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Linear fractional dynamic equations: Hyers–Ulam stability analysis on time scale

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ABSTRACT

The article's purpose is to examine ăthe Hyers–Ulam stability (HUS) for some linear fractional dynamic equations (FDEs) with the Caputo Δ–derivative on time scale. If we swap out a certain FDE for a fractional dynamical inequality, we want to know how close the solutions of the fractional dynamical inequality are to the solutions of the exact FDEs. Meanwhile, the generalized HUS result is obtained as a direct corollary. To achieve this goal, we solve the aforementioned equations utilizing the time scale version of the Laplace transform. Subsequently, the HUS is investigated in accordance with theseăsolutions.

1. Introduction

The term "Ulam stability" was born in 1940 through a novel question asked by Ulam at Wisconsin University. This question about the stability problem of functional equations is succinctly stated as follows: "Under what conditions does there exist an additive mapping near an approximately additive mapping?". In Banach spaces, Hyers [1] provided an answer to the problem of Ulam for additive functions:

"Let Ξ_1, Ξ_2 be two real Banach space and $\varepsilon > 0$. Then for every mapping $\Lambda : \Xi_1 \to \Xi_2$ satisfying

$$\|\Lambda(x+y) - \Lambda(x) - \Lambda(y)\| \leq \varepsilon, \tag{1.1}$$

for all $x, y \in \Xi_1$ there exists a unique additive mapping $B : \Xi_1 \to \Xi_2$ with the property

$$\|\Lambda(x) - B(x)\| \leq \varepsilon, \quad \forall x \in \Xi_1.\varepsilon \tag{1.2}$$

This is the beginning point of the HUS theory of functional equations. Rassias [2] presented an impressive generalization of the HUS of mappings by taking into account variables. However, the stability properties of all types of equations have become of interest to numerous mathematicians. Over the next two decades, virtually all research on this stability concentrated on various kinds of functional equations and various abstract spaces [3,4].

Recently, it was suggested to generalize Ulam's problem by substituting differential equations in place of functional equations. Obloza [5] was the first person to discuss the stability of differential equations via the concept of Ulam stability. Many researchers have been concentrating on the study of the HUS of differential equations [6–15].

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The field of fractional calculus encompasses the examination of integrals and derivatives of fractional orders. The phenomenon of fractionalization is of significant importance in bridging the gap between the classical model and quantum physics, enabling a holistic understanding of the universe at various levels, including subatomic particles and cosmic structures. The primary aim of this project is to enhance the accuracy in representing the physical realm. Fractional calculus has experienced a surge in significance in recent times due to its efficacy as a robust tool for accurately and successfully modeling intricate phenomena across various seemingly disparate domains in the realms of science and engineering.

In addition, fractional differential equations (FDEs) have been utilized in diverse domains, such as economics, various epidemiological, electrical engineering, conducted research on thermal modeling, electrochemistry, conducted research on networks, and among others [16–36]. In the three decades prior, FDEs have become increasingly popular and significant. The Ulam stability of ${}^{C}D^{\alpha}_{\Delta,r_0}u(r) = f(r,u(r))$ with the Caputo derivative was first introduced by Wang et al. [37] via the fixed point theorem. Also, Wang et al. [38] looked at the Ulam stability of the same equation under impulsive settings that same year. In 2015, Jiang et al. [39] discuss the Ulam stability terminology for a kind of operator with appropriate conditions of ${}^{C}D^{\alpha}_{\Delta,r_0}u(z) = (Qu)(r)$ together with the causal operator Q. Cuong [40] investigated the HUS for multi-order FDEs with Riemann–Liouville derivative using the Banach fixed point theorem with Bielecki's type norm.

Hilger [41,42] proposed time scale calculus to unify and generalize the study of theories of discrete and continuous differential equations and to extend these theories to other types of equations known as dynamic equations, which have recently gained a lot of attention. The extension and unification of discrete and continuous equations are the two main features of time scale calculus. Numerous results for continuous dynamic equations transfer pretty readily to analogous results for discrete dynamic equations, although sometimes the results for discrete dynamic equations can seem to be at odds with those of continuous ones. In order to avoid having to repeat the proof of results twice for discrete and continuous dynamic equations, one can study dynamic equations on time scales. Many contributions and developments in time scale, applications of the theory, and methods have been made by many scholars in various fields [43–48]. Nevertheless, there are few studies on the Ulam stability of dynamic equations on time scales. To the best of our knowledge, [49] was the first to study the Ulam stability of several linear and nonlinear dynamic equations as well as integral equations on time scales using direct and operational methods.

