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Article *in* Applied Mathematics in Science and Engineering · February 2024

DOI: 10.1080/27690911.2024.2312803

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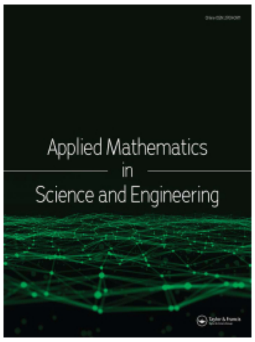

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Applied Mathematics in Science and Engineering

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/gipe21

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To cite this article: Sarem H. Hadi, Maslina Darus, Badriah Alamri, Şahsene Altınkaya & Abdullah Alatawi (2024) On classes of ζ-uniformly *q*-analogue of analytic functions with some subordination results, Applied Mathematics in Science and Engineering, 32:1, 2312803, DOI: 10.1080/27690911.2024.2312803

To link to this article: https://doi.org/10.1080/27690911.2024.2312803

Published online: 19 Feb 2024.

APPLIED MATHEMATICS IN SCIENCE AND ENGINEERING
2024, VOL. 32, NO. 1, 2312803
https://doi.org/10.1080/27690911.2024.2312803

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On classes of ζ-uniformly q-analogue of analytic functions with some subordination results

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ABSTRACT
This article presents new classes of ζ-uniformly q-starlike and q-convex analytic functions of order υ using the principle of q-calculus. The renowned classes of starlike and convex functions are utilized to establish these classes. The investigation of these uniformly classes leads to the study of some geometric notions, including coefficient estimates, sharpness, distortion and growth theorems, and convex linear combinations. Furthermore, some subordination properties involving integral means inequalities and subordinate sequences are examined.

ARTICLE HISTORY
Received 19 September 2023
Accepted 25 January 2024

KEYWORDS
Univalent functions; q-calculus; uniformly q-analogue analytic functions; integral means inequalities; subordinate factor sequence

MATHS
30C20; 30C45; 30C50; 30C55

1. Introduction

We represent the set of analytic functions f in $\mathbb{U}$ as $\mathscr{A}$, where

$$\mathrm{f}(z) = z + \sum_{\mathrm{i}=2}^{\infty} a_{\mathrm{i}} z^{\mathrm{i}} \tag{1}$$

and

$$\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$$

is the open unit disc. The set $\mathcal{S} \subset \mathscr{A}$ comprises of all functions that are both normalized and univalent.

We also consider the subset $\mathcal{T}$ of the set $\mathscr{A}$ of the following analytic function:

$$\mathrm{f}(z) = z - \sum_{\mathrm{i}=2}^{\infty} |a_{\mathrm{i}}| z^{\mathrm{i}}. \tag{2}$$

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If the functions f and h belong to the class of holomorphic functions (i.e. they are both analytic), then f is subordinate to h ($\mathsf{f} \prec \mathsf{h}$), if a Schwarz function ϑ exists and satisfying certain conditions

$$\vartheta(0) = 0 \quad \text{and} \quad |\vartheta(z)| < 1,$$

such that

$$\mathsf{f}(z) = \mathsf{h}(\vartheta(z)).$$

Equivalently, if the inequality below holds:

$$\mathsf{f}(z) \prec \mathsf{h}(z) \Leftrightarrow \mathsf{f}(0) = \mathsf{h}(0) \text{ and } \mathsf{f}(\mathbb{U}) \subset \mathsf{h}(\mathbb{U}),$$

then the functions f and h will be subordinated.

The convolution of any analytic functions $\mathsf{f}, \hbar \in \mathscr{A}$, expressed by $\mathsf{f} * \hbar$, is given as

$$\left(\mathsf{f} * \hbar\right)(z) := z + \sum_{\mathrm{i}=2}^{\infty} a_{\mathrm{i}} d_{\mathrm{i}} z^{\mathrm{i}} := \left(\hbar * \mathsf{f}\right)(z),$$

where $\mathsf{f}(z)$ have been aforementioned in (1) and

$$\hbar(z) = z + \sum_{\mathrm{i}=2}^{\infty} d_{\mathrm{i}} z^{\mathrm{i}}, \quad (z \in \mathbb{U}).$$

The renowned starlike function class, denoted by $\mathcal{S}^*$, is formulated by

$$\mathcal{R}\left(\frac{z\mathsf{f}'(z)}{\mathsf{f}(z)}\right) > 0, \quad (z \in \mathbb{U}). \tag{3}$$

Moreover, the convex function class, denoted by $\mathcal{C}$, is formulated by

$$\mathcal{R}\left(1 + \frac{z\mathsf{f}''(z)}{\mathsf{f}'(z)}\right) > 0, \quad (z \in \mathbb{U}). \tag{4}$$

Noting that from inequalities (3) and (4) (see [1]), we have

$$z\mathsf{f}'(z) \in \mathcal{S}^* \Leftrightarrow \mathsf{f}(z) \in \mathcal{C}.$$

For $\rho \in \mathbb{C}\backslash\{0\}$, Nasr and Aouf [2] established the subsequent class of starlike functions $\mathcal{S}^*(\rho)$ of order ρ

$$\mathcal{S}^*(\rho) = \left\{\mathsf{f} \in \mathcal{A} : \mathcal{R}\left(1 + \frac{1}{\rho}\left(\frac{z\mathsf{f}'(z)}{\mathsf{f}(z)} - 1\right)\right) > 0, \quad (z \in \mathbb{U})\right\}, \tag{5}$$

while Wiatrowski [3] defined and verified the class of convex functions $\mathcal{C}(\rho)$ of order ρ as follows:

$$\mathcal{C}(\rho) = \left\{f \in \mathcal{A} : \mathcal{R}\left(1 + \frac{1}{\rho}\frac{z\mathsf{f}''(z)}{\mathsf{f}'(z)}\right) > 0, \quad (z \in \mathbb{U})\right\}. \tag{6}$$

The ζ-uniformly of starlike functions $\mathcal{US}^*(\upsilon,\zeta)$ of order υ is given by

$$\mathcal{R}\left\{\frac{z\mathsf{f}'(z)}{\mathsf{f}(z)}-\upsilon\right\}>\zeta\left|\frac{z\mathsf{f}'(z)}{\mathsf{f}(z)}-1\right|, \qquad (0\le\upsilon<1,\zeta\ge 0), \tag{7}$$

and the ζ-uniformly of convex functions $\mathcal{UC}(\upsilon,\zeta)$ of order υ is given by

$$\mathcal{R}\left\{\left(1+\frac{z\mathsf{f}''(z)}{\mathsf{f}'(z)}\right)-\upsilon\right\}>\zeta\left|\frac{z\mathsf{f}''(z)}{\mathsf{f}'(z)}\right|, \qquad (0\le\upsilon<1,\zeta\ge 0), \tag{8}$$

where the classes $\mathcal{US}^*(\upsilon,\zeta)$ and $\mathcal{UC}(\upsilon,\zeta)$ with their generalizations provided by Shams et al. [4] (also see [5–10], for related findings).

