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Partial Sums of a Generalized q-Differential Operator

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Abstract. This article discusses on a new q-differential operator defined by a generalized q-Mittag-Leffler function. Motivated by this operator, we state a new subclass for λ-uniformly starlike of order ϖ. Further, we study coefficient estimate and partial sums for this class.

INTRODUCTION

Let $\mathscr{A}$ be the class of analytic and univalent functions $f(z)$ in the open unit disk $\mathscr{H} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized form

$$f(z) = z + \sum_{l=2}^{\infty} a_l z^l. \tag{1}$$

Also, we consider the subclass of the class $\mathscr{A}$ denoted by $\mathscr{T}$ generated of the following functions:

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \ (a_j \geq 0). \tag{2}$$

The Hadamard product of two functions f and $\mathscr{F}$ denoted by $f * \mathscr{F}$ is defined by

$$(f * \mathscr{F})(z) := z + \sum_{l=2}^{\infty} a_l d_l z^l =: (\mathscr{F} * f)(z),$$

where $f(z)$ has been mentioned in (1) and $\mathscr{F}(z) = z + \sum_{l=2}^{\infty} d_l z^l$.

For $0 \leq \zeta < 1$, the subclasses $\mathscr{S}^*(\zeta)$ and $\widehat{\mathscr{C}}(\zeta)$ of the class $\mathscr{A}$ which are called starlike and convex functions of order ζ, respectively and given by

$$\mathscr{S}^*(\zeta) = \left\{ f \in \mathscr{A} : \mathscr{R}\left(\frac{zf'(z)}{f(z)}\right) > \zeta \right\},$$

and

$$\widehat{\mathscr{C}}(\zeta) = \left\{ f \in \mathscr{A} : \mathscr{R}\left(\frac{zf''(z)}{f(z)} + 1\right) > \zeta \right\}.$$

Lately, many authors have focused on the field of q-calculus; it is well known as q-analysis. This interest is due to the importance of its applications in various mathematics and quantum physics fields. In 1908, Jackson [1]the first to introduced and developed the concepts of q-derivative and q-integral. Also, the geometries for q-analysis have been found in many studies presented on quantum groups. It has also been identified that there is a relationship between q-integral and q-derivative. Recently, certain classes of functions that are analytic in $\mathscr{H}$ using fractional q-calculus operators have been investigated by many authors (for examples, see [2, 3, 4, 5, 6]).

Definition .1. [1] For $0 < q < 1$, the q-factorial denoted by $[n]_q!$ is defined by:

$$[j]_q! := \begin{cases} [j]_q[j-1]_q \dots [2]_q[1]_q, & j = 1, 2, 3, \dots \\ 1, & j = 0. \end{cases}$$

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Then the q-numbers $[j]_q$ is given by

$$[j]_q := \begin{cases} \dfrac{1-q^j}{1-q} & (j \in \mathbb{C}) \\ \sum\limits_{j=0}^{n-1} q^j & (j = n \in \mathbb{N}) \\ 0 & (j = 0). \end{cases}$$

Definition .2. [1] The q-generalized Pochhammer symbol $[\kappa]_{l,q}$, $\kappa \in \mathbb{C}$ is given by

$$[\kappa]_{l,q} = [\kappa]_q[\kappa+1]_q[\kappa+2]_q \ldots [\kappa+l-1]_q,$$

and the q-Gamma function is given as below

$$\Gamma_q(\kappa+1) = [\kappa]_q \Gamma_q(\kappa) \text{ and } \Gamma_q(1) = 1.$$

Definition .3. [1] The q-derivative operator is defined as below

$$\mathfrak{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.(z \neq 0)$$

To calculate the q-derivative of the function $f(z)$ in (1), we have

$$\mathfrak{D}_q f(z) = 1 + \sum_{l=2}^{\infty} [l]_q a_l z^{l-1}.$$

We note that

$$lim_{q \to 1-} \mathfrak{D}_q f(z) = f'(z).$$

In 2019, [4] defined the classes of starlike and convex functions of the derivative function $\mathfrak{D}_q f(z)$ as below

$$\ddot{\mathscr{S}}_q^*(\zeta) = \mathscr{R}\left(1 + \frac{1}{\zeta}\left(\frac{z\mathfrak{D}_q f(z)}{f(z)} - 1\right)\right) > 0,$$

and

$$\widehat{\mathscr{C}_q}(\zeta) = \mathscr{R}\left(1 + \frac{1}{\zeta}\left(\frac{\mathfrak{D}_q(z\mathfrak{D}_q f(z))}{\mathfrak{D}_q f(z)}\right)\right) > 0,$$

where $\zeta \in \mathbb{C} \setminus \{0\}$ and $0 < q < 1$.

