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Convergence Theorems for Derivatives of a New Baskakov – Type Operators

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ABSTRACT: The motive of this paper is to introduce and investigate the derivative of some Baskakov operators. Firstly, we treated with the convergence theorems on some certain type of Baskakov operators, and then we found the derivative of these operators. Also, we introduced the convergence direct theorems on the derivatives of the same operators. We found the convergence of the derivative of $\mathcal{H}_\ell$. Finally, we established and proved a Voronovskaya-type theorem for the derivative of $\mathcal{H}_\ell$.

Keywords: Baskakov type operators, Korovkin's theorem, Voronovskaya theorem, linear positive operators (L.P.O.), derivative of linear positive operators, exponential weighted space.

1. INTRODUCTION

To approximate some real valued functions and continuous on the interval $[0,\infty)$, Baskakov [1] in 1968 defined a certain sequence of L.P.O. $\{L_n\}$. $L_n: C[\,0\,,\infty) \to C[\,0\,,A]$ where $0 \leq x < \infty$ as:

$$L_n(f;x) = \sum_{\iota=0}^{\infty}(-1)^{\iota}\frac{\varphi_n^{(\iota)}(x)}{\iota!}(x)^{\iota} f\left(\frac{\iota}{n}\right), \text{ for } n = 1,2,\dots \quad (1)$$

Also, Baskakov in the same paper redefined these operators when he defined $\varphi_n^{(\iota)}(x)$ as the following formula,

$$L_n(\,f\,;\,x\,) = \frac{1}{(\,1+x\,)^n}\sum_{\iota=0}^{\infty}\frac{n\,(\,n+1\,)\dots(\,n+\iota-1\,)}{\iota!}\left(\frac{x}{1+x}\right)^{\iota} f\left(\frac{\iota}{n}\right) \quad (2)$$

Which led to the bellow formula

$$L_n(f;x) = \sum_{\iota=0}^{\infty}\binom{n+\iota-1}{\iota} x^{\iota}(1+x)^{-n-\iota} f\left(\frac{\iota}{n}\right). \quad (3)$$

A relation between two generalizations of Baskakov type operators which depending on a some parameters interduced in 1982 by J. A. H. Alkemade[2], also, V. Gupta[3] studied a local and global direct results in ordinary and simultaneous for some modified Baskakov type operators.

Ulrich Abel, M. Ivan and Hongkai Li [4] 2007 intoduced the local approximation by generalized Baskakov –Durrmeyer type operators.

A. Wafi and S. Khatoon [5] 2008 etablished the convergence order for the generalization of Baskakov – type operators. also, they found the derivatives of generalized Baskakov- type operators and then they established a Voronovskaya – type theorem for the derivative for these operators.

Lots of researchers interested in the well known theorem (the Voronoviskaya –type theorem),and they studied the rate of convergence in weighted space, they introduced and proved them for some oeperators one of them the modifieds Baskakov operators [6] and [7].
In 2020 H. J. Sadiq [8] introduced a new Baskakov operators defined as follow

$$\mathcal{H}_\ell(f(u);x) := \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty} \mathcal{k}_{\ell,s,\iota}(x), \tag{4}$$

$$\text{for, } \mathcal{k}_{\ell,s,\iota}(x) = \binom{\ell+\iota+s-1}{\iota+s} x^{\iota+s}(1+x)^{-\ell-\iota-s}, \tag{5}$$

$\eta(x,s) \coloneqq \sum_{\iota=0}^{\infty} \mathcal{k}_{\ell,s,\iota}(x)$, $\eta(x,s) \in (0,1)$, $x \in (\mathbb{R}^+)$.

