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On Truncated of classical Beta Operators

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Abstract

In this paper, we study the truncated of classical Beta operators $\beta_{n,N}(x)$ and the error occurs by the approximation. We estimate the truncated error in terms of modulus of continuity and the weighted norm of the function being approximated.

Keywords: MKZ Truncated operators, approximated order, Beta operators, modulus of continuity.

Introduction

A study of truncated of sequence of linear positive operators is a branch of approximation theory. In [4],[9],[12],[11] and [5] there is a review of truncated of some especially Szãsz –Mirakyan operators. Here, we study the truncated of the sequences of linear positive operators (Beta operators$\beta_{n,N}(x)$).Then, we find the error occurs by this approximation of truncated of Beta operator in terms of modulus of continuity of a function being approximated. For more properties of Beta operators you can read the articles [1],[2],[3],[6] and [7].

In the beginning, we give some preliminaries as follow:

$$\beta_n(f;x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\, f(\frac{k}{n}), \tag{1}$$

where

$$b_{n,k}(x) := n\binom{n+k}{k} x^k (1+x)^{-n-k-1} \tag{2}$$

$n \in \mathbb{N}$ and $x \in \mathbb{R}_0 = [0,\infty)$.

For the functions f belonging to the weighted space C_p,

$$C_p = \{f \in C[0,\infty) : f(x) = O(1+x^p) \text{for some } p \in \mathbb{N}\},$$

$$\|f\|_p \equiv \|f(.)\|_p = sup_{x\in\mathbb{R}_0} u_p(x)|f(x)|,$$

$$u_p(x)\beta_n(|f(t)-f(x)|;x) \le K_1(p)\omega_2\left(f; C_p; \sqrt{\frac{x(1+x)}{n} + \frac{x(2x+1)}{n^2}}\right), \tag{3}$$

for $f \in C_p, n \in \mathbb{N}, x \in \mathbb{R}_0$ where $\omega_2(f;C_p)$ is the modulus of continuity of f, we can get the properties of $\beta_{n,N}(f)$ as follows:

$$\beta_{n,N}(f;x) = \sum_{k=0}^{N} b_{n,k}(x) f\left(\frac{k}{n}\right).$$

We can evaluate that:

$$\sum_{k=0}^{\infty} b_{n,k}(x) = 1, \text{for } x \in [0,1) \ , n \in N, \tag{4}$$

$$\beta_n(f(t);0) = f(0) = \beta_{n,N}(f(t);0)$$

and

$$\beta_{n,N}(f(t);x) - f(x) = \beta_{n,N}(f(t)-f(x);x) - f(x)\textstyle\sum_{k=N+1}^{\infty} b_{n,k}(x) \tag{5}$$

for every $f \in C(I), x \in (0,1)$ and $n, N \in \mathbb{N}$.

In some papers ([4],[9],[11]and[12]) the authors investigated the truncated Szãsz-Mirakyan operators defined as:

$$S_{n,N}(f;x) = e^{-nx} \sum_{k=0}^{N} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \; x \in \mathbb{R}_0, n \in \mathbb{N},$$

where $f: \mathbb{R}_0 \rightarrow \mathbb{R}$, and $N = N(n,x)$ are positive integers depending on n and x. The results presented in some papers [4],[9]and [12] show that if $x > 0$ is a fixed point and $N = N(n,x)$ are integers such that $N > nx$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{N-nx}{\sqrt{n}} = \infty$, then

$$\lim_{n\to\infty} S_{n,N}(f;x) = f(x) \text{ for every } f \in C_p, \; p \in \mathbb{N}_0.$$

The aim of this note is to derive similar results for the truncated MKZ operators.

L. Rempulska and M. Skorupka in [10] are considered a strong approximation of $f \in C_p$for some linear positive operators.