The Mittag-Leffler function has a large role in solving differential and integral equations. Recently, it has become a topic of great interest in the field in a calculus and its applications. So, some mathematicians consider the classic Mittag-Leffler function to be the Queen function of fractional calculus (for example, see [7]).

Mittag Leffler in 1903 [8] established the function $E_\tau(z)$ as follows

$$E_\tau(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\tau l + 1)}, (\tau, z \in \mathbb{C}, \mathscr{R}(\tau) > 0).$$

Wiman in 1905 [9] presented the generalization of Mittag-Leffler function given by

$$E_{\tau,\mu}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\tau l + \mu)} (\tau, \mu, z \in \mathbb{C}, [\mathscr{R}(\tau), \mathscr{R}(\mu)] > 0).$$

Prabhakar in 1971 [10] provided the function $E^\rho_{\tau,\mu}(z)$ as the following form

$$E^\rho_{\tau,\mu}(z) = \sum_{l=0}^{\infty} \frac{(\rho)_l}{\Gamma(\mu + \tau l)} \frac{z^l}{l!} \ \ (\tau, \mu, \rho, z \in \mathbb{C}, [\mathscr{R}(\tau), \mathscr{R}(\mu), \mathscr{R}(\rho)] > 0).$$

Likewise, Shukla and Prajapati [11] (also see [12]) presented a generalization of Mittag-Leffler function

$$E_{\tau,\mu}^{\rho,k}(z)=\sum_{l=0}^{\infty}\frac{(\rho)_{kl}}{\Gamma(\tau l+\mu)}\frac{z^l}{l!},\ (\tau,\mu,\rho,z\in\mathbb{C},[\mathscr{R}(\tau),\mathscr{R}(\mu),\mathscr{R}(\rho)]>0)$$

where $k\in(0,1)\bigcup\mathbb{N}$ and $(\rho)_{kl}=\frac{\Gamma(\rho+kl)}{\Gamma(\rho)}$ is the generalized Pochhammer symbol particularly reduce to

$$k^{kl}\prod_{m=1}^{k}\left(\frac{\rho+m-1}{k}\right)_l \text{ if } k\in\mathbb{N}.$$

Many studies have focused on the study of the Mittag-Leffler function and its generalizations (see, [13, 14, 15, 16, 17, 18, 19]).
Mansour [20] introduced a new concept of the q-Mittag-Leffler function as follows

$$E_{\tau,\mu}(q;z)=\sum_{l=0}^{\infty}\frac{z^l}{\Gamma_q(\tau l+\mu)},(\mu\in\mathbb{C},\tau>0;|z|<(1-q)^{-\tau}).$$

Hadi et al. [21] defined a generalization of q-Mittag-Leffler function given by

$$ML_{\tau,\mu}^{\rho}(q;z)=z+\sum_{l=2}^{\infty}\frac{(\rho)_{kl}}{\Gamma_q(\mu+\tau l)}\frac{z^l}{l!}\ (\tau,\mu,\rho\in\mathbb{C},[\mathscr{R}(\tau),\mathscr{R}(\mu),\mathscr{R}(\rho)]>0) \tag{3}$$

where $k\in(0,1)\bigcup\mathbb{N}$ and $(\rho)_{kl}=\frac{\Gamma(\rho+kl)}{\Gamma(\rho)}$ is the generalized Pochhammer symbol.
Next, we define a new q-differential operator $M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z):\mathscr{A}\to\mathscr{A}$ depending on the q-Mittag-Leffler in (3) as follows

$$M_{\tau,\mu,\rho}^{\eta}f(z)=ML_{\tau,\mu}^{\rho}(q;z)*f(z), \tag{4}$$

$$M_{\tau,\mu,\rho}^{\eta,\delta,1}f(z)=(1-(\eta-\delta))\left(ML_{\tau,\mu}^{\rho}(q;z)*f(z)\right)+(\eta-\delta)z\mathfrak{D}_q(ML_{\tau,\mu}^{\rho}(q;z)*f(z)), \tag{5}$$

$$\vdots$$

$$M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z)=M_{\tau,\mu,\rho}^{\eta,\delta,1}\left(M_{\tau,\mu,\rho}^{\eta,\delta,j-1}(q;z)f(z)\right). \tag{6}$$