Note that $\mathcal{k}_{\ell,s,0}(x) = \binom{\ell+s-1}{s} x^s(1+x)^{-\ell-s}$

Let $\alpha > 0$, $C_\alpha(\mathbb{R}^+) = \{f \in C(\mathbb{R}^+);\ f$ is continuous real value function, $S_\alpha(x)$ is bounded, continuous in $\mathbb{R}^+\}$, [9] and [10]

where $S_\alpha(x) = \begin{cases} 1 & if\ \alpha = 0 \\ (1+x^\alpha)^{-1} & if\ \alpha \in N \end{cases}$

The norm which normed the polynomial weight space is

$$\| f(x)\|_\alpha \coloneqq sup_{x\in(\mathbb{R}^+)} S_\alpha(x)|f(x)|.$$

Also, we have $C_\alpha^m[0,\infty) = \{f \in C_\alpha[0,\infty);\ f^{(\iota)} \in C_\alpha[0,\infty), \text{for } f = 1,2,\dots,m\}$, $m \in N$.

Then study some approximation properties also prove the convergence theorems for the operators defined in eq. (4).

In 2008 A. Wafi and Salma Khatoon [5] introduced the derivative of some generalized Baskakov operators $B_n^a(f;x)$.

$$B_n^a(f;x) = \sum_{\iota=0}^{\infty} P_{n,\iota}(x,a) f(\iota/a),\ x \in R_0,\ \iota = 0,1,2,\dots\ a = 1,2,\dots \tag{6}$$

$$\text{Where } P_{n,\iota}(x,a) = e^{\frac{-ax}{1+x}} \frac{P_\iota(n,a)}{\iota!} \frac{x^\iota}{(1+x)^{n+\iota}},\ \text{for } a \ge 0. \tag{7}$$

Some papers were recruit to find the derivatives of a sequence of positive and linear operators and to study them properties by some authors, it is possible to review the references ([11]-[14]).

2. PRELIMINARIES AND NOTATIONS

The operators $\mathcal{H}_\ell(.;x)$ have proved the Korovkin's theorem in [8], and the following results were obtained.
THEOREM 2.1 (BOHMAN- KOROVKIN THEOREM): The operators $\mathcal{H}_\ell(f(u);x)$ satisfy the bellow conditions for every $x \in [0,\infty)$ and $f \in C_\alpha$.

$$1-\ \mathcal{H}_\ell(1;x) = 1, \tag{8}$$

$$2-\mathcal{H}_\ell(u;x) = x + \frac{s(1+x)}{\ell\eta(x,s)}\mathcal{k}_{\ell,s,0}(x), \tag{9}$$

$$3-\ \mathcal{H}_\ell(u^2;x) = x^2\left(1+\frac{1}{\ell}\right) + \frac{x}{\ell} + \frac{s(1+x)}{\ell\eta(x,s)}\mathcal{k}_{\ell,s,0}(x)\left\{\frac{1+x(\ell+1)}{\ell}\right\}, \tag{10}$$

$$4\text{-}\ \mathcal{H}_\ell(u^3;x) = \frac{x^3(\ell+1)(\ell+2)}{\ell^2} + \frac{3x^2(\ell+1)}{\ell} + \frac{x}{\ell} + \frac{s(1+x)\mathcal{k}_{\ell,s,0}(x)}{\ell\eta(x,s)}\left[\frac{x(2\ell+1)+x(\ell+2)[x(\ell+1)+s]+s^2}{\ell^2}\right], \tag{11}$$

$$5\text{-}\ \mathcal{H}_\ell(u^4;x) == \frac{x^4(\ell+1)(\ell+2)(\ell+3)}{\ell^3} + \frac{6x^3(\ell+1)(\ell+2)}{\ell^3} + x^2\frac{(\ell+3)+2(3\ell+2)}{\ell^3} + \frac{x}{\ell^3} + \frac{s(1+x)\mathcal{k}_{\ell,s,0}(x)}{\ell\eta(x,s)}\left[\frac{s^3}{\ell^3} + \frac{x(2\ell+1)+x(\ell+2)[x(\ell+1)+s]+s^2}{\ell^2} + \frac{3x^2(\ell+1)^2+x(6\ell+4)}{\ell^3}\right].$$

(12)