The fractional and time scales calculus have been mixed by Bastos's Ph.D. thesis [50], to introduce fractional calculus on time scales. Georgiev [51] created the fundamentals of fractional dynamic calculus and took into account the resolution of FDEs on time scales. The study of FDEs on time scale has attracted the attention of many researchers [52–56]. There are only a few papers which consider the HUS for FDEs on time scale [57–62]. Despite this, the Ulam stability of FDEs with Caputo Δ–derivatives on time scale is still rare.

This paper's purpose is to discuss the general solution and HUS for some linear FDEs with the Caputo Δ–derivative on time scale.

2. Preliminaries

This section covers some fundamental time-scale calculus concepts.

Definition 2.1 (*[45]*). The time scale $\mathbb{T}$ is defined as a non-empty arbitrary subset of $\mathbb{R}$ that is closed and non-empty.

For examples, $\mathbb{C}$, $\mathbb{Q}$, and $[0,1)\,,(0,1]\,,(0,1)\,,\ (0,1] \cup \{2,6\}$ do not represent $\mathbb{T}$. Whereas $\mathbb{Z}$, any closed interval $[a,b] \in \mathbb{R}$, the set $[0,1] \cup [4,5]$, $\mathbb{N}$, and $\mathbb{R}$ represent $\mathbb{T}$.

Definition 2.2 (*[43]*). At $\ell \in \mathbb{T}$, the operator $\sigma : \mathbb{T} \to \mathbb{T}$ is

$$\sigma(\ell) = \inf\{r \in \mathbb{T} : r > \ell\},$$

it is known as a forward jump operator. If $\sigma(\ell) = \ell$, then ℓ is right-dense.

Definition 2.3 (*[45]*). At $\ell \in \mathbb{T}$, the operator $\rho : \mathbb{T} \to \mathbb{T}$ is

$$\rho(\ell) = \sup\{\eta \in \mathbb{T} : \eta < \ell\},$$

it is known as a backward jump operator. If $\rho(\ell) = \ell$, and $\ell > \inf\mathbb{T}$, then point ℓ is called left-dense.

Definition 2.4 (*[43]*). The function $\mu : \mathbb{T} \to [0,\infty)$ is a graininess function, and is given by:

$$\mu(\ell) = \sigma(\ell) - \ell, \quad \forall \ell \in \mathbb{T}.$$

Definition 2.5 (*[43]*). A time scale's derived form, denoted as:

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup(\mathbb{T}) < \infty, \\ \mathbb{T} & \text{if } \sup(\mathbb{T}) = \infty. \end{cases}$$

Definition 2.6 (*[45]*). Let $\varphi : \mathbb{T} \to \mathbb{R}$ at all $\eta \in \mathbb{T}^{\kappa}$. The Hilger or delta derivative is represented by $\varphi^{\Delta}(\eta)$ as follows: $\forall \varepsilon > 0$, a neighborhood exists $\mathcal{M}_{\mathbb{T}}$ of η, $\mathcal{M}_{\mathbb{T}} = (\eta - \delta, \eta + \delta) \cap \mathbb{T}$ for some $\delta > 0$, we have

$$\left|\varphi\,(\sigma(\eta)) - \varphi(\kappa) - \varphi^{\Delta}(\eta)\,(\sigma(\eta) - \kappa)\right| \le \varepsilon\,|\sigma(\eta) - \kappa|,$$

at $\kappa \in \mathcal{M}_{\mathbb{T}},\ \ \kappa \neq \sigma(\eta)$.

Definition 2.7 ([45]). The definition of the time scale monomials function $h_\eta(r, r_0) : \mathbb{T} \times \mathbb{T} \to \mathbb{R},\ \eta \in \mathbb{N}_0$ is

$$h_0(r, r_0) = 1, \quad \forall\, r, r_0 \in \mathbb{T},$$

and

$$h_{\eta+1}(r, r_0) = \int_{r_0}^{r} h_\eta(r, r_0)\Delta r,\ \ \forall r, r_0 \in \mathbb{T}.$$

Theorem 2.1 ([45]). *Let $r, r_0 \in \mathbb{T}$, $\eta \in \mathbb{N}$. Then*

(1) $h_\eta(r, r) = 0$,
(2) $h_1(r, r_0) = r - r_0$,
(3) $h_\eta^{\Delta}(r, r_0) = h_{\eta-1}(r, r_0),\ \ \forall\, r \in \mathbb{T}$.