Now, by (5) and (6), we show that

$$\begin{aligned}M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z)&=z+\sum_{l=2}^{\infty}[1+([l]_q-1)(\eta-\delta)]^j\frac{(\rho)_{kl}}{\Gamma_q(\mu+\tau l)}\frac{a_l z^l}{l!}\\&=z+\sum_{l=2}^{\infty}\Xi_j a_l z^l,\end{aligned} \tag{7}$$

where $\Xi_j=[1+([l]_q-1)(\eta-\delta)]^j\frac{(\rho)_{kl}}{\Gamma_q(\mu+\tau l)l!}$, $j\in\mathbb{N}\bigcup\{0\}$, and $0\leq\delta\leq\eta$.
Making use of the new q-differential operator $M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z)$ and the class of λ-uniformly starlike functions of order ϖ defined by [22], we introduce the following subclass:

Definition .4. For $0\leq\alpha\leq1,\ 0\leq\varpi<1,\ \lambda\geq0$, a function $f(z)$ belongs to the subclass $\mathscr{G}_{\tau,\mu,\eta,\delta}^{\rho,\varpi,\lambda,\alpha}(q,j;z)$ of λ-uniformly starlike function of order ϖ if and only if

$$\mathscr{R}\left(\frac{z\mathfrak{D}_q(M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z))}{(1-\alpha)z+\alpha(M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z))}-\varpi\right)>\lambda\left|\frac{z\mathfrak{D}_q(M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z))}{(1-\alpha)z+\alpha(M_{\tau,\mu,\rho}^{\eta,\delta,j}f(z))}-1\right|\ z\in\mathscr{H}. \tag{8}$$

We also set

$$\mathscr{T}\mathscr{G}_{\tau,\mu,\eta,\delta}^{\rho,\varpi,\lambda,\alpha}(q,j;z)=\mathscr{G}_{\tau,\mu,\eta,\delta}^{\rho,\varpi,\lambda,\alpha}(q,j;z)\bigcap\mathscr{T}.$$

SET OF MAIN RESULTS

In the first theorem, we present the coefficient estimate for the function $f(z)$ in (1) to be in the subclass $\mathscr{G}^{\rho,\varpi,\lambda,\alpha}_{\tau,\mu,\eta,\delta}(q,j;z)$.

Theorem .1. *For* $0 \le \alpha \le 1, 0 \le \varpi < 1,$ and $\lambda \ge 0$. *A class* $\mathscr{G}^{\rho,\varpi,\lambda,\alpha}_{\tau,\mu,\eta,\delta}(q,j;z)$ *contains the function* $f(z)$ *if it satisfies the condition below*

$$\sum_{l=2}^{\infty} \left([l]_q(1+\lambda) - \alpha(\varpi+\lambda)\right)\Xi_j |a_l| \le 1-\varpi. \tag{9}$$

Proof. To show that the function $f(z) \in \mathscr{G}^{\rho,\varpi,\lambda,\alpha}_{\tau,\mu,\eta,\delta}(q,j;z)$, *we just need to prove that*

$$\lambda \left| \frac{z\mathfrak{D}_q(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))}{(1-\alpha)z+\alpha(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))} - 1 \right| - \mathscr{R}\left(\frac{z\mathfrak{D}_q(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))}{(1-\alpha)z+\alpha(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))} - 1 \right) \le 1-\varpi.$$

Since $-\mathscr{R}(z) \le |z|$, *we note that*

$$\begin{aligned}
&\lambda \left| \frac{z\mathfrak{D}_q(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))}{(1-\alpha)z+\alpha(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))} - 1 \right| - \mathscr{R}\left(\frac{z\mathfrak{D}_q(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))}{(1-\alpha)z+\alpha(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))} - 1 \right) \\
&\le (1+\lambda)\left| \frac{z\mathfrak{D}_q(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))}{(1-\alpha)z+\alpha(M^{\eta,\delta,j}_{\tau,\mu,\rho}f(z))} - 1 \right| \le (1+\lambda)\frac{\sum_{l=2}^{\infty}([l]_q-\alpha)\Xi_j|a_l||z|^l}{|z|-\alpha\sum_{l=2}^{\infty}\Xi_j|a_l||z|^l} \\
&= (1+\lambda)\frac{\sum_{l=2}^{\infty}([l]_q-\alpha)\Xi_j|a_l|}{1-\alpha\sum_{l=2}^{\infty}\Xi_j|a_l|} \le 1-\varpi.
\end{aligned}$$

where we have employed the inequalities $|z| < 1$ *and* (9). *This completes the proof of Theorem .1.*