2. Quantum calculus

Quantum calculus is a fascinating mathematical framework that extends traditional calculus by incorporating principles from quantum mechanics. It introduces a new perspective on differentiation and integration, allowing for the analysis of functions and operators in a quantum-like context. The principle of q-calculus has profoundly influenced the investigation of Geometric Function Theory (GFT) and its significant applications in various areas, such as mathematical science and quantum physics. This principle presents a resemble framework to ordinary calculus but eliminates the need for limits. The notion of q-calculus, involving the derivative and integral, was primarily provided by Jackson [11]. The investigation of q-calculus has led to the exploration of other pertinent aspects, such as the examination of special functions (for a more comprehensive review, see [12–21]).

Ismail et al. [22] proposed the notion of q-calculus in the context of GFT. This work has led to the identification of various classes of analytic functions in the disc $\mathbb{U}$, known as the Ma and Minda classes, which exhibit a strong connection to the concept of subordination. Furthermore, the use of q-calculus, specifically fractional q-integral operators, has been employed to build a multitude of analytic functions. Moreover, a considerable body of literature has been dedicated to the study of certain categories of analytic functions in $\mathbb{U}$ through the utilization of q-calculus (e.g. see [23–32]).

More recently, numerous studies have examined certain uniformly classes of analytic functions by employing the notion of q-calculus (for instance, see [33–37]).

This investigation begins with the essential notions and consequently, the analysis for our proposed study of the q-calculus. This article makes the assumption that $0<q<1$. The next definitions are necessary for a univalent function f:

Definition 2.1 ([11]): *The q-number $[\mu]_q$ can be represented by the following expression:*

$$[\mu]_q := \begin{cases} \sum_{\mu=0}^{\mathrm{m}-1} q^{\mu} & (\mu=\mathrm{m}\in\mathbb{N}) \\[2ex] \dfrac{1-q^{\mu}}{1-q} & (\mu\in\mathbb{C}). \end{cases}$$

Definition 2.2 ([11]): *The operator denoted as* $\mathfrak{D}_q$ *represents the q-derivative and given by*

$$\mathfrak{D}_q \mathsf{f}\,(z) = \frac{\mathsf{f}(qz) - \mathsf{f}(z)}{\left(q-1\right) z} \quad (z \neq 0)\,.$$

Then the q-derivative of f *in* (1) *is supplied by*

$$\mathfrak{D}_q \mathsf{f}(z) = 1 + \sum_{\mathrm{i}=2}^{\infty} [\mathrm{i}]_q a_{\mathrm{i}} z^{\mathrm{i}-1}.$$

In addition, the notion of q-generalized Pochhammer is supplied by

$$[\gamma;\mathrm{i}]_q = [\gamma]_q[\gamma+1]_q[\gamma+2]_{q\ldots}\ldots[\gamma+\mathrm{i}-1]_q$$

and the definition of the q-Gamma function is

$$\Gamma_q\,(\gamma+1) = [\gamma]_q \Gamma_q(\gamma) \quad \textit{and} \quad \Gamma_q(1) = 1.$$

It follows that $\Gamma_q(\mathrm{i}+1) = [\mathrm{i}]_q!$.

Given that the q-derivative is a generalized version of the ordinary derivative. Thus, considering the classes (5)–(8), we introduce novel classes of ζ-uniformly analytic functions that exhibit q-starlikeness and q-convexity in $\mathbb{U}$. We extensively investigate various aspects including coefficient estimates ($|a_{\mathrm{i}}|$), growth and distortion theorems, Hadamard products property, convex linear combinations, and several subordination properties encompassing integral means inequalities and subordinate factor sequences as well.

3. The uniformly classes $\mathcal{S}_q^*(\rho, \upsilon, \zeta)$ and $\mathcal{C}_q(\rho, \upsilon, \zeta)$

The following definitions introduce classes of ζ-uniformly q-starlike $\mathcal{S}_q^*(\rho, \upsilon, \zeta)$ and q-convex $\mathcal{C}_q(\rho, \upsilon, \zeta)$ univalent functions.

Definition 3.1: *The function* $\mathsf{f} \in \mathcal{A}$ *belongs to* $\mathcal{S}_q^*(\rho, \upsilon, \zeta)$ *if it fulfils the subsequent condition*

$$\mathcal{R}\left[1 + \frac{1}{\rho}\left(\frac{z\mathfrak{D}_q \mathsf{f}(z)}{\mathsf{f}(z)} - \upsilon\right)\right] > \zeta \left|1 + \frac{1}{\rho}\left(\frac{z\mathfrak{D}_q \mathsf{f}(z)}{\mathsf{f}(z)} - 1\right)\right|, \tag{9}$$

where $0 < \upsilon \leq 1,\ \rho \in \mathbb{C}\backslash\{0\}$ *and* $\zeta \geq 0$.

Definition 3.2: *The function* $\mathsf{f} \in \mathcal{A}$ *belongs to* $\mathcal{C}_q(\rho, \upsilon, \zeta)$ *if it fulfils the subsequent condition*

$$\mathcal{R}\left[1 + \frac{1}{\rho}\left(\frac{\mathfrak{D}_q(z\mathfrak{D}_q \mathsf{f}(z))}{\mathfrak{D}_q \mathsf{f}(z)} - \upsilon\right)\right] > \zeta \left|1 + \frac{1}{\rho}\left(\frac{\mathfrak{D}_q(z\mathfrak{D}_q \mathsf{f}(z))}{\mathfrak{D}_q \mathsf{f}(z)} - 1\right)\right|, \tag{10}$$

where $0 < \upsilon \leq 1,\ \rho \in \mathbb{C}\backslash\{0\}$ *and* $\zeta \geq 0$.