Example 2.1 ([51]). Consider some elucidatory time scales

(1) Let $\mathbb{T} = \mathbb{R}$, we have

$$h_\eta(r, r_0) = \frac{(r - r_0)^\eta}{\eta!},\ \ \forall\, r, r_0 \in \mathbb{T},\ \eta \in \mathbb{N}.$$

(2) Let $\mathbb{T} = \mathbb{Z}$, we have

$$h_\eta(r, r_0) = \frac{(r - r_0)^{(\eta)}}{\eta!} = \begin{pmatrix} r - r_0 \\ \eta \end{pmatrix},\ \ \forall\, r, r_0 \in \mathbb{T},\ \eta \in \mathbb{N},$$

where $r^{(0)} = 1$, and $r^{(\eta)} = \prod_{i=0}^{\eta-1}(r - i)$.

Definition 2.8 ([43]). The definition of the time scale Laplace transform of a function $\varphi : \mathbb{T} \to \mathbb{R}$ at all $r \in \mathbb{T}$, is

$$\mathcal{L}_\Delta\{\varphi(r)\}(s) = \int_0^\infty \varphi(r) e^{\sigma}_{\ominus s}(r, 0)\Delta r,$$

for $s \in \mathcal{D}\{\varphi\}$, and $\mathcal{D}\{\varphi\}$ includes every complex numbers $s \in \mathbb{C}$ with an improper integral. Inverse Laplace transform for time scale is

$$\varphi(r) = \frac{1}{2\pi i}\int_\chi \mathcal{L}_\Delta\{\varphi(r)\}(s)\prod_{\eta=0}^{\ell-1}(1 + \mu(r_\eta)s)ds,\ \ \forall\, \ell \in \mathbb{N}_0,$$

where χ is any positively oriented closed curve.

Theorem 2.2 ([43]). *At all $s \in \mathbb{C}\backslash\{0\}$, let $1 + s\mu(r) \neq 0$ and $\eta \in \mathbb{N}_0$, we have*

$$\mathcal{L}_\Delta(h_\eta(r, 0))(s) = \frac{1}{s^{\eta+1}}, \quad \forall r \in \mathbb{T}_0,$$

and

$$\lim_{r\to\infty}(h_\eta(r, 0)e_{\ominus s}(r, 0)) = 0.$$

Definition 2.9 ([51]). For given function $\varphi, u : \mathbb{T} \to \mathbb{R}$, their convolution $\varphi * u$ is defined by

$$(\varphi * u)(r) = \int_{r_0}^{r} \hat{\varphi}(r, \sigma(s))u(s)\Delta s,\ \ \forall\, r \in \mathbb{T},\ \ r \geq r_0,$$

where $\hat{\varphi}$ is the shift or delay of φ.

Theorem 2.3 ([43]). *If $\mathcal{L}_\Delta\{\varphi\}(s)$ and $\mathcal{L}_\Delta\{u\}(s)$ be Laplace transform of the functions $\varphi, u : \mathbb{T} \to \mathbb{R}$, respectfully, and $\mathcal{L}_\Delta\{\varphi * u\}(s)$ exist for $s \in \mathbb{C}$. Then we have*

$$\mathcal{L}_\Delta\{\varphi * u\}(s) = \mathcal{L}_\Delta\{\varphi\}(s)\mathcal{L}_\Delta\{u\}(s).$$

Definition 2.10 ([51]). The definition of the generalized fractional Δ−power function is

$$h_\alpha(r, r_0) = \mathcal{L}_\Delta^{-1}\left(\frac{1}{s^{\alpha+1}}\right)(r), \quad \forall r \geq r_0,$$

at all $s \in \mathbb{C}\backslash\{0\}$ is given by

$$h_\alpha(r, \eta) = \widehat{h_\alpha(., r_0)}(r, \eta), \quad \forall \eta, r \in \mathbb{T}, \quad r \geq \eta \geq r_0.$$