Theorem .2. *For* $0 \le \alpha \le 1, 0 \le \varpi < 1,$ and $\lambda \ge 0$. *A function* $f(z)$ *belongs to the class* $\mathscr{T}\mathscr{G}^{\rho,\varpi,\lambda,\alpha}_{\tau,\mu,\eta,\delta}(q,j;z)$ *if and only if*

$$\sum_{l=2}^{\infty} \left([l]_q(1+\lambda) - \alpha(\varpi+\lambda)\right)\Xi_j |a_l| \le 1-\varpi. \tag{10}$$

Proof. Since $\mathscr{T}$ *is the subclass of* $\mathscr{A}$, *then the sufficient condition holds by Theorem .1 and the functions* (2). *Now, we have to prove the necessity condition.*

Suppose that the function $f(z) \in \mathscr{T}\mathscr{G}^{\rho,\varpi,\lambda,\alpha}_{\tau,\mu,\eta,\delta}(q,j;z)$. *Then, from the fact* $|\mathscr{R}(z)| \le |z|$ *for* $z \in \mathbb{U}$, *we obtain*

$$\left| \frac{z-\sum_{l=2}^{\infty}[l]_q\Xi_j a_l z^l}{z-\sum_{l=2}^{\infty}\alpha\Xi_j a_l z^l} - \varpi \right| > \lambda \left| \frac{\sum_{l=2}^{\infty}([l]_q-\alpha)\Xi_j a_l z^l}{z-\sum_{l=2}^{\infty}\alpha\Xi_j a_l z^l} \right|. \tag{11}$$

In the real axis, we select the z values for which $\mathfrak{D}_q f(z)$ *is real. Putting* $z \to 1-$ *in real values and upon clearing the denominator of* (11),

$$\sum_{l=2}^{\infty} \left([l]_q(1+\lambda) - \alpha(\varpi+\lambda)\right)\Xi_j |a_l| - 1 - \varpi \le 0.$$

Corollary .1. *Let* $f(z) \in \mathscr{T}\mathscr{G}^{\rho,\varpi,\lambda,\alpha}_{\tau,\mu,\eta,\delta}(q,j;z)$, *then*

$$|a_l| \le \frac{1-\varpi}{\phi_l^j}, \tag{12}$$

for $0 \le \alpha \le 1,\ 0 \le \varpi < 1,$ and $\lambda \ge 0$, *with*

$$f(z) = z + \frac{1-\varpi}{\phi_l^j} z^l,$$

where $\phi_l^j = ([l]_q(1+\lambda) - \alpha(\varpi+\lambda))\,\Xi_j$.
If $l = 2$, *then*

$$|a_2| \leq \frac{1-\varpi}{\phi_2^j},$$

where $\phi_2^j = ([2]_q(1+\lambda) - \alpha(\varpi+\lambda))\,\Xi_j$.

PARTIAL SUMS FOR THE CLASS $\mathscr{TG}^{\rho,\varpi,\lambda,\alpha}_{\tau,\mu,\eta,\delta}(q,j;z)$

Lately, many results were given related to partial sums of Mittag-Leffler function and its generalizations (see for example, [23]).
Previous works were establish by Silverman [24] and many authors (see, [25, 26, 27]) on partial sums of analytic functions. In this part, we study the partial sums property for the function defined by (1) in relation to its sequence of partial sums of the form

$$f_m(z) = z + \sum_{l=2}^{m} a_l z^l.$$

Theorem .3. *If the function* $f(z)$ *satisfies the inequality* (9)*, then*

$$\mathscr{R}\left(\frac{f(z)}{f_m(z)}\right) \geq \frac{\phi_{m+1}^j - 1 + \varpi}{\phi_{m+1}^j}, \quad (z \in \mathscr{H}) \tag{13}$$

where

$$\phi_l^j \geq \begin{cases} 1-\varpi, & \text{if } l = 2,3,\ldots,m \\ \phi_{m+1}^j, & \text{if } l = m+1, m+2,\ldots \end{cases}. \tag{14}$$