We also define

$$\mathcal{T}\mathcal{S}_q^*(\rho, \upsilon, \zeta) = \mathcal{S}_q^*(\rho, \upsilon, \zeta) \cap \mathcal{T}$$

and

$$\mathcal{T}\mathcal{C}_q(\rho, \upsilon, \zeta) = \mathcal{C}_q(\rho, \upsilon, \zeta) \cap \mathcal{T}.$$

Example 3.3: *The classes $\mathcal{S}_q^*(\rho,\upsilon,\zeta)$ and $\mathcal{C}_q(\rho,\upsilon,\zeta)$ reduce to various subclasses as follows:*

(1) *If $q \to 1^-$, then $\mathcal{S}_q^*(\rho,\upsilon,\zeta) = \mathcal{S}^*(\rho,\upsilon,\zeta)$ and $\mathcal{C}_q(\rho,\upsilon,\zeta) = \mathcal{C}(\rho,\upsilon,\zeta)$ as follows:*

$$\mathcal{R}\left[1+\frac{1}{\rho}\left(\frac{z\mathrm{f}'(z)}{\mathrm{f}(z)}-\upsilon\right)\right] > \zeta\left|1+\frac{1}{\rho}\left(\frac{z\mathrm{f}'(z)}{\mathrm{f}(z)}-1\right)\right|$$

and

$$\mathcal{R}\left[1+\frac{1}{\rho}\left(\frac{z\mathrm{f}''(z)}{\mathrm{f}'(z)}-\upsilon\right)\right] > \zeta\left|1+\frac{1}{\rho}\left(\frac{z\mathrm{f}''(z)}{\mathrm{f}'(z)}-1\right)\right|,$$

respectively.

(2) $\mathcal{S}_q^*(\rho,1,0) = \mathcal{S}_q^*(\rho)$ *and* $\mathcal{C}_q(\rho,0,0) = \mathcal{C}_q(\rho)$ $(\rho \in \mathbb{C}\backslash\{0\})$, *which have been established by Mahmood et al.* [38], *which are formulated by*

$$\mathcal{S}_q^*(\rho) := \left\{f \in \mathcal{A} : \mathcal{R}\left[1+\frac{1}{\rho}\left(\frac{z\mathfrak{D}_q\mathrm{f}(z)}{\mathrm{f}(z)}-1\right)\right] > 0\right\}$$

and

$$\mathcal{C}_q(\rho) := \left\{f \in \mathcal{A} : \mathcal{R}\left[1+\frac{1}{\rho}\left(\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathrm{f}(z))}{\mathfrak{D}_q\mathrm{f}(z)}\right)\right] > 0\right\}.$$

(3) *If $\rho \in \mathbb{R}$, then $\mathcal{S}_q^*(1-\eta,1,0) = \mathcal{S}_q^*(\eta)$ and $\mathcal{C}_q(1-\eta,0,0) = \mathcal{C}_q(\eta)$ $(0 \le \eta < 1)$, which have been defined by Seoudy and Aouf* [39], *which are formulated by*

$$\mathcal{S}_q^*(\eta) := \left\{f \in \mathcal{A} : \mathcal{R}\left[\frac{z\mathfrak{D}_q\mathrm{f}(z)}{\mathrm{f}(z)}\right] > \eta\right\}$$

and

$$\mathcal{C}_q(\eta) := \left\{f \in \mathcal{A} : \mathcal{R}\left[\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathrm{f}(z))}{\mathfrak{D}_q\mathrm{f}(z)}\right] > \eta\right\}.$$

4. A set of main results

The following theorems determine coefficient estimates $|a_{\mathrm{i}}|$ for the classes $\mathcal{S}_q^*(\rho,\upsilon,\zeta)$ and $\mathcal{C}_q(\rho,\upsilon,\zeta)$ as outlined below:

Theorem 4.1: *For $-1 \le \upsilon < 1, \zeta \ge 0$, and $\rho \in \mathbb{C}\backslash\{0\}$. Then $\mathrm{f}(z) \in \mathcal{TS}_q^*(\rho,\upsilon,\zeta)$ if and only if*

$$\sum_{\mathrm{i}=2}^{\infty}\left[\left(|\rho| + [\mathrm{i}]_q\right)(1-\zeta) + \zeta - \upsilon\right]|a_{\mathrm{i}}| \le 1 - \upsilon + |\rho|(1-\zeta). \tag{11}$$

Proof: Let $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$, then from (9) we have

$$\mathcal{R}\left[1 + \frac{1}{\rho}\left(\frac{z\mathfrak{D}_q\mathsf{f}(z)}{\mathsf{f}(z)} - \upsilon\right)\right] > \zeta\left|1 + \frac{1}{\rho}\left(\frac{z\mathfrak{D}_q\mathsf{f}(z)}{\mathsf{f}(z)} - 1\right)\right|$$

$$= \mathcal{R}\left[1 + \frac{1}{\rho}\left(\frac{z(1-\upsilon) - \sum_{i=2}^{\infty}([i]_q - \upsilon)a_i z^i}{z - \sum_{i=2}^{\infty} a_i z^i}\right)\right]$$

$$> \zeta\left|1 - \frac{1}{\rho}\left(\frac{\sum_{i=2}^{\infty}([i]_q - 1)\, a_i z^i}{z - \sum_{i=2}^{\infty} a_i z^i}\right)\right|.$$

From the fact $-\mathcal{R}(z) \le |z|$, we attain

$$1 + \frac{(1-\upsilon) - \sum_{i=2}^{\infty}([i]_q - \upsilon)\,|a_i|\,|z|^i}{|\rho|(1 - \sum_{i=2}^{\infty}|a_i|\,|z|^i)} > \zeta\left[1 - \frac{\sum_{i=2}^{\infty}([i]_q - 1)\,|a_i|\,|z|^i}{|\rho|\,(1 - \sum_{i=2}^{\infty}|a_i|\,|z|^i)}\right]$$

$$= 1 + \frac{(1-\upsilon) - \sum_{i=2}^{\infty}([i]_q - \upsilon)\,|a_i|}{|\rho|(1 - \sum_{i=2}^{\infty}|a_i|)} > \zeta\left[1 - \frac{\sum_{i=2}^{\infty}([i]_q - 1)\,|a_i|}{|\rho|\,(1 - \sum_{i=2}^{\infty}|a_i|)}\right]$$

$$= \frac{|\rho| - |\rho|\sum_{i=2}^{\infty}|a_i| + (1-\upsilon) - \sum_{i=2}^{\infty}([i]_q - \upsilon)\,|a_i|}{|\rho| - |\rho|\sum_{i=2}^{\infty}|a_i|}$$

$$> \zeta\left[\frac{|\rho| - |\rho|\sum_{i=2}^{\infty}|a_i| - \sum_{i=2}^{\infty}([i]_q - 1)\,|a_i|}{|\rho| - |\rho|\sum_{i=2}^{\infty}|a_i|}\right].$$