Definition 2.11 ([51]). At all $r \in \mathbb{T}$, and $\alpha > 0$. The Riemann–Liouville fractional $\Delta-$integral for $\varphi : \mathbb{T} \to \mathbb{R}$ is

$$I^0_{\Delta,r_0}\varphi(r) = \varphi(r),$$

$$\begin{aligned}(I^\alpha_{\Delta,r_0}\varphi)(r) &= (h_{\alpha-1}(\cdot, r_0) * \varphi)(r)\\ &= \int_{r_0}^{r} \widehat{h_{\alpha-1}(\cdot, r_0)}(r, \sigma(\upsilon))\varphi(\upsilon)\Delta\upsilon\\ &= \int_{r_0}^{r} h_{\alpha-1}(r, \sigma(\upsilon))\varphi(\upsilon)\Delta\upsilon.\end{aligned}$$

Definition 2.12 ([51]). At all $r, r_0 \in \mathbb{T}$, and $\alpha \geq 0$. The Riemann–Liouville fractional $\Delta-$derivative for $\varphi : \mathbb{T} \to \mathbb{R}$ is

$$D^\alpha_{\Delta,r_0}\varphi(r) = D^\eta_\Delta I^{\eta-\alpha}_{\Delta,r_0}\varphi(r), \quad \forall r \in \mathbb{T},$$

where $\eta = -[-\alpha]$.

Definition 2.13 ([51]). At all $r, r_0 \in \mathbb{T}$, and $\alpha \geq 0$. The Caputo fractional $\Delta-$derivative for $\varphi : \mathbb{T} \to \mathbb{R}$ is

$${}^{C}D^\alpha_{\Delta,r_0}\varphi(r) = D^\alpha_{\Delta,r_0}\left(\varphi(r) - \sum_{\ell=0}^{\eta-1} h_\ell(r, r_0)\varphi^{\Delta^\ell}(r_0)\right), \quad \forall r > 0,$$

where $\eta = [\alpha] + 1$.

Theorem 2.4 ([51]). *Let $\varphi(r) \in C^\eta_{rd}([0,\infty)_\mathbb{T}, \mathbb{R})$ for all $r \in \mathbb{T}$, $\eta \in \mathbb{N}$, $\eta - 1 < \alpha \leq \eta$ and $\alpha > 0$. Then*

$$\mathcal{L}_\Delta\left({}^{C}D^\alpha_{\Delta,r_0}u(r)\right)(s) = s^\alpha \mathcal{L}_\Delta(\varphi(r))(s) - \sum_{\varpi=0}^{\eta-1} s^{\alpha-\varpi-1}\varphi^{\Delta^\varpi}(r_0),$$

at all $s \in \mathbb{C}$ for which

$$\lim_{r\to\infty}\left(\varphi^{\Delta^\varpi}(r)\, e_{\ominus s}(r, 0)\right) = 0, \quad \varpi \in \{0, \dots, \eta-1\}.$$

Definition 2.14 ([51]). The $\Delta-$Mittag-Leffler function is described as

$${}_\Delta F_{\alpha,\beta}(\lambda, r, r_0) = \sum_{\ell=0}^{\infty} \lambda^\ell h_{\ell\alpha+\beta-1}(r, r_0), \tag{2.1}$$

where $\alpha, \beta > 0$, $\lambda \in \mathbb{R}$.

Theorem 2.5 ([51]). *Let $\alpha, \beta > 0$ and $\ell \in \mathbb{N}$, we have*

$$\mathcal{L}_\Delta\left\{{}_\Delta F_{\alpha,\beta}(\lambda, r, r_0)\right\}(s, r_0) = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \tag{2.2}$$

$$\mathcal{L}_\Delta\left\{\frac{\partial^\ell}{\partial\lambda^\ell}\, {}_\Delta F_{\alpha,\beta}(\lambda, r, r_0)\right\}(s, r_0) = \frac{\ell!\, s^{\alpha-\beta}}{(s^\alpha - \lambda)^{\ell+1}}, \tag{2.3}$$

where $|\lambda| < |s|^\alpha$.