It is the suitable chance to give the definition of the σ −th order moment for the operators $\mathcal{H}_\ell(.;x)$ as the formula $\Upsilon_{\ell,s,\sigma}(x) := \mathcal{H}_\ell((u-x)^\sigma;x)$, $\sigma \in N^0$, $N^0 = 0,1,2,\ldots$. By returning to [8] Sadiq found it as the following:

for every $x \in [0,\infty)$ and $f \in C_\alpha$,

$$\Upsilon_{\ell,s,\sigma}(x) := \mathcal{H}_\ell((u-x)^\sigma;x) := \frac{1}{\eta(x,s)}\sum_{k=0}^{\infty} \hbar_{\ell,s,k}(x)\left(\frac{k+s}{\ell}-x\right)^\sigma.$$

LEMMA 2.2 Let $\Upsilon_{\ell,s,\sigma}(x) := \mathcal{H}_\ell((u-x)^\sigma;x)$, $\sigma \in N^0$. Then we get

1- $\Upsilon_{\ell,s,0}(x) = 1,$ (13)

2- $\Upsilon_{\ell,s,1}(x) = \frac{s(1+x)}{\ell\eta(x,s)}\hbar_{\ell,s,0}(x),$ (14)

3- $\Upsilon_{\ell,s,2}(x) = \frac{x^2}{\ell} + \frac{x}{\ell} + \frac{s(1+x)}{\ell\eta(x,s)}\hbar_{\ell,s,0}(x)\left\{\frac{x}{\ell} + (x-1)s\right\},$ (15)

4- $\Upsilon_{\ell,s,3}(x) = \frac{2x^3}{\ell^2} + \frac{3x^2}{\ell^2} + \frac{x}{\ell^2} + \frac{s(1+x)}{\ell\eta(x,s)}\hbar_{\ell,s,0}(x)\left\{2x\left(\frac{\ell^2-\ell+1}{\ell^2}\right) + \frac{x}{\ell}\left(\frac{\ell(4-3s)+1}{\ell}\right) + \frac{s^2+\ell(1+x)}{\ell}\right\},$ (16)

5- $\Upsilon_{\ell,s,4}(x) = x^4\left\{\frac{3(\ell+2)}{\ell^3}\right\} + x^3\left\{\frac{4(4\ell+3)}{\ell^3}\right\} + x^2\left\{\frac{3\ell+7}{\ell^3}\right\} + \frac{x}{\ell^3}$

$+\frac{s(1+x)}{\ell\eta(x,s)}\hbar_{\ell,s,0}(x)\left\{x^3\left[\frac{(\ell+3)}{\ell} - \frac{4(\ell+1)(\ell+2)}{\ell^3}\right] + x^2\left[\frac{3(\ell+1)^2}{\ell^3} + \frac{(\ell+1)(\ell+2)-4(2\ell+1)-4s(\ell+2)}{\ell^2} + \frac{6}{\ell}\right] + x\frac{(2\ell+1)-4s^2+s(\ell+2)}{\ell^2} + \frac{s^2}{\ell^2}\left(\frac{s}{\ell}+1\right)\right\}.$ (17)

3. AUXILIARY RESULTS: CONVERGENCE THEOREMS FOR DERIVATIVE OF $\mathcal{H}_\ell(f;x)$

Firstly, we need to study the properties of the $\sigma - th$ moment of the operators $\mathcal{H}_\ell(.,x)$.

LEMMA 3.1 For $x \geq 0$, $\ell \in N$ we have

1- $\lim_{\ell\to\infty} \mathcal{H}_\ell(u-x;x) = 0$, $\lim_{\ell\to\infty}\mathcal{H}_\ell((u-x)^2;x) = 0,$ (18)

2- $\lim_{\ell\to\infty} \ell\mathcal{H}_\ell(u-x;x) = \frac{s(1+x)}{\eta(x,s)}\hbar_{\ell,s,0}(x),$ (19)

3- $\lim_{\ell\to\infty} \ell\,\mathcal{H}_\ell((u-x)^2;x) = x(x+1) + \frac{s(1+x)}{\eta(x,s)}\hbar_{\ell,s,0}(x)\{(x-1)s\}.$ (20)

Here, we discuss the uniform convergence for the operators $\mathcal{H}_\ell(.,x)$.

THEOREM3.2 For the operators $\mathcal{H}_n$ and $h \in C[0,\infty)$ satisfying for $x > 0$, $\frac{h(x)}{1+x^2}$ is convergent as $x \to \infty$. Then the operators $\mathcal{H}_\ell$ has a uniform convergence to f, in symbols $\mathcal{H}_\ell \rightrightarrows f$.