The inequality in (13) *is sharp with the function defined by*

$$f(z) = z + \frac{1-\varpi}{\phi_{m+1}^j} z^{m+1}. \tag{15}$$

Proof. Let the function $u(z)$ *be given by*

$$\frac{1+u(z)}{1-u(z)} = \frac{\phi_{m+1}^j}{1-\varpi}\left[\frac{f(z)}{f_m(z)} - \frac{\phi_{m+1}^j - 1 + \varpi}{\phi_{m+1}^j}\right] = \frac{1+\sum_{l=2}^{m} a_l z^{l-1} + \left(\frac{\phi_{m+1}^j}{1-\varpi}\right)\sum_{l=m+1}^{\infty} a_l z^{l-1}}{1+\sum_{l=2}^{m} a_l z^{l-1}}. \tag{16}$$

It is necessary to prove that $|u(z)| \leq 1$.
Next, based on (16) *we can define*

$$u(z) = \frac{\left(\frac{\phi_{m+1}^j}{1-\varpi}\right)\sum_{l=m+1}^{\infty} a_l z^{l-1}}{2+2\sum_{l=2}^{m} a_l z^{l-1} + \left(\frac{\phi_{m+1}^j}{1-\varpi}\right)\sum_{l=m+1}^{\infty} a_l z^{l-1}}.$$

So we get

$$|u(z)| \leq \frac{\left(\frac{\phi_{m+1}^j}{1-\varpi}\right)\sum_{l=m+1}^{\infty} |a_l|}{2-2\sum_{l=2}^{m} |a_l| - \left(\frac{\phi_{m+1}^j}{1-\varpi}\right)\sum_{l=m+1}^{\infty} |a_l|}.$$

Now $|u(z)| \leq 1$ *if*

$$2\left(\frac{\phi_{m+1}^{j}}{1-\varpi}\right)\sum_{l=m+1}^{\infty}|a_l| \leq 2-2\sum_{l=2}^{m}|a_l|,$$

and equivalently

$$\sum_{l=2}^{m}|a_l|+\sum_{l=m+1}^{\infty}\frac{\phi_{m+1}^{j}}{1-\varpi}|a_l| \leq 1.$$

From the inequality (9)*, it suffices to prove that*

$$\sum_{l=2}^{m}|a_l|+\sum_{l=m+1}^{\infty}\frac{\phi_{m+1}^{j}}{1-\varpi}|a_l| \leq \sum_{l=2}^{\infty}\frac{\phi_{l}^{j}}{1-\varpi}|a_l|,$$

which is equal to

$$\sum_{l=2}^{m}\left(\frac{\phi_{m}^{j}-1+\varpi}{1-\varpi}\right)|a_l|+\sum_{l=m+1}^{\infty}\left(\frac{\phi_{m}^{j}-\phi_{m+1}^{j}}{1-\varpi}\right)|a_l| \geq 0.$$

Theorem .4. *If the function* $f(z)$ *satisfies the inequality* (9)*, then*

$$\mathscr{R}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{\phi_{m+1}^{j}}{\phi_{m+1}^{j}+1-\varpi}, \quad (z\in\mathscr{H}) \tag{17}$$

where $\phi_{m+1}^{j} \geq 1-\varpi$ *with*

$$\phi_{l}^{j} \geq \begin{cases} 1-\varpi, & \text{if } l=2,3,\ldots,m \\ \phi_{m+1}^{j}, & \text{if } l=m+1,m+2,\ldots \end{cases}. \tag{18}$$