Theorem 2.6 ([51]). *Let $\eta - 1 < \alpha \leq \eta\, (\eta \in \mathbb{N})$, and $\lambda \in \mathbb{R}$. Then the functions*

$$x_\kappa(r) = {}_\Delta F_{\alpha,\kappa+1}(\lambda, r, r_0)\ (\kappa = 0, \dots, \eta-1), \tag{2.4}$$

yield the fundamental system of solutions to

$${}^{C}D^\alpha_{\Delta,r_0}x(r) - \lambda x(r) = 0.$$

3. Main results

In this section, we will discuss general solutions to a class of linear nonhomogeneous FDEs with the Caputo $\Delta-$derivative. Then, we proceed to analyze Ulam–Hyers's stability.

3.1. Nonhomogeneous FDEs with the Caputo $\Delta-$derivative

In [51], the particular solutions of the following nonhomogeneous equation with the Caputo $\Delta-$derivative have been derived

$$\sum_{\kappa=1}^{\omega} A_\kappa\, {}^{C}D^{\alpha_\kappa}_{\Delta,r_0}x(r) + A_0 x(r) = f(r), \quad \forall r \in \mathbb{T}, \tag{3.1}$$

where $\omega \in \mathbb{N}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\omega$, $A_0, A_\kappa \in \mathbb{R}$.

By using the following Laplace fractional analog of the Green function:

$$G_{\alpha_1,\alpha_2,\ldots,\alpha_\omega}(r) = \mathcal{L}_\Delta^{-1}\left(\frac{1}{P_{\alpha_1,\alpha_2,\ldots,\alpha_\omega}(s)}\right)(r), \quad \forall\, r \in \mathbb{T}, \tag{3.2}$$

where $P_{\alpha_1,\alpha_2,\ldots,\alpha_\omega}(s) = A_0 + \sum_{\kappa=1}^{\omega} A_\kappa s^{\alpha_\kappa}$.

For a particular solution $x_p(r)$ of Eq. (3.1) with the initial conditions

$$x_p^{\Delta^\ell}(r_0) = 0, \ \forall \ell = 0, \ldots, \omega_\kappa - 1.$$

By using the Laplace transform of both side of Eq. (3.1), we get

$$\mathcal{L}_\Delta\left\{x_p(r)\right\}(s) = \frac{\mathcal{L}_\Delta\{f(r)\}(s)}{P_{\alpha_1,\ldots,\alpha_\omega}(s)}.$$

Then

$$x_p(r) = \left(G_{\alpha_1,\ldots,\alpha_\omega} * f\right)(r). \tag{3.3}$$

Here, we apply this method to find particular solutions to a class of linear nonhomogeneous FDEs on time scale.

It is important to state the following theorem to complete our result.

Theorem 3.1. *Let $\eta \in \mathbb{N}$, $\eta - 1 < \alpha \le \eta$, $0 < \beta < \alpha$, and $\lambda, \mu \in \mathbb{R}$. The equation*

$${}^{C}D^{\alpha}_{\Delta,r_0}x(r) - \lambda\, {}^{C}D^{\beta}_{\Delta,r_0}x(r) - \mu x(r) = 0, \tag{3.4}$$

with initial conditions

$$x^{\Delta^i}(r_0) = b_i, \ \forall\, i = 0, \ldots, m-1.$$

has its fundamental system of solutions given by

$$\begin{aligned} x_i &= \sum_{\ell=0}^{\infty} \frac{\mu^\ell}{\ell!}\frac{\partial^\ell}{\partial\lambda^\ell}\, {}_\Delta F_{\alpha-\beta,\beta\ell+i+1}(\lambda, r, r_0) \\ &\quad - \lambda \sum_{\ell=0}^{\infty} \frac{\mu^\ell}{\ell!}\frac{\partial^\ell}{\partial\lambda^\ell}\, {}_\Delta F_{\alpha-\beta,\beta\ell+i+1+\alpha-\beta}(\lambda, r, r_0), \end{aligned} \tag{3.5}$$

for $i = 0, \ldots, m-1$,

$$x_i = \sum_{\ell=0}^{\infty} \frac{\mu^\ell}{\ell!}\frac{\partial^\ell}{\partial\lambda^\ell}\, {}_\Delta F_{\alpha-\beta,\beta\ell+i+1}(\lambda, r, r_0), \tag{3.6}$$

for $i = m, \ldots, \eta - 1$, and $\left|\frac{\mu s^{-\beta}}{s^{\alpha-\beta}-\lambda}\right| < 1, \quad \forall s \in \mathbb{C}$. Provided that the series in Eqs. (3.5) and (3.6) are convergent.