The well-known Meyer-König and Zeller operators that we called MKZ is given by

$$M_n(f;x) = \begin{cases} \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n+k}\right) & \text{if } \; 0 \le x < 1 \\ f(1) & \text{if } \; x = 1 \end{cases} \tag{6}$$

$$p_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1} \tag{7}$$

Where $f \in C(I), x \in I = [0,1]$ and $n \in \mathbb{N} = \{1,2,\ldots\}, C(I)$is the space of continuous functions on I with the fallowing norm

$$\|f\| = \sup\{|f(x)|: x\epsilon I\}$$

$$\text{and } \|M_n(f) - f\| \le \tfrac{31}{27}\omega\left(f, \tfrac{1}{\sqrt{n}}\right), n \in N. \tag{8}$$

In [10] Rempulska were examined the strong approximation of by MKZ operators and he proved the inequality

$$M_n(|f(t) - f(x)|; x) \le \tfrac{3}{2}\omega(f; \tfrac{1}{\sqrt{n}}) \tag{9}$$

for $x \in [0,1]$, $n \in \mathbb{N}$.

In this paper, we show that an analog of MKZ operators connected with Beta operators.

Theorem (1). [5]

Let $x \in (0,1)$ and $n \in \mathbb{N}$ be fixed, and let $N = N(n,x)$ be an integer such that

$N > (n+1)\frac{x}{1-x}$. Then

$$\left|M_{n,N}(f(t);x) - f(x)\right| \le \tfrac{3}{2}\omega\left(f; \tfrac{1}{\sqrt{n}}\right) + |f(x)| \tfrac{1}{\sqrt{2\pi nx}} \tfrac{N}{(1-x)N - nx + 1 - x} \tag{10}$$

holds for every $f \in C(I)$.

Corollary(1). [5]

Let $x_0 \in (0,1)$ and let $N = N(n, x_0)$ be integers such that

(i) $N > \frac{(n+1)x_0}{(1-x_0)}$ for $n \in \mathbb{N}$,

(ii) $(N/n)_{n=1}^{\infty}$ is a bounded sequence,

(iii) $\lim_{n\to\infty} \frac{N - \frac{(n+1)x_0}{1-x_0}}{\sqrt{n}} = \infty$

Then

$$\lim_{n\to\infty} M_{n,N}(f; x_0) = f(x_0) \text{holds for every } f \in C(I). \tag{11}$$

Corollary(2). [5]

Let $x_0 \in (0,1)$and let $N = \left[\frac{n+x_0}{1-x_0}\right]$ for $n \in \mathbb{N}$.($[y]$ denotes the integral part of $y \in \mathbb{R}$). Then the conditions (i)-(iii) are satisfied, and consequently the convergence (8) holds for every $f \in C(I)$. Moreover, we have

$$\left|M_{n,N}(f; x_0) - f(x_0)\right| \leq \frac{3}{2}\omega\left(f; \frac{1}{\sqrt{n}}\right) + K_2(x_0)|f(x_0)|\frac{1}{\sqrt{n}}$$

for $f \in C(I)$ and $n \in \mathbb{N}$, where $K_2(x_0) = const. > 0$.

Corollary(3). [5]

If $x_0 \in (0,1)$ *and* $N = \left[(n+1)\frac{x_0}{1-x_0} + \sqrt{n\alpha_n}\right]$ for $n \in \mathbb{N}$, where (α_n) is a non- bounded sequence of positive numbers such that $(\frac{\alpha_n}{\sqrt{n}})_1^{\infty}$ is bounded , then (i)-(iii) and

(11) are satisfied , and the inequality

$$\left|M_{n,N}(f; x_0) - f(x_0)\right| \leq \frac{3}{2}\omega\left(f; \frac{1}{\sqrt{n}}\right) + K_3(x_0)|f(x_0)|\frac{1}{\alpha_n}$$

holds for every x $f \in C(I)$ and $n \in \mathbb{N}$, where $K_3(x_0) = const. > 0$.

Here, we want to prove the above results in the space C_p with weighted norm:

Theorem (2).

Suppose that $p \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $x \in (0, \infty)$. For $N = N(n, x)$ be an integer such that $N > nx$, then the inequality

$$u_p(x)\left|\beta_{n,N}(f(t,x) - f(x); x)\right| \leq K_1(p)\omega_2\left(f; C_p; \sqrt{\frac{x(1+x)}{n} + \frac{x(2x+1)}{n^2}}\right)$$

$$+\|f\|_p \frac{\sqrt{n(1+x)+1}}{\sqrt{2\pi(n^2x - nx)}} \frac{(x+1)N}{N+nx}, \tag{12}$$

holds for every $f \in C_p$, where $K_1(p) > 0$ is constant and is given in (3).