Using theorem of maximum modulus, we conclude that

$$\sum_{i=2}^{\infty}\left[\left(|\rho| + [i]_q\right)(1-\zeta) + \zeta - \upsilon\right]|a_i| \le 1 - \upsilon + |\rho|\,(1-\zeta).$$

Conversely, let the inequality (11) be correct for $z \in \mathbb{U}$, then

$$\mathcal{R}\left[1 + \frac{1}{\rho}\left(\frac{z\mathfrak{D}_q\mathsf{f}(z)}{\mathsf{f}(z)} - \upsilon\right)\right] - \zeta\left|1 + \frac{1}{\rho}\left(\frac{z\mathfrak{D}_q\mathsf{f}(z)}{\mathsf{f}(z)} - 1\right)\right| > 0,$$

then

$$1 + \frac{1}{|\rho|}\left(\frac{(1-\upsilon) - \sum_{i=2}^{\infty}([i]_q - \upsilon)\,|a_i|\,|z|^{j-1}}{1 - \sum_{i=2}^{\infty}|a_i|\,|z|^{j-1}}\right)$$
$$-\zeta\left\{1 - \frac{1}{|\rho|}\left(\frac{\sum_{i=2}^{\infty}([i]_q - 1)\,|a_i|\,|z|^{j-1}}{1 - \sum_{i=2}^{\infty}|a_i|\,|z|^{j-1}}\right)\right\} > 0.$$

If we let $z \to 1^-$ along the real axis, select the values of z for which $\mathfrak{D}_q\mathsf{f}(z)$ is real, and then eliminate the denominator from the above inequality, we can observe that

$$\sum_{i=2}^{\infty}\left[\left(|\rho| + [i]_q\right)(1-\zeta) + \zeta - \upsilon\right]\left|a_j\right| \le 1 - \upsilon + |\rho|\,(1-\zeta).$$

■

Corollary 4.2: *Let* $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$*, then*

$$|a_{\mathrm{i}}| \leq \frac{1 - \upsilon + |\rho| (1 - \zeta)}{[(|\rho| + [\mathrm{i}]_q) (1 - \zeta) + \zeta - \upsilon]} \tag{12}$$

for $\mathrm{i} \geq 2,\ -1 \leq \upsilon < 1,\ \zeta \geq 0$*, and* $\rho \in \mathbb{C}\backslash\{0\}$ *with the sharpness of* $\mathsf{f}(z)$ *given by*

$$\mathsf{f}(z) = z - \frac{1 - \upsilon + |\rho| (1 - \zeta)}{[(|\rho| + [\mathrm{i}]_q) (1 - \zeta) + \zeta - \upsilon]} z^j.$$

Theorem 4.3: *For* $-1 \leq \upsilon < 1, \zeta \geq 0$*, and* $\rho \in \mathbb{C}\backslash\{0\}$*. Then* $\mathsf{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$ *if and only if*

$$\sum_{\mathrm{i}=2}^{\infty} [\mathrm{i}]_q [(|\rho| + [\mathrm{i}]_q) (1 - \zeta) + \zeta - \upsilon] |a_{\mathrm{i}}| \leq 1 - \upsilon + |\rho| (1 - \zeta), \tag{13}$$

and we find that the sharpness of f *as below*

$$\mathsf{f}(z) = z - \frac{1 - \upsilon + |\rho| (1 - \zeta)}{[\mathrm{i}]_q [(|\rho| + [\mathrm{i}]_q) (1 - \zeta) + \zeta - \upsilon]} z^j.$$

Proof: Let $\mathsf{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$, then from (9) we have

$$\begin{aligned}
&\mathcal{R}\left[1 + \frac{1}{\rho}\left(\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathsf{f}(z))}{\mathfrak{D}_q\mathsf{f}(z)} - \upsilon\right)\right] > \zeta \left|1 + \frac{1}{\rho}\left(\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathsf{f}(z))}{\mathfrak{D}_q\mathsf{f}(z)} - 1\right)\right| \\
&= \mathcal{R}\left[1 + \frac{1}{\rho}\left(\frac{(1 - \upsilon) - \sum_{\mathrm{i}=2}^{\infty} [\mathrm{i}]_q([\mathrm{i}]_q - \upsilon) a_{\mathrm{i}} z^{\mathrm{i}-1}}{1 - \sum_{\mathrm{i}=2}^{\infty} a_{\mathrm{i}} z^{\mathrm{i}-1}}\right)\right] \\
&> \zeta \left|1 - \frac{1}{\rho}\left(\frac{\sum_{\mathrm{i}=2}^{\infty} [\mathrm{i}]_q([\mathrm{i}]_q - 1)\, a_{\mathrm{i}} z^{\mathrm{i}}}{1 - \sum_{\mathrm{i}=2}^{\infty} a_{\mathrm{i}} z^{\mathrm{i}}}\right)\right|,
\end{aligned}$$

we can obtain the desired result utilizing an approach resembling that of Theorem 4.1. ■

Theorems 4.1 and 4.3 lead to the subsequent corollaries, when $q \to 1^-$.

Corollary 4.4: *For* $-1 \leq \upsilon < 1, \zeta \geq 0$*, and* $\rho \in \mathbb{C}\backslash\{0\}$*. Then* $\mathsf{f}(z) \in \mathcal{TS}^*(\rho, \upsilon, \zeta)$ *if and only if*

$$\sum_{\mathrm{i}=2}^{\infty} [(|\rho| + \mathrm{i}) (1 - \zeta) + \zeta - \upsilon] |a_{\mathrm{i}}| \leq 1 - \upsilon + |\rho| (1 - \zeta).$$

Corollary 4.5: *For* $-1 \leq \upsilon < 1, \zeta \geq 0$*, and* $\rho \in \mathbb{C}\backslash\{0\}$*. Then* $\mathsf{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$ *if and only if*

$$\sum_{\mathrm{i}=2}^{\infty} \mathrm{i}[(|\rho| + \mathrm{i}) (1 - \zeta) + \zeta - \upsilon] |a_{\mathrm{i}}| \leq 1 - \upsilon + |\rho| (1 - \zeta).$$