Proof. From the uniformity for the operators $\mathcal{H}_\ell$, by using the Korovkin's conditions [15], [16]. Then it is enough to prove that

$\mathcal{H}_\ell(u^m;x) \longrightarrow x^m$ for $m = 0,1,2$.

using theorem (2.1), then we get $\mathcal{H}_\ell(u^m;x) \longrightarrow x^m$ as $\ell \longrightarrow \infty$ for $m = 0,1,2$.

So, we arrive at the required. □

Next, the most important theorem we based on in this paper is to find the derivative of the operators $\mathcal{H}_\ell(f;x)$.

THEOREM 3.3 For $f \in C_\alpha[0,\infty)$, $x \geq 0$ then the operators $\mathcal{H}_\ell(f;x)$ have the following derivative

$$(\mathcal{H}_\ell(f))'_x = \frac{\ell}{x(1+x)}\Upsilon_{\ell,s,1}(x) - \frac{(\eta(x,s))'_x}{\eta(x,s)}. \tag{21}$$

Proof. $\mathcal{H}_\ell(f(u);x) := \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x)\, f(u)$ (from Eq. (6))

for, $k_{\ell,s,\iota}(x) = \binom{\ell+\iota+s-1}{\iota+s} x^{\iota+s}(1+x)^{-\ell-\iota-s}$,

$\eta(x,s) \coloneqq \sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x)$, $\eta(x,s) \in (0,1)$, $x \in [0,\infty)$. Therefore, we get

$$(\mathcal{H}_\ell(f(u);x))'_x = \frac{d}{dx}\mathcal{H}_\ell(f(u);x)$$

$$= \frac{1}{\eta(x,s)}\frac{d}{dx}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x)\, f(u) + \sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)\frac{d}{dx}\frac{1}{\eta(x,s)}$$

$$= \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty}\binom{\ell+\iota+s-1}{\iota+s} f(u)\left\{x^{\iota+s}\frac{d}{dx}(1+x)^{-\ell-\iota-s} + (1+x)^{-\ell-\iota-s}\frac{d}{dx}x^{\iota+s}\right\} - \sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)\frac{(\eta(x,s))'_x}{(\eta(x,s))^2}$$

$$= \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty}\binom{\ell+\iota+s-1}{\iota+s} f(u)\{x^{\iota+s}(-n-\iota-s)(1+x)^{-\ell-\iota-s-1} + (1+x)^{-\ell-\iota-s}(\iota+s)x^{\iota+s-1}\}$$
$$- \sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)\frac{(\eta(x,s))'_x}{(\eta(x,s))^2}$$

$$(\mathcal{H}_\ell(f(u);x))'_x = \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)\frac{x(-n-\iota-s)+(\iota+s)(1+x)}{1+x} - \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)\frac{(\eta(x,s))'_x}{\eta(x,s)}$$

$$x(1+x)(\mathcal{H}_\ell(f(u);x))'_x = \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)(\iota+s-\ell x) - \frac{x(1+x)}{\eta(x,s)}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)\frac{(\eta(x,s))'_x}{\eta(x,s)}$$

$$\frac{x(1+x)}{\ell}(\mathcal{H}_\ell(f(u);x))'_x = \frac{1}{\eta(x,s)}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)(\frac{\iota+s}{\ell} - x) - \frac{x(1+x)}{\ell\,\eta(x,s)}\sum_{\iota=0}^{\infty} k_{\ell,s,\iota}(x) f(u)\frac{(\eta(x,s))'_x}{\eta(x,s)}$$

$$\frac{x(1+x)}{\ell}(\mathcal{H}_\ell(f(u);x))'_x = \mathcal{H}_\ell\big((u-x);x\big) - \frac{x(1+x)}{\ell}\mathcal{H}_\ell(1;x)\frac{(\eta(x,s))'_x}{\eta(x,s)}$$