The inequality in (17) *is sharp with the function defined by*

$$f(z)=z+\frac{1-\varpi}{\phi_{m+1}^{j}}z^{m+1}. \tag{19}$$

Proof. From definition $u(z)$ *by*

$$\frac{1+u(z)}{1-u(z)}=\frac{\phi_{m+1}^{j}+1-\varpi}{1-\varpi}\left[\frac{f_m(z)}{f(z)}-\frac{\phi_{m+1}^{j}}{\phi_{m+1}^{j}+1-\varpi}\right],$$

and following the same method in Theorem (.3)*.*

Theorem .5. *Let the function* $f(z)$ *satisfies the inequality* (9)*, then*

$$\mathscr{R}\left(\frac{\mathfrak{D}_q(f(z))}{\mathfrak{D}_q(f_m(z))}\right) \geq \frac{\phi_{m+1}^{j}-[m+1]_q(1-\varpi)}{\phi_{m+1}^{j}} \quad (z\in\mathscr{H}) \tag{20}$$

and

$$\mathscr{R}\left(\frac{\mathfrak{D}_q(f_m(z))}{\mathfrak{D}_q(f(z))}\right) \geq \frac{\phi_{m+1}^{j}}{\phi_{m+1}^{j}+[m+1]_q(1-\varpi)} \quad (z\in\mathscr{H}) \tag{21}$$

where $\phi_{m+1}^{j} \geq [m+1]_q(1-\varpi)$ *with*

$$\phi_{l}^{j} \geq \begin{cases} [l]_q(1-\varpi), & \text{if } \ l=2,3,\ldots,m \\ [l]_q\left(\frac{\phi_{m+1}^{j}}{[m+1]_q}\right), & \text{if } \ l=m+1,m+2,\ldots \end{cases}. \tag{22}$$

The inequalities in (20) *and* (21) *are sharp with the function defined by* (15).

Proof. Let the function $u(z)$ be given by

$$\frac{1+u(z)}{1-u(z)}=\frac{\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)}\left[\frac{\mathfrak{D}_q(f(z))}{\mathfrak{D}_q(f_m(z))}-\frac{\phi_{m+1}^{j}-[m+1]_q(1-\varpi)}{\phi_{m+1}^{j}}\right], \tag{23}$$

where

$$u(z)=\frac{\left(\frac{\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)}\right)\sum_{l=m+1}^{\infty}[l]_q a_l z^{l-1}}{2+2\sum_{l=2}^{m}[l]_q a_l z^{l-1}+\left(\frac{\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)}\right)\sum_{l=m+1}^{\infty}[l]_q a_l z^{l-1}}.$$

Now, $|u(z)|\le 1$ *if and only if*

$$\sum_{l=2}^{m}[l]_q|a_l|+\frac{\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)}\sum_{l=m+1}^{\infty}[l]_q|a_l|\le 1.$$

From the inequality (9)*, we have to prove that*

$$\sum_{l=2}^{m}[l]_q|a_l|+\frac{\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)}\sum_{l=m+1}^{\infty}[l]_q|a_l|\le\sum_{l=2}^{\infty}\frac{\phi_l^{j}}{1-\varpi}|a_l|,$$

which is equivalent to

$$\sum_{l=2}^{m}\left(\frac{\phi_l^{j}-[l]_q(1-\varpi)}{1-\varpi}\right)|a_l|+\sum_{l=m+1}^{\infty}\frac{[m+1]_q\phi_l^{j}-[l]_q\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)}|a_l|\ge 0.$$

To prove the condition (21)*, define the function $u(z)$ as follows:*

$$\frac{1+u(z)}{1-u(z)}=\frac{[m+1]_q(1-\varpi)+\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)}\left[\frac{\mathfrak{D}_q(f_m(z))}{\mathfrak{D}_q(f(z))}-\frac{\phi_{m+1}^{j}}{[m+1]_q(1-\varpi)+\phi_{m+1}^{j}}\right]$$

and we get the desired result by using similar claims in the first part.

CONCLUSION

Recently, q-calculus has received a great attention due to wide applications in many fields, especially mathematical and quantum physics. In this article, we defined a new q-differential operator associated with q-Mittag-Leffler function. Using this operator, we introduced a new subclass of λ-uniformly starlike functions of order ϖ. Moreover, we studied the partial sums of this subclass.