Theorem 4.6 (Distortion Theorem): *If* $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$, *then*

$$1 - \frac{[2]_q(1-\upsilon+|\rho|(1-\zeta))r}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]} \leq \left|\mathfrak{D}_q(\mathsf{f}(z))\right| \leq 1 + \frac{[2]_q(1-\upsilon+|\rho|(1-\zeta))r}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

Proof: Let $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$, we note from Theorem 4.1 that

$$\sum_{\mathrm{i}=2}^{\infty} |a_{\mathrm{i}}| \leq \frac{1-\upsilon+|\rho|(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

From the fact $|z| = r < 1$, we get

$$\begin{aligned}\left|\mathfrak{D}_q(\mathsf{f}(z))\right| &\leq 1 + \sum_{\mathrm{i}=2}^{\infty} [\mathrm{i}]_q |a_{\mathrm{i}}| \, |z|^{\mathrm{i}-1} \leq 1 + r\sum_{\mathrm{i}=2}^{\infty} [2]_q |a_{\mathrm{i}}| \\ &\leq 1 + \frac{[2]_q(1-\upsilon+|\rho|(1-\zeta))r}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\left|\mathfrak{D}_q(\mathsf{f}(z))\right| &\geq 1 - \sum_{\mathrm{i}=2}^{\infty} [\mathrm{i}]_q |a_{\mathrm{i}}| \, |z|^{\mathrm{i}-1} \geq 1 - r\sum_{\mathrm{i}=2}^{\infty} [2]_q |a_{\mathrm{i}}| \\ &\geq 1 - \frac{[2]_q(1-\upsilon+|\rho|(1-\zeta))r}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.\end{aligned}$$

Here, the proof completes. ■

Theorem 4.7 (Distortion Theorem): *If* $\mathsf{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$, *then*

$$1 - \frac{(1-\upsilon+|\rho|(1-\zeta))r}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]} \leq \left|\mathfrak{D}_q(\mathsf{f}(z))\right| \leq 1 + \frac{(1-\upsilon+|\rho|(1-\zeta))r}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

Proof: Our argument of this theorem is similar to that of Theorem 4.6. ■

Theorem 4.8 (Growth Theorem): *Let* $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$, *then*

$$r - \frac{1-\upsilon+|\rho|(1-\zeta)\,r^2}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]} \leq \left|\mathsf{f}(z)\right| \leq r + \frac{1-\upsilon+|\rho|(1-\zeta)\,r^2}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

Proof: Let $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$, we note from Theorem 4.1 that

$$\sum_{\mathrm{i}=2}^{\infty} |a_\mathrm{i}| \leq \frac{1-\upsilon+|\rho|(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

From the fact $|z| = r < 1$, we get

$$\left|\mathsf{f}(z)\right| \leq |z| + \sum_{\mathrm{i}=2}^{\infty} |a_\mathrm{i}|\,|z|^\mathrm{i} \leq r + r^2 \sum_{\mathrm{i}=2}^{\infty} |a_\mathrm{i}| \leq r + \frac{(1-\upsilon+|\rho|(1-\zeta))r^2}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

Similarly, we obtain

$$\left|\mathsf{f}(z)\right| \geq |z| - \sum_{\mathrm{i}=2}^{\infty} |a_\mathrm{i}|\,|z|^\mathrm{i} \geq r - r^2 \sum_{\mathrm{i}=2}^{\infty} |a_\mathrm{i}| \geq r - \frac{(1-\upsilon+|\rho|(1-\zeta))r^2}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

Hence, the desired outcome can be inferred. ■

Similarly, we can establish the subsequent growth bounds:

Theorem 4.9: *Let* $\mathsf{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$, *then*

$$r - \frac{(1-\upsilon+|\rho|(1-\zeta))r^2}{[2]_q\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]} \leq \left|\mathsf{f}(z)\right| \leq r + \frac{(1-\upsilon+|\rho|(1-\zeta))r^2}{[2]_q\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

Theorem 4.10: *Put* $\mathsf{f}_1(z) = z$ *and*

$$\mathsf{f}_\mathrm{i}(z) = z - \frac{1-\upsilon+|\rho|(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]} z^\mathrm{i}, \quad \mathrm{i} \geq 2, \tag{14}$$

then $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$ *if and only if* $\mathsf{f}(z)$ *is given by*

$$\mathsf{f}(z) = \sum_{\mathrm{i}=1}^{\infty} J_\mathrm{i}\mathsf{f}_\mathrm{i}(z),$$

where $J_\mathrm{i} \geq 0$ *and* $\sum_{\mathrm{i}=1}^{\infty} J_\mathrm{i} = 1$.

Proof: Assume that $\mathrm{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$, then Corollary 4.2 gives

$$|a_{\mathrm{i}}| \leq \frac{1-\upsilon+|\rho|\,(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]}.$$

Setting $J_{\mathrm{i}} = \frac{[(|\rho|+[\mathrm{i}]_q)(1-\zeta)+\zeta-\upsilon]}{1-\upsilon+|\rho|(1-\zeta)} a_{\mathrm{i}}$ with $J_1 = 1 - \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}}$. Then

$$\begin{aligned} \mathrm{f}(z) &= z - \sum_{\mathrm{i}=2}^{\infty} a_{\mathrm{i}} z^{\mathrm{i}} = z - \sum_{\mathrm{i}=2}^{\infty} \frac{1-\upsilon+|\rho|\,(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]} J_{\mathrm{i}} z^{\mathrm{i}} \\ &= z - \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}} z + \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}} \mathrm{f}_{\mathrm{i}}(z) = z\left[1 - \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}}\right] + \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}} \mathrm{f}_{\mathrm{i}}(z) \\ &= J_1 \mathrm{f}_1(z) + \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}} \mathrm{f}_{\mathrm{i}}(z) = \sum_{\mathrm{i}=1}^{\infty} J_{\mathrm{i}} \mathrm{f}_{\mathrm{i}}(z). \end{aligned}$$

Conversely, let $\mathrm{f}(z)$ be defined as

$$\mathrm{f}(z) = \sum_{\mathrm{i}=1}^{\infty} J_{\mathrm{i}} \mathrm{f}_{\mathrm{i}}(z).$$