Proof. Let $m - 1 < \beta \le m\,(m \le \eta; \eta \in \mathbb{N})$. Using time scale Laplace transform of Eq. (3.4), we have

$$\mathcal{L}_\Delta\{x(r)\}(s) = \sum_{i=0}^{m-1} b_i \frac{s^{\alpha-i-1}}{s^\alpha - \lambda s^\beta - \mu} - \lambda \sum_{i=0}^{m-1} b_i \frac{s^{\beta-i-1}}{s^\alpha - \lambda s^\beta - \mu}. \tag{3.7}$$

For $s \in \mathbb{C}$ and $\left|\frac{\mu s^{-\beta}}{s^{\alpha-\beta}-\lambda}\right| < 1$, we have

$$\begin{aligned} \frac{1}{s^\alpha - \lambda s^\beta - \mu} &= \frac{s^{-\beta}}{s^{\alpha-\beta}-\lambda}\frac{1}{1 - \frac{\mu s^{-\beta}}{s^{\alpha-\beta}-\lambda}} \\ &= \frac{s^{-\beta}}{s^{\alpha-\beta}-\lambda}\sum_{\ell=0}^{\infty}\frac{\mu^\ell s^{-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^\ell} \\ &= \sum_{\ell=0}^{\infty}\frac{\mu^\ell s^{-\beta-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}}. \end{aligned} \tag{3.8}$$

Form Eqs. (3.7) and (3.8), we obtain

$$\begin{aligned} \mathcal{L}_\Delta\{x(r)\}(s) &= \sum_{i=0}^{\eta-1} b_i s^{\alpha-i-1}\left(\sum_{\ell=0}^{\infty}\frac{\mu^\ell s^{-\beta-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}}\right) - \lambda\sum_{i=0}^{m-1} b_i s^{\beta-i-1}\left(\sum_{\ell=0}^{\infty}\frac{\mu^\ell s^{-\beta-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}}\right) \\ &= \sum_{i=0}^{\eta-1} b_i\left(\sum_{\ell=0}^{\infty}\frac{\mu^\ell s^{\alpha-i-1-\beta-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}}\right) - \lambda\sum_{i=0}^{m-1} b_i\left(\sum_{\ell=0}^{\infty}\frac{\mu^\ell s^{\beta-i-1-\beta-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}}\right). \end{aligned} \tag{3.9}$$

In addition, for $s \in \mathbb{C}$ and $|\lambda s^{\beta-\alpha}| < 1$, we get

$$\frac{s^{\alpha-i-1-\beta-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}} = \frac{s^{(\alpha-\beta)-(\beta\ell+i+1)}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}} = \frac{1}{\ell!}\mathcal{L}_{\Delta}\left\{\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\, {}_{\Delta}F_{\alpha-\beta,\beta\ell+i+1}(\lambda,r,r_0)\right\}(s), \tag{3.10}$$

and

$$\frac{s^{\beta-i-1-\beta-\ell\beta}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}} = \frac{s^{(\alpha-\beta)-(\beta\ell+i+1+\alpha-\beta)}}{\left(s^{\alpha-\beta}-\lambda\right)^{\ell+1}} = \frac{1}{\ell!}\mathcal{L}_{\Delta}\left\{\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\, {}_{\Delta}F_{\alpha-\beta,\beta\ell+i+1+\alpha-\beta}(\lambda,r,r_0)\right\}(s). \tag{3.11}$$

From Eqs. (3.9), (3.10), and (3.11), we have

$$x(r) = \sum_{i=0}^{\eta-1} b_i x_i(r), \tag{3.12}$$

where $x_i(r)\,(i = 0,\dots,\eta-1)$ are given by Eq. (3.5) for $i = 0,\dots,m-1$ and by Eq. (3.6) for $i = m,\dots,\eta-1$. For $\vartheta = 0,\dots,\eta-1$, we have