$$\frac{x(1+x)}{\ell}(\mathcal{H}_\ell(f(u);x))'_x = \Upsilon_{\ell,s,1}(x) - \frac{x(1+x)}{\ell}\frac{(\eta(x,s))'_x}{\eta(x,s)}$$

$$(\mathcal{H}_\ell(f(u);x))'_x = \frac{\ell}{x(1+x)}\Upsilon_{\ell,s,1}(x) - \frac{(\eta(x,s))'_x}{\eta(x,s)}$$

$$(\mathcal{H}_\ell(f(u);x))'_x = \frac{s}{x\eta(x,s)} k_{\ell,s,0}(x) - \frac{(\eta(x,s))'_x}{\eta(x,s)}.$$

we can prove the convergence of the first derivative for the operators $\mathcal{H}_\ell(.;x)$ at the next theorem

THEOREM 3.4 For $f \in C^1_\alpha(\mathbb{R}^+)$, so we have for every $x(0,\infty)$;

$$\lim_{\ell\to\infty}\big(\mathcal{H}_\ell(f)\big)'_x(x) = f(x)\frac{s}{\eta(x,s)} k_{\ell,s,0}(x) + f'(x)\left\{x + \frac{s^2(x-1)}{\eta(x,s)} k_{\ell,s,0}(x)\right\} - \frac{(\eta(x,s))'_x}{\eta(x,s)}. \tag{22}$$

Proof. For $x \in (0,\infty)$, $f \in C^1_\alpha[0,\infty)$ by Taylor formula we get

$$f(u) = f(x) + f'(x)(u) + \varphi(u,x)(u),\ u \in (\mathbb{R}^+). \tag{23}$$

where $(u) = (u - x)$for the function $\varphi(u) = \varphi(u, x) \in C_\alpha[0, \infty)$ and
$\lim_{\ell\to\infty} \varphi(u) = 0.$

Using Eq. (18) and (23), we get for $\ell \in \mathbb{N}$

$$\left(\mathcal{H}_\ell(f)\right)'_x(x) = \frac{\ell}{(1+x)}\mathcal{H}_\ell((u-x)\{f(x) + f'(x)(u) + \varphi(u,x)(u)\}; x) - \frac{\left(\eta(x,s)\right)'_x}{\eta(x,s)}$$

$$= \frac{\ell}{(1+x)} f(x)\Upsilon_{\ell,s,1}(x) + \frac{\ell}{(1+x)} f'(x)\Upsilon_{\ell,s,2}(x) + \frac{\ell}{(1+x)}\mathcal{H}_\ell((u)^2\varphi(u,x); x) - \frac{\left(\eta(x,s)\right)'_x}{\eta(x,s)}. \quad (24)$$

Applying Eq. (18), we get

$$= f(x)\frac{s}{\eta(x,s)}\mathcal{k}_{\ell,s,0}(x) + f'(x)\left\{x + \frac{s^2(x-1)}{\eta(x,s)}\mathcal{k}_{\ell,s,0}(x)\right\} + \frac{\ell}{(1+x)}\mathcal{H}_\ell((u)^2\varphi(t,x); x) - \frac{\left(\eta(x,s)\right)'_x}{\eta(x,s)}.$$

Depending on some attributes of $\varphi(u, x)$, lemma (2.2) and (2.3), we get the following

$$\lim_{\ell\to\infty} \mathcal{H}_\ell\big((u)\varphi(u,x); x\big) = 0, \text{ and } \lim_{\ell\to\infty} \mathcal{H}_\ell(\varphi^2(u, x); x) = 0 \quad (25)$$

Applying the Hölder inequality, we get

$$\left|\mathcal{H}_\ell\big((u)\varphi(u,x); x\big)\right| \le [\mathcal{H}_\ell(\varphi^2(u,x); x)]^{\frac{1}{2}}[\Upsilon_{\ell,s,4}(x)]^{\frac{1}{2}}$$

Using Eq. (25) in above, we get

$$(26)\lim_{\ell\to\infty} \mathcal{H}_\ell((u)^2\varphi(u,x); x) = 0$$

By using Eq. (18, 19, 20, 25) and (26) we obtain the required. □

Next, we establish and prove a main result it is a Voronoviskaja-type theorem for the first derivatives of the operators $\mathcal{H}_\ell(.; x)$.

