from the first order.

Definition 4.1. (modulus of continuity) ([9],[10] and [2]).The modulus of continuity $\omega(f;\delta)$ for $f\in C[a,b]$ and if $\delta > 0$, can be defined by

$$\omega(f;\delta)=\sup_{\substack{x,y\,\in\,[a,b],\\ |x-y|\leq\delta}}|f(x)-f(y)|,$$

Below, we deal with this theorem to study the order of approximation.

Theorem 4.1. Suppose $f\in C_{\sigma,\mathrm{K}}(\mathbb{R}_0)$, $\upsilon\in\mathbb{N}$, then

$$\left|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(.\,;\upsilon;x)-f(x)\right|\leq 2\omega(f;\delta) \tag{34}$$

where $\delta=\sqrt{\frac{2-\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)}x^2+\left\{\frac{2\mathfrak{n}+2}{(\mathfrak{n}-1)(\mathfrak{n}-2)}+\frac{2-\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right\}x+\frac{\upsilon+\mathfrak{n}+2+2g(\mathfrak{n}x;\upsilon)(\upsilon-1)!}{(\upsilon-1)!(\mathfrak{n}-1)(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)}}$.

Proof: Depending on the most popular property of modulus of continuity

$$|f(\mathrm{t})-f(x)|\leq\omega(f;\delta)\left(\frac{|\mathrm{t}-x|}{\delta}+1\right) \tag{35}$$

By reference to the fact that the family $\left\{\widetilde{A_{\mathfrak{n}}^{\upsilon}}(.\,,\upsilon;x)\right\}$ is L. P. O., so we have

$$\left|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(.\,,\upsilon;x)-f(x)\right|\leq\left(\frac{1}{\delta}\widetilde{A_{\mathfrak{n}}^{\upsilon}}(|\mathrm{t}-x|,\upsilon;x)+1\right)\omega(f;\delta)$$

Enforcement Cauchy- Schwartz inequality and Eq. (26) we obtain

$$\left|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(.\,,\upsilon;x)-f(x)\right|\leq\omega(f;\delta)\left(\frac{1}{\delta}\sqrt{\left(\widetilde{A_{\mathfrak{n}}^{\upsilon}}(\mathrm{t}-x)^2,\upsilon;x\right)}+1\right)$$

$$\leq\omega(f;\delta)\left(\frac{1}{\delta}\left(\sqrt{\left|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(\widetilde{\psi_{\mathfrak{n}}^{2}};\upsilon;x)\right|}\right)+1\right)$$

$$\leq\omega(f;\delta)\times\left(\frac{1}{\delta}\sqrt{\frac{2-\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)}x^2+\left\{\frac{2\mathfrak{n}+2}{(\mathfrak{n}-1)(\mathfrak{n}-2)}+\frac{2-\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right\}x+\frac{\upsilon+\mathfrak{n}+2+2g(\mathfrak{n}x;\upsilon)(\upsilon-1)!}{(\upsilon-1)!(\mathfrak{n}-1)(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)}}+1\right)$$

Optionally

$$\delta=\sqrt{\frac{2-\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)}x^2+\left\{\frac{2\mathfrak{n}+2}{(\mathfrak{n}-1)(\mathfrak{n}-2)}+\frac{2-\mathfrak{n}}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\upsilon-1)!g(\mathfrak{n}x;\upsilon)}\right\}x+\frac{\upsilon+\mathfrak{n}+2+2g(\mathfrak{n}x;\upsilon)(\upsilon-1)!}{(\upsilon-1)!(\mathfrak{n}-1)(\mathfrak{n}-2)g(\mathfrak{n}x;\upsilon)}},$$

Therefore, we get $\left|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(f,\upsilon;x)-f(x)\right|\leq 2\omega(f;\delta)$. ■

Theorem 4.2. For $\in C_{\sigma,\mathrm{K}}(\mathbb{R}_0)$, $\upsilon\in\mathbb{N}$ and $x\in[0,\infty)$. Then we get for $\xi\in\mathbb{R}^+,(\xi>0)$, the following relation

$$\lim_{\mathfrak{n}\to\infty} \widetilde{A_{\mathfrak{n}}^{\upsilon}}(f,\upsilon;x) = f(x), \tag{36}$$
satisfies uniformly on $[0,\xi]$.

Proof. Depending on the Eq. (19), (20) and (21) respectively, we obtain
$$\left\|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(1;x) - 1\right\|_{C[0,\xi]} = 0, \tag{37}$$
$$\left\|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(\mathrm{t};x) - x\right\|_{C[0,\xi]} = \max_{x\in[0,\xi]} \left\{\frac{2}{\mathfrak{n}-2}\ x + \frac{1}{\mathfrak{n}-2} + \frac{1}{(\mathfrak{n}-2)(\upsilon-1)!\,g(\mathfrak{n}x;\upsilon)}\right\}, \tag{38}$$
$$\leq \frac{2}{\mathfrak{n}-2}\xi + \frac{1}{\mathfrak{n}-2} + \frac{1}{(\mathfrak{n}-2)(\upsilon-1)!\,g(\mathfrak{n}\xi;\upsilon)} \to 0 \text{ as } \mathfrak{n}\to\infty.$$
$$\left\|\widetilde{A_{\mathfrak{n}}^{\upsilon}}(\mathrm{t}^2;x) - x^2\right\|_{C[0,\xi]}$$
$$= \max_{x\in[0,\xi]} \left\{\frac{3\mathfrak{n}-2}{(\mathfrak{n}-1)(\mathfrak{n}-2)} x^2 + \frac{\mathfrak{n}x}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\left(4 + \frac{1}{(\upsilon-1)!\,g(\mathfrak{n}x;\upsilon)}\right) + \frac{\upsilon+3}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\upsilon-1)!\,g(\mathfrak{n}x;\upsilon)} + \frac{2}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\right\}, \tag{39}$$
$$\leq \frac{3\mathfrak{n}-2}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\xi^2 + \frac{\mathfrak{n}\xi}{(\mathfrak{n}-1)(\mathfrak{n}-2)}\left(4 + \frac{1}{(\upsilon-1)!\,g(\mathfrak{n}\xi;\upsilon)}\right) + \frac{\upsilon+3}{(\mathfrak{n}-1)(\mathfrak{n}-2)(\upsilon-1)!\,g(\mathfrak{n}\xi;\upsilon)} + \frac{2}{(\mathfrak{n}-1)(\mathfrak{n}-2)} \to 0 \text{ for sufficiently large } \mathfrak{n}.$$
So, by Eq. (37), (38) and (39) we got the required proof by reference to P. P. Korovkin [11]. ■

CONCLUSION

The motive of the present paper is to define a new sequence of L.P.O constructing from Szãsz type operators and Beta function and discused some approximation properties of these operators. So, we investigated some approximation results. Also, we estabished a Voronoviskaja theorem and finally, we find the rate of convergent, and found the uniformly convergent on it.

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