$$\begin{aligned} x_i^{\Delta^{\vartheta}} &= \left(\sum_{\ell=0}^{\infty}\frac{\mu^{\ell}}{\ell!}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\, {}_{\Delta}F_{\alpha-\beta,\beta\ell+i+1+\alpha-\beta}(\lambda,r,r_0) \right.\\ &\left. -\lambda\sum_{\ell=0}^{\infty}\frac{\mu^{\ell}}{\ell!}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\, {}_{\Delta}F_{\alpha-\beta,\beta\ell+i+1+\alpha-\beta}(\lambda,r,r_0)\right)^{\Delta^{\vartheta}} \\ &= \sum_{\ell=0}^{\infty}\frac{\mu^{\ell}}{\ell!}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\left(\sum_{\upsilon=0}^{\infty}\lambda^{\upsilon}h_{\upsilon(\alpha-\beta)+\beta\ell+i}(r,r_0)\right)^{\Delta^{\vartheta}} \\ &- \lambda\sum_{\ell=0}^{\infty}\frac{\mu^{\ell}}{\ell!}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\left(\sum_{\upsilon=0}^{\infty}\lambda^{\upsilon}h_{\upsilon(\alpha-\beta)+\beta\ell+i+\alpha-\beta}(r,r_0)\right)^{\Delta^{\vartheta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} x_i^{\Delta^{\vartheta}} &= \sum_{\ell=0}^{\infty}\frac{\mu^{\ell}}{\ell!}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\sum_{\upsilon=0}^{\infty}\lambda^{\upsilon}h_{\upsilon(\alpha-\beta)+\beta\ell+i-\vartheta}(r,r_0) \\ &- \lambda\sum_{\ell=0}^{\infty}\frac{\mu^{\ell}}{\ell!}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\sum_{\upsilon=0}^{\infty}\lambda^{\upsilon}h_{\upsilon(\alpha-\beta)+\beta\ell+i+\alpha-\beta-\vartheta}(r,r_0), \end{aligned}$$

for $i = 0,\dots,m-1$, and

$$x_i^{\Delta^{\vartheta}} = \sum_{\ell=0}^{\infty}\frac{\mu^{\ell}}{\ell!}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\sum_{\upsilon=0}^{\infty}\lambda^{\upsilon}h_{\upsilon(\alpha-\beta)+\beta\ell+i-\vartheta}(r,r_0),$$

for $i = n,\dots,m-1$.

For $i > \vartheta$, $x_i^{\Delta^{\vartheta}}(r_0) = 0$, and for $i = \vartheta$, $x_i^{\Delta^{\vartheta}}(r_0) = 1$.

Corollary 3.1. *Let $\eta \in \mathbb{N}$, $\eta-1 < \alpha \le \eta$, $0 < \beta < \alpha$, and $\lambda \in \mathbb{R}$. The following equation*

$${}^{C}D^{\alpha}_{\Delta,r_0}x(r) - \lambda\, {}^{C}D^{\beta}_{\Delta,r_0}x(r) = 0, \tag{3.13}$$

with initial conditions

$$x^{\Delta^{i}}(r_0) = b_i,\ \forall\, i = 0,\dots,\eta-1.$$

has its fundamental system of solutions given by

$$x_i = {}_{\Delta}F_{\alpha-\beta,i+1}(\lambda,r,r_0) - \lambda\, {}_{\Delta}F_{\alpha-\beta,i+1+\alpha-\beta}(\lambda,r,r_0), \tag{3.14}$$

for $i = 0,\dots,m-1$,

$$x_i = {}_{\Delta}F_{\alpha-\beta,i+1}(\lambda,r,r_0), \tag{3.15}$$

for $i = m,\dots,\eta-1$.

Theorem 3.2. *Let $f : \mathbb{T} \to \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\eta-1 < \alpha \le \eta\,(\eta \in \mathbb{N})$. The FDE*

$${}^{C}D^{\alpha}_{\Delta,r_0}x(r) - \lambda x(r) = f(r),\ \forall\, r \in \mathbb{T}, \tag{3.16}$$

is solvable, and has the following general solution

$$x(r) = \left(G_\alpha * f\right)(r) + \sum_{\kappa=0}^{m-1} b_\kappa x_\kappa(r), \tag{3.17}$$

where

$$G_\alpha(r) = {}_\Delta F_{\alpha,\alpha}(\lambda, r, r_0),$$

and $x_\kappa(r)$ is given by Eq. (2.4), and b_κ are arbitrary real constants.

Proof. Eq. (3.16) is Eq. (3.1) with $\omega = 1$, $\alpha_1 = \alpha$, $A_1 = 1$, $A_0 = -\lambda$ and using Eq. (3.2), we get

$$G_\alpha(r) = \mathcal{L