THEOREM 3.5 For $f \in C_\alpha^3(\mathbb{R}^+)$ then for every $x \in (0, \infty)$; we get

$$\lim_{\ell\to\infty}\ell\left[\left(\mathcal{H}_\ell(f)\right)'_x(x) - f'(x)\right] = f''(x)\left(\frac{2x^2+3x+1}{2(1+x)}\right) + f'''(x)\left\{\frac{3x^3+16x^2+3x}{6(1+x)}\right\} - \frac{\left(\eta(x,s)\right)'_x}{\eta(x,s)}. \quad (27)$$

Proof. For $x \in (0, \infty)$, $f \in C_\alpha^3[0, \infty)$ by Taylor's expansion we get

$$f(u) = \sum_{k=0}^{3}\frac{(u-x)^k}{k!}f^{(k)}(x) + \varphi(u,x)(u-x)^3, u \in (\mathbb{R}^0) \quad (28)$$

where the function $\varphi(u) \equiv \varphi(u, x) \in C_\alpha[0, \infty)$ and $\lim_{\ell\to\infty} \varphi(u) = 0$

since $(\mathcal{H}_\ell(f(u); x))'_x = \frac{\ell}{x(1+x)}\Upsilon_{\ell,s,1}(x) - \frac{(\eta(x,s))'_x}{\eta(x,s)}$

$$(\mathcal{H}_\ell(f(u); x))'_x$$

$$= \frac{\ell}{x(1+x)}\mathcal{H}_\ell\left((u-x)\left\{\sum_{k=0}^{3}\frac{(u-x)^k}{k!}f^{(k)}(x) + \varphi(u,x)(u-x)^3\right\}; x\right) - \frac{\left(\eta(x,s)\right)'_x}{\eta(x,s)}$$

$$(\mathcal{H}_\ell(f(u); x))'_x$$

$$= \frac{\ell}{x(1+x)}\mathcal{H}_\ell\left((u-x)\left\{f(x) + (u-x)f'(x) + \frac{(u-x)^2}{2!}f''(x) + \frac{(u-x)^3}{3!}f'''(x) + \varphi(u,x)(u-x)^3\right\}; x\right)$$

$$- \frac{\left(\eta(x,s)\right)'_x}{\eta(x,s)}$$

$$(\mathcal{H}_\ell(f(u);x))'_x = \frac{\ell}{x(1+x)} f(x)\,\mathcal{H}_\ell\big((u-x);x\big) + \frac{\ell}{x(1+x)} f'(x)\,\mathcal{H}_\ell((u-x)^2;x)$$
$$+\frac{\ell}{x(1+x)} f''(x)\,\mathcal{H}_\ell\left(\frac{(u-x)^3}{2!};x\right) + \frac{\ell}{x(1+x)} f'''(x)\,\mathcal{H}_\ell\left(\frac{(u-x)^4}{3!};x\right)$$
$$+\frac{\ell}{x(1+x)}\,\mathcal{H}_\ell((u-x)^4\varphi(u,x);x) - \frac{(\eta(x,s))'_x}{\eta(x,s)}$$