Then

$$\begin{aligned} \mathrm{f}(z) &= J_1 \mathrm{f}_1(z) + \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}} \mathrm{f}_{\mathrm{i}}(z) = J_1 z + \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}} \left[z - \frac{1-\upsilon+|\rho|\,(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]} z^{\mathrm{i}}\right] \\ &= z - \sum_{\mathrm{i}=2}^{\infty} \frac{1-\upsilon+|\rho|\,(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]} J_{\mathrm{i}} z^{\mathrm{i}}. \end{aligned}$$

Consequently, it may be inferred that

$$\begin{aligned} &\sum_{\mathrm{i}=2}^{\infty} \frac{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]}{1-\upsilon+|\rho|\,(1-\zeta)} \cdot \frac{1-\upsilon+|\rho|\,(1-\zeta)}{\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]} J_j \\ &\quad = \sum_{\mathrm{i}=2}^{\infty} J_{\mathrm{i}} = \sum_{\mathrm{i}=1}^{\infty} J_{\mathrm{i}} - J_1 = 1 - J_1 \leq 1. \end{aligned}$$

Hence $\mathrm{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$. ■

Theorem 4.11: *Put* $\mathrm{f}_1(z) = z$ *and*

$$\mathrm{f}_{\mathrm{i}}(z) = z - \frac{1-\upsilon+|\rho|\,(1-\zeta)}{[\mathrm{i}]_q\left[\left(|\rho|+[\mathrm{i}]_q\right)(1-\zeta)+\zeta-\upsilon\right]} z^{\mathrm{i}}, \quad \mathrm{i} \geq 2, \tag{15}$$

then $\mathrm{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$ *if and only if* $\mathrm{f}(z)$ *is defined by*

$$\mathrm{f}(z) = \sum_{\mathrm{i}=1}^{\infty} J_{\mathrm{i}} \mathrm{f}_{\mathrm{i}}(z),$$

where $J_{\mathrm{i}} \geq 0$ *and* $\sum_{\mathrm{i}=1}^{\infty} J_{\mathrm{i}} = 1$.

Proof: By employing the identical technique as demonstrated in Theorem 4.10, we can attain the necessary finding of Theorem 4.11. ■

4.1. Integral means inequalities

To investigate our main findings, we must mention the following Littlewood's Lemma:

Lemma 4.12 (*[40]*): *Suppose* $\mathsf{f}(z)$ *and* $\hbar(z)$ *are two analytic functions and* $\mathsf{f}(z) \prec \hbar(z)$*, then*

$$\int_0^{2\pi} \left|\mathsf{f}\left(re^{i\theta}\right)\right|^{\chi} \mathrm{d}\theta \le \int_0^{2\pi} \left|\hbar\left(re^{i\theta}\right)\right|^{\chi} \mathrm{d}\theta, \quad (0 \le r < 1, \chi > 0).$$

Equality holds if and only if $r = 0$ *or if* $r \neq 0$ *but* $\hbar$ *constant, or* $\Phi(z) = e^{i\upsilon} z$*.*

Theorem 4.13: *Suppose that* $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$ *and*

$$\mathsf{f}_2(z) = z - \frac{1 - \upsilon + |\rho|(1-\zeta)}{\left[\left(|\rho| + [2]_q\right)(1-\zeta) + \zeta - \upsilon\right]} z^2. \tag{16}$$

For $z = re^{i\theta}$*, we deduce that*

$$\int_0^{2\pi} |\mathsf{f}(z)|^{\chi} \mathrm{d}\theta \le \int_0^{2\pi} \left|\mathsf{f}_2(z)\right|^{\chi} \mathrm{d}\theta, \quad (0 \le r < 1).$$

Proof: Since $\mathsf{f}(z) = z - \sum_{\mathrm{i}=2}^{\infty} |a_{\mathrm{i}}| z^{\mathrm{i}}$, then

$$\int_0^{2\pi} |z - \sum_{\mathrm{i}=2}^{\infty} a_{\mathrm{i}} z^{\mathrm{i}}|^{\chi} \mathrm{d}\theta \le \int_0^{2\pi} \left| z - \frac{[1 - \upsilon + |\rho|(1-\zeta)]z^2}{\left[\left(|\rho| + [2]_q\right)(1-\zeta) + \zeta - \upsilon\right]} \right|^{\chi} \mathrm{d}\theta.$$

By Lemma 4.12, we have

$$1 - \sum_{\mathrm{i}=2}^{\infty} |a_{\mathrm{i}}| z^{\mathrm{i}-1} \prec 1 - \frac{1 - \upsilon + |\rho|(1-\zeta)}{\left[\left(|\rho| + [2]_q\right)(1-\zeta) + \zeta - \upsilon\right]} z.$$

Setting

$$1 - \sum_{\mathrm{i}=2}^{\infty} |a_{\mathrm{i}}| z^{\mathrm{i}-1} = 1 - \frac{1 - \upsilon + |\rho|(1-\zeta)}{\left[\left(|\rho| + [2]_q\right)(1-\zeta) + \zeta - \upsilon\right]} \vartheta(z).$$

We get $\vartheta(z) = \sum_{\mathrm{i}=2}^{\infty} \frac{[(|\rho|+[2]_q)(1-\zeta)+\zeta-\upsilon]}{1-\upsilon+|\rho|(1-\zeta)} a_{\mathrm{i}} z^{\mathrm{i}-1}$ and $\vartheta(z)$ is analytic in $\mathbb{U}$ with $\vartheta(0) = 0$.

It suffices to demonstrate that $\vartheta(z)$ satisfies $|\vartheta(z)| < 1$.

That is,

$$\begin{aligned} |\vartheta(z)| &= \left| \sum_{\mathrm{i}=2}^{\infty} \frac{\left[\left(|\rho| + [\mathrm{i}]_q\right)(1-\zeta) + \zeta - \upsilon\right]}{1 - \upsilon + |\rho|(1-\zeta)} a_{\mathrm{i}} z^{\mathrm{i}-1} \right| \\ &\le \sum_{\mathrm{i}=2}^{\infty} \frac{\left[\left(|\rho| + [2]_q\right)(1-\zeta) + \zeta - \upsilon\right]}{1 - \upsilon + |\rho|(1-\zeta)} |a_{\mathrm{i}}| \, |z| < 1. \end{aligned}$$

Hence, $|\vartheta(z)| < 1$. ■

Theorem 4.14: *Suppose that* $\mathsf{f}(z) \in \mathcal{TS}_q^*(\rho, \upsilon, \zeta)$ *and*

$$\mathsf{f}_3\,(z) = z - \frac{1 - \upsilon + |\rho|\,(1 - \zeta)}{[\left(|\rho| + [3]_q\right)(1 - \zeta) + \zeta - \upsilon]} z^3. \tag{17}$$

For $z = re^{i\theta}$, *we deduce that*

$$\int_0^{2\pi} |\mathsf{f}(z)|^{\chi}\, \mathrm{d}\theta \le \int_0^{2\pi} \left|\mathsf{f}_3(z)\right|^{\chi}\, \mathrm{d}\theta,\ (0 \le r < 1).$$

The proof of the desired result readily follows from the application of Theorem 4.13.