Proof:

For given, $f \in C(I), x \in (0,1)$ and $n, N \in \mathbb{N}$ we get by (1),(3),(4) and (5):

$$\beta_{n,N}(f(t);x) - f(x) = \beta_{n,N}(f(t) - f(x);x) - f(x) \quad (13)$$

$$u_p(x)\left|\beta_{n,N}(f(t);x) - f(x)\right|$$

$$\leq u_p(x)\beta_{n,N}(|f(t) - f(x)|;x) + \left|u_p(x)f(t)\sum_{k=N+1}^{\infty} b_{n,k}(x)\right|$$

$$\leq K_1(p)\omega\left(f; C_p; \sqrt{\frac{x(x+1)}{n} + \frac{x(2x+1)}{n^2}}\right) + u_p(x)|f(x)|\sum_{k=1}^{\infty} b_{n,N+k}(x)$$

$$\leq K_1(p)\omega\left(f; C_p; \sqrt{\frac{x(x+1)}{n} + \frac{x(2x+1)}{n^2}}\right)$$

$$+u_p(x)|f(x)|n(1+x)^{-n-1}\sum_{k=1}^{\infty}\binom{n+N+k}{N+k}x^{N+k}(1+x)^{-k}$$

$$\leq K_1(p)\omega\left(f; C_p; \sqrt{\frac{x(x+1)}{n} + \frac{x(2x+1)}{n^2}}\right)$$

$$+u_p(x)\,|f(x)|b_{n,N}(x)\sum_{k=1}^{\infty}\frac{(n+N+1)(n+N+2)\ldots(n+N+k)}{(N+1)(N+2)\ldots(N+k)}\left(\frac{x}{1+x}\right)^k$$

$$\leq K_1(p)\omega\left(f; C_p; \sqrt{\frac{x(x+1)}{n} + \frac{x(2x+1)}{n^2}}\right)$$

$$+u_p(x)|f(x)|b_{n,N}(x)\sum_{k=1}^{\infty}\left(1+\frac{n}{N+1}\right)\left(1+\frac{n}{N+2}\right)\ldots\left(1+\frac{n}{N+k}\right)\left(\frac{x}{1+x}\right)^k$$

$$\leq K_1(p)\omega\left(f; C_p; \sqrt{\frac{x(x+1)}{n} + \frac{x(2x+1)}{n^2}}\right)$$

$$+u_p(x)|f(x)|b_{n,N}(x)\sum_{k=1}^{\infty}\left[\left(1+\frac{n}{N+1}\right)\frac{x}{1+x}\right]^k. \quad (14)$$

By the assumptions: $x \in (0,1)$ and $N > nx$ we can write

$$\therefore \sum_{k=N+1}^{\infty} b_{n,k}(x) \leq b_{n,N}(x)\sum_{k=1}^{\infty}\left(\left(1+\frac{n}{N+1}\right)\frac{x}{1+x}\right)^k$$

$$< b_{n,N}(x)\sum_{k=1}^{\infty}\left(\left(1+\frac{n}{N}\right)\frac{x}{1+x}\right)^{k}$$

$$= b_{n,N}(x)\,\frac{(N+n)x}{N+nx}.$$

If $N > nx$ and $x > 0$, we have

$$b_{n,N}(x) < b_{n,N}\left(\frac{N}{n}\right) = n\binom{n+N}{N}\left(\frac{N}{n}\right)^{N}\left(1+\frac{N}{n}\right)^{-n-N-1}$$

$$= \frac{n(n+N)!}{N!\,(n-1)!}\,\frac{N^{N}e^{-N}}{N!}\left(\frac{(n+N)^{-n-N-1}e^{-n-N-1}}{(n+N)!}\right)^{-1}\left(\frac{n^{n}e^{-n-1}}{n!}\right).$$

By using the Stirling formula, consequently

$n! \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$, we have

$$b_{n,N}(x) < \frac{n+1}{N+n+1}\left(\frac{1}{\sqrt{2\pi N}}\right)^{-1}\left(\frac{1}{\sqrt{2\pi(n-1)}}\right)^{-1}\frac{1}{\sqrt{2\pi(n+N+1)}}$$

$$= \frac{\sqrt{2\pi(n+N+1)}}{\sqrt{4\pi N(n-1)}} = \frac{\sqrt{n+N+1}}{\sqrt{2\pi N(n-1)}}.$$