$$\left[(\mathcal{H}_\ell(f(u);x))'_x(x) - f'(x)\right] = f(x)\frac{1}{\eta(x,s)}\mathcal{k}_{\ell,s,0}(x) + f'(x)\frac{s}{x\eta(x,s)}\mathcal{k}_{\ell,s,0}(x)\left\{\frac{x+\ell(x-1)s}{\ell}\right\} + \frac{1}{(1+x)}\frac{f''(x)}{2!}\left(\frac{2x^2+3x+1}{\ell}\right) + \frac{f''(x)}{2!}\left\{\frac{s}{\eta(x,s)}\mathcal{k}_{\ell,s,0}(x)\left\{2\left(\frac{\ell^2-\ell+1}{\ell^2}\right) + \frac{1}{\ell}\left(\frac{\ell(4-3s)+1}{\ell}\right) + \frac{s^2+\ell(1+x)}{x\ell}\right\}\right\} + \frac{1}{(1+x)}\frac{f'''(x)}{6}\left\{x^3\left\{\frac{(3\ell+2)}{\ell^2}\right\} + x^2\left\{\frac{(12\ell^2+27\ell+21)}{\ell^2}\right\} + x\left\{\frac{2\ell+4}{\ell^2}\right\} + \frac{1}{\ell^2}\right\} + \frac{f'''(x)}{6}\frac{s}{\eta(x,s)}\mathcal{k}_{\ell,s,0}(x)\left\{x^2\left[\frac{(\ell+3)}{\ell} - \frac{4(\ell+1)(\ell+2)}{\ell^3}\right] + x\left[\frac{3(\ell+1)^2}{\ell^3} + \frac{(\ell+1)(\ell+2)-4(2\ell+1)-4s(\ell+2)}{\ell^2} + \frac{6}{\ell}\right] + \frac{(2\ell+1)-4s^2+s(\ell+2)}{\ell^2} + \frac{s^2}{x\ell^2}\left(\frac{s}{\ell}+1\right)\right\} + \frac{\ell}{x(1+x)}\,\mathcal{H}_\ell((u-x)^4\varphi(u,x);x) - \frac{(\eta(x,s))'_x}{\eta(x,s)}.$$

Observe that $\lim_{\ell\to\infty}\frac{1}{\eta(x,s)}\mathcal{k}_{\ell,s,0}(x) \to 0$ when $\ell\to\infty$.

So, we get

$$\lim_{\ell\to\infty}\ell\left[(\mathcal{H}_\ell(f(u);x))'_x(x) - f'(x)\right] = \frac{f''(x)}{2}\left(\frac{2x^2+3x+1}{(1+x)}\right) + \frac{1}{(1+x)}\frac{f'''(x)}{6}\left\{x^3\left\{\frac{3(\ell+2)}{\ell}\right\} + x^2\left\{\frac{4(4\ell+3)}{\ell}\right\} + x\left\{\frac{3\ell+7}{\ell}\right\} + \frac{1}{\ell}\right\} + \frac{\ell}{x(1+x)}\,\mathcal{H}_\ell((u-x)^4\varphi(u,x);x) - \frac{(\eta(x,s))'_x}{\eta(x,s)}. \quad (29)$$

By the features of $\varphi(u,x)$, lemma (2.1) and theorem (3.2) when ℓ approach to ∞, we get

$$\lim_{\ell\to\infty}\mathcal{H}_\ell(\varphi^2(u,x);x) = 0, \quad \lim_{\ell\to\infty}\mathcal{H}_\ell((u-x)^4\varphi(u,x);x) = 0 \quad (30)$$

According to the Holder inequality we get

$$|\,\mathcal{H}_\ell((u-x)^2\varphi(u,x);x)| \le \big(\mathcal{H}_\ell((u-x)^4;x)\big)^{1/2}\big(\mathcal{H}_\ell(\varphi^2(u,x);x)\big)^{1/2}.$$

Using the hypothesis in equation (30) in above, we have

$$\lim_{\ell\to\infty}\ell\mathcal{H}_\ell((u-x)^2\varphi(u,x);x) = 0 \quad (31)$$

Substitute the hypothesis in (31) we have when ℓ approach to ∞.

$$\lim_{\ell\to\infty}\ell\left[(\mathcal{H}_\ell(f(u);x))'_x(x) - f'(x)\right] = f''(x)\left(\frac{2x^2+3x+1}{2(1+x)}\right) + f'''(x)\left(\frac{3x^3+16x^2+3x}{6(1+x)}\right) - \frac{(\eta(x,s))'_x}{\eta(x,s)}. \quad \square$$

4. CONCLUSIONS

In the present paper we have found a derivative of some new family of modified Baskakov – type operators. We have introduced some convergence results for the first derivatives of $\mathcal{H}_\ell(.;x)$. Moreover, we have investigated some results about convergence properties of the derivative of $\mathcal{H}_\ell(.;x)$. Finally, we have introduced the Voronovskaya theorem for the first derivatives of $\mathcal{H}_\ell(.;x)$ for $f \in C_\alpha^3(\mathbb{R}^+)$.

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