With the same approach used in the proofs of Theorems 4.13 and 4.14, we find Theorem 4.15.

Theorem 4.15: *Suppose that* $\mathsf{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$ *and*

$$\mathsf{f}_s\,(z) = z - \frac{1 - \upsilon + |\rho|\,(1 - \zeta)}{[s]_q[\left(|\rho| + [s]_q\right)(1 - \zeta) + \zeta - \upsilon]} z^s \quad (s = 2, 3). \tag{18}$$

For $z = re^{i\theta}$, *we deduce that*

$$\int_0^{2\pi} |\mathsf{f}(z)|^{\chi}\, \mathrm{d}\theta \le \int_0^{2\pi} \left|\mathsf{f}_s(z)\right|^{\chi}\, \mathrm{d}\theta \quad (s = 2, 3;\ 0 \le r < 1).$$

If $q \to 1^-$ in Theorems 4.13, 4.14 and 4.15, we get

Corollary 4.16: *Suppose that* $\mathsf{f}(z) \in \mathcal{TS}^*(\rho, \upsilon, \zeta)$ *and*

$$\mathsf{f}_s\,(z) = z - \frac{1 - \upsilon + |\rho|\,(1 - \zeta)}{[(|\rho| + s)\,(1 - \zeta) + \zeta - \upsilon]} z^s.$$

For $z = re^{i\theta}$, *we deduce that*

$$\int_0^{2\pi} |\mathsf{f}(z)|^{\chi}\, \mathrm{d}\theta \le \int_0^{2\pi} \left|\mathsf{f}_s(z)\right|^{\chi}\, \mathrm{d}\theta, \quad (s = 2, 3; 0 \le r < 1).$$

Corollary 4.17: *Suppose that* $\mathsf{f}(z) \in \mathcal{TC}(\rho, \upsilon, \zeta)$ *and*

$$\mathsf{f}_s\,(z) = z - \frac{1 - \upsilon + |\rho|\,(1 - \zeta)}{s[(|\rho| + s)\,(1 - \zeta) + \zeta - \upsilon]} z^s.$$

For $z = re^{i\theta}$, *we deduce that*

$$\int_0^{2\pi} |\mathsf{f}(z)|^{\chi}\, \mathrm{d}\theta \le \int_0^{2\pi} \left|\mathsf{f}_s(z)\right|^{\chi}\, \mathrm{d}\theta, \quad (s = 2, 3; 0 \le r < 1).$$

4.2. Subordinate sequence properties

This section explores interesting subordination results within the classes of ζ-uniformly of q-starlike $\mathcal{S}_q^*(\rho,\upsilon,\zeta)$ and q-convex $\mathcal{C}_q(\rho,\upsilon,\zeta)$ of analytic functions.

Definition 4.18 ([41]): *A complex sequence $\{\eta_i\}_{i=1}^{\infty}$ is named a subordinating factor sequence if $f(z)=z+\sum_{i=2}^{\infty}a_iz^i$ is univalent and convex in $\mathbb{U}$, we obtain*

$$\sum_{i=1}^{\infty}\eta_i a_i z^i \prec f(z).$$

Lemma 4.19 ([41]): *A complex sequence $\{\eta_i\}_{i=1}^{\infty}$ is said to be subordinating factor sequence iff*

$$\mathcal{R}\left\{1+2\sum_{i=1}^{\infty}\eta_i z^i\right\}>0,\quad (z\in\mathbb{U}).$$

Theorem 4.20: *Suppose $f(z)\in\mathcal{TS}_q^*(\rho,\upsilon,\zeta)$ and $\hbar(z)$ are any other function in the class of convex functions, then*

$$\frac{\Omega_{i=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\Omega_{i=2}(\rho,\zeta,\upsilon)]}\left(f*\hbar\right)(z)\prec\hbar(z),\tag{19}$$

where $\Omega_{i=2}(\rho,\zeta,\upsilon)=[(|\rho|+[2]_q)(1-\zeta)+\zeta-\upsilon]$ and

$$\mathcal{R}\left\{f(z)\right\}>-\frac{1-\upsilon+|\rho|(1-\zeta)+\Omega_{i=2}(\rho,\zeta,\upsilon)}{\Omega_{i=2}(\rho,\zeta,\upsilon)}.\tag{20}$$

The constant $\frac{\Omega_{i=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\Omega_{i=2}(\rho,\zeta,\upsilon)]}$ is the best estimate.

Proof: Assume $f(z)\in\mathcal{TS}_q^*(\rho,\upsilon,\zeta)$ and $\hbar(z)=z+\sum_{i=2}^{\infty}u_iz^i\in\mathbb{U}$, then

$$\frac{\Omega_{i=2}(\rho,\zeta,\upsilon)}{2\left[1-\upsilon+|\rho|(1-\zeta)+\Omega_{i=2}(\rho,\zeta,\upsilon)\right]}\left(f*\hbar\right)(z)$$
$$=\frac{\Omega_{i=2}(\rho,\zeta,\upsilon)}{2\left[1-\upsilon+|\rho|(1-\zeta)+\Omega_{i=2}(\rho,\zeta,\upsilon)\right]}\left[z+\sum_{i=2}^{\infty}a_iu_iz^i\right].$$

Thus, by Definition 4.18, Theorem 4.20 is correct if the following sequence is a subordinating factor sequence, with $a_1=1$.

$$\left\{\frac{\Omega_{i=2}(\rho,\zeta,\upsilon)}{2\left[1-\upsilon+|\rho|(1-\zeta)+\Omega_{i=2}(\rho,\zeta,\upsilon)\right]}a_i\right\}_{i=1}^{\infty}.$$