Therefore,

$$u_p(x)|f(x)|\sum_{k=N+1}^{\infty} b_{n,k}(x) \le \|f\|_p\,\frac{\sqrt{n+N+1}}{\sqrt{2\pi N(n-1)}}\,\frac{(N+n)x}{N+nx}. \qquad (15)$$

Putting $N = nx$ then we get $x = \frac{N}{n}, x > 0$

$$u_p(x)|f(x)|\sum_{k=N+1}^{\infty} b_{n,k}(x) \le \|f\|_p\,\frac{\sqrt{n+nx+1}}{\sqrt{2\pi nx(n-1)}}\,\frac{(xn+n)N}{(N+nx)n}$$

$$\le \|f\|_p\,\frac{\sqrt{n(1+x)+1}}{\sqrt{2\pi(n^2x-nx)}}\,\frac{(xn+n)N}{(N+nx)n}$$

$$\le \|f\|_p\,\frac{\sqrt{n(1+x)+1}}{\sqrt{2\pi(n^2x-nx)}}\,\frac{(x+1)N}{(N+nx)}.$$

from (13) ,(14), (15) we have results (12) ■

Corollary(4).

Let $x_0 \in (0,\infty)$ and let $N = (n, x_0)$be positive integers such that for every $n \in \mathbb{N}$:

(i) $N > nx_0$,
(ii) $(N/n)_{n=1}^{\infty}$ *is* a bounded sequence,

(iii) $\lim_{n\to\infty}\frac{N-nx}{\sqrt{n}}=\infty.$

Then the convergence

$$\lim_{n\to\infty}\beta_{n,N}(f;x_0)=f(x_0) \qquad (16)$$

holds for every $f\in C_p$, $p\in\mathbb{N}_0$.

Corollary(5)

For a fixed $x_0>0$ let $N=[n\,(x_0+1)]$for $n\in\mathbb{N}$. Then the conditions (i), (ii) and (iii) are satisfied, and hence (16) holds for $f\in C_p$. Also, we have

$$u_p(x_0)\left|\beta_{n,N}(f;x_0)-f(x_0)\right|\le K_4(\omega(f;C_\rho;\sqrt{\frac{x_0^2+x_0}{n}+\frac{x_0(2x_0+1)}{n^2}}+\|f\|_\rho\frac{1}{\sqrt{n}}))$$

for every $f\in C_p$and $n\in\mathbb{N}$, where $K_4=K_4(p,x_0)$ is a positive constant depending on ρ and x_0.

Corollary(6)

Let $x_0\in(0,\infty)$ and let $N=[nx_0+\sqrt{n}\alpha_n]$ for $n\in\mathbb{N}$, where (α_n) is a non-bounded sequence of numbers $\alpha_n\ge 1$ such that $(\frac{\alpha_n}{\sqrt{n}})_{n=1}^{\infty}$is bounded. Then (i), (ii) and (iii) are satisfied and hence (12) holds for every $f\in C_p$. Moreover, we have

$$u_p(x)\left|\beta_{n,N}(f;x_0)-f(x_0)\right|$$
$$\le K_5(\rho,x_0)(\omega(f;C_\rho;\sqrt{\frac{x_0^2+x_0}{n}+\frac{x_0(2x_0+1)}{n^2}},\frac{1}{\alpha_n})+\|f\|_\rho)$$

for $n\in\mathbb{N}$, where $K_5=K_5(p,x_0)=\text{constant}>0.$

There are similar conditions of convergence of partial sums of truncated MKZ operators and Beta operator from theorem (1) and theorem (2). In our opinion, this is an expected result because both sequences are convergent to the function as $n\to\infty$.

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حول قطع مؤثر بيتا الاعتيادي

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المستخلص

في هذا البحث؛ درسنا قطع مؤثر بيتا العادي $\beta_{n,N}(x)$ والخطأ الناتج من التقريب. وضمنا خطأ القطع موصوفا" بمقياس الاستمرارية والمعيار الوزني لفضاء الدالة المقربة.

الكلمات المفتاحية : قطع مؤثر MKZ, رتبة التقريب , مؤثر بيتا , مقياس التقارب .