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REFERENCES

1. F. H. Jackson, "On q-functions and a certain difference operator," Trans. R. Soc. Edinburgh **46**, 253–281 (1908).

2. K. Vijaya, G. Murugusundaramoorthy, and S. Yalcin, "Certain class of analytic functions involving Salagean type q-difference operator," Konuralp. J. Math. **6**, 264–271 (2018).
3. S. Araci, U. Duran, M. Acikgoz, and H. M. Srivastava, "A certain (p,q)-derivative operator and associated divided differences," J. Ineq. Appl. **2016**, art. 301 (2016).
4. S. Mahmood, N. Raza, E. S. A. Abujarad, G. Srivastava, H. M. Srivastava, and S. N. Malik, "Geometric properties of certain classes of analytic functions associated with a q-integral operator," Symmetry **11**, 1–14 (2019).
5. S. H. Hadi and M. Darus, "Differential subordination and superordination of a q-derivative operator connected with the q-Exponential function," Int. J. Nonlinear Anal. Appl. **2022**, Article ID 5865416 (2022).
6. S. H. Hadi and M. Darus, "(p,q)-Chebyshev polynomials for the families of biunivalent function associating a new integral operator with (p,q)-Hurwitz Zeta function," Turk. J. Math. **46**, 24115–2429 (2022).
7. F. Mainardi and R. Gorenflo, "Time-fractional derivatives in relaxation processes: a Tutorial Survey," Fract. Calc. Appl. Anal. **10**, 269–308 (2007).
8. G. M. Mittag-Leffler, "Sur la nouvelle function $e\alpha(x)$," C. R .Acad. Sci Paris **137**, 554–558 (1903).
9. A. Wiman, "Uber den fundamental satz in der theory der funktionen," Acta Math. **29**, 191–201 (1905).
10. T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernal," (1971).
11. A. Shukla and J. Prajapati, "On a generalization of Mittag-Leffler function and its properties," J. Math. Ann. Appl. **336**, 797–811 (2017).
12. H. Srivastava and Z. Tomovski, "Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernal," Appl. Math. Comp. **211**, 198–210 (2009).
13. A. A. Attiya, "Some applications of Mittag-Leffler function in the unit disk," Filomat **30**, 2075–2081 (2016).
14. V. Kiryakova, "The multi-index Mittag-Leffler functions as important class of specialfunctions of fractional calculus," Comput. Math. Appl. **59**, 1885–1895 (2010).
15. J. P. Konovska, "Convergence of series in three parametric Mittag-Leffler functions," Math. Slovaca **64**, 73–84 (2014).
16. S. Horrigue and S. M. Madian, "Some inclusion relationships of meromorphic functions associated to new generalization of Mittag-Leffler function," Filomat **34**, 1545–1556 (2020).
17. S. Elhaddad and M. Darus, "Differential subordination and superordinationfor a new differential operator containing Mittag-Leffler function," Kragujevac Journal of Mathematics **45**, 699–708 (2021).
18. Y. L. Cang and J. L. Liu, "A family of multivalent analytic functions associated with Srivastava-Tomovski generalization of the Mittag-Leffler function," Filomat **23**, 4619–4625 (2018).
19. Z. Jia, N. Khan, S. Khan, and B. Khan, "polynomial coefficients estimates for certain subclasses of q-Mittag-Leffler-type analytic and bi-univalent functions," AIMS Mathematics **7**, 2512–2528 (2021).
20. Z. S. I. Mansour, "Linear sequential q-difference equations of fractional order," Fract. Calc. Appl. Anal. **12**, 159–178 (2009).
21. S. H. Hadi, M. Darus, C. Park, and J. R. Lee, "Some geometric properties of multivalent functions associated with a new generalized q-Mittag-Leffler function," AIMS Mathematics **7**, 11772–11783 (202).
22. E. S. A. AbuJarad, M. H. A. AbuJarad, T. Abdeljawad, and F. Jarad, "Certain subclasses of β-uniformly q-starlike and β-uniformly q-convex functions," Journal of Function Spaces **2020**, Article ID 5865416 (2020).
23. M. S. U. Rehman, Q. Z. Ahmad, H. M. Srivastava, B. Khan, and N. Khan, "Partial sums of generalized q-Mittag-Leffler functions," AIMS Mathematics **5**, 408–420 (2020).
24. H. Silverman, "Univalent functions with negative coefficients," Proc. Amer. Math. Soc. **51**, 109–116 (1975).
25. H. Silverman, "Partial sums of starlike and convex functions," J. Math Anal. App. **209**, 221–227 (1997).
26. B. A. Frasin and G. Murugusundaramoothy, "Partial sums of certain analytic functions," Mathematica **53**, 131–142 (2011).
27. T. Rosy, K. G. Subramanian, and G. Murugusundaramoorthy, "Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives," J. Ineq. Pure Appl. Math. **4**, art. 64 (2003).