By Lemma 4.19, equivalently

$$\mathcal{R}\left\{1+\sum_{i=1}^{\infty}\frac{\Omega_{i=2}(\rho,\zeta,\upsilon)}{\left[1-\upsilon+|\rho|(1-\zeta)+\Omega_{i=2}(\rho,\zeta,\upsilon)\right]}a_iz^i\right\}>0.\tag{21}$$

Now, for $|z| = r < 1$, then

$$\begin{aligned}
&\mathcal{R}\left\{1+\sum_{\mathrm{i}=1}^{\infty}\frac{\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}a_{\mathrm{i}}z^{\mathrm{i}}\right\}\\
&=\mathcal{R}\left\{1+\frac{\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}z\right.\\
&\quad\left.+\frac{\sum_{\mathrm{i}=1}^{\infty}\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)a_{\mathrm{i}}z^{\mathrm{i}}}{1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}\right\}.
\end{aligned}$$

Since $\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)$ is an increasing function of i, then

$$\begin{aligned}
&\geq 1-\{\frac{\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}r-\frac{\sum_{\mathrm{i}=1}^{\infty}\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)a_{\mathrm{i}}r^{\mathrm{i}}}{1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}\}\\
&>1-\frac{\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}r-\frac{1-\upsilon+|\rho|(1-\zeta)}{[1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)]}r\\
&=1-r>0.
\end{aligned}$$

This establishes the validity of inequality (21) and confirms the subordinating result (19) as proven in Theorem 4.20. The inequality (20) follows from (19) by taking

$$\hbar(z)=\frac{z}{1-z}=z+\sum_{\mathrm{i}=2}^{\infty}z^{\mathrm{i}}\in\mathbb{U}.$$

Now, we express the function $\mathcal{P}(z)$ as below

$$\mathcal{P}(z)=z-\frac{1-\upsilon+|\rho|(1-\zeta)}{\left[\left(|\rho|+[2]_q\right)(1-\zeta)+\zeta-\upsilon\right]}z^2\in\mathcal{TS}_q^{*}(\rho,\upsilon,\zeta).$$

Then from (19), we have

$$\frac{\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2\left[1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)\right]}\mathcal{P}(z)\prec\frac{z}{1-z}.$$

We observe that

$$min\left\{\mathcal{R}\left(\frac{\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2\left[1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)\right]}\mathcal{P}(z)\right)\right\}=-\frac{1}{2}.$$

This demonstrates the best possible of the following constant:

$$\frac{\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2\left[1-\upsilon+|\rho|(1-\zeta)+\Omega_{\mathrm{i}=2}(\rho,\zeta,\upsilon)\right]}.$$

■

Theorem 4.21: *Suppose* $\mathsf{f}(z) \in \mathcal{TC}_q(\rho, \upsilon, \zeta)$ *and* $\hbar(z)$ *be any other function in the class of convex functions, then*

$$\frac{\overline{\Omega}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\overline{\Omega}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)]}\left(\mathsf{f}*\hbar\right)(z)\prec\hbar(z), \tag{22}$$

where $\overline{\Omega}_{\mathrm{i}=2}(\rho,\zeta,\upsilon) = [2]_q[(|\rho|+[2]_q)(1-\zeta)+\zeta-\upsilon]$ *and*

$$\mathcal{R}\left\{\mathsf{f}(z)\right\} > -\frac{1-\upsilon+|\rho|(1-\zeta)+\zeta+\overline{\Omega}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{\overline{\Omega}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}. \tag{23}$$

The constant $\frac{\overline{\Omega}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\overline{\Omega}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)]}$ *is the best estimate.*

Proof: The proof is similar to that of Theorem 4.20. ∎

Theorems 4.20 and 4.21 yield to the following outcomes, when $q \to 1^-$.

Corollary 4.22: *Suppose* $\mathsf{f}(z) \in \mathcal{TS}^*(\rho, \upsilon, \zeta)$ *and* $\hbar(z)$ *are any other function in the class convex of functions, then*

$$\frac{\Xi_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\Xi_{\mathrm{i}=2}(\rho,\zeta,\upsilon)]}\left(\mathsf{f}*\hbar\right)(z)\prec\hbar(z),$$

where $\Xi_{\mathrm{i}=2}(\rho,\zeta,\upsilon) = [(|\rho|+2)(1-\zeta)+\zeta-\upsilon]$ *and*

$$\mathcal{R}\left\{\mathsf{f}(z)\right\} > -\frac{1-\upsilon+|\rho|(1-\zeta)+\Xi_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{\Xi_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}.$$

The constant $\frac{\Xi_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\Xi_{\mathrm{i}=2}(\rho,\zeta,\upsilon)]}$ *is the best estimate.*

Corollary 4.23: *Suppose* $\mathsf{f}(z) \in \mathcal{TC}(\rho, \upsilon, \zeta)$ *and* $\hbar(z)$ *be any other function in the class of convex functions, then*

$$\frac{\overline{\Xi}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\overline{\Xi}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)]}\left(\mathsf{f}*\hbar\right)(z)\prec\hbar(z), \tag{24}$$

where $\overline{\Xi}_{\mathrm{i}=2}(\rho,\zeta,\upsilon) = 2[(|\rho|+2)(1-\zeta)+\zeta-\upsilon]$ *and*

$$\mathcal{R}\left\{\mathsf{f}(z)\right\} > -\frac{1-\upsilon+|\rho|(1-\zeta)+\zeta+\overline{\Xi}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{\overline{\Xi}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}. \tag{25}$$

The constant $\frac{\overline{\Xi}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)}{2[1-\upsilon+|\rho|(1-\zeta)+\overline{\Xi}_{\mathrm{i}=2}(\rho,\zeta,\upsilon)]}$ *is the best estimate.*

5. Conclusion

Having as inspiration the results obtained by applying q-calculus on certain analytic functions, we have established in this investigation new classes of ζ-uniformly q-starlike and q-convex functions of order υ. Additionally, we have derived a range of geometric properties, including estimates for coefficients, the sharpness, distortion and growth theorems, and convex linear combinations. Furthermore, we have leveraged the classes of uniformly q-starlike and q-convex functions to investigate various subordination findings, such as integral means inequalities and subordinate sequences. This study serves as a valuable reference for future works and sheds light on new concepts within geometric function theory. Moreover, numerous other geometrical consequences such as the convolution, radius theorems, closer property, partial sums and neighbourhood, which have not yet been addressed and are still open to other researchers.

Acknowledgments

The authors would like to express their heartfelt gratitude to the dedicated editorial teams and thoughtful reviewers for their invaluable contributions to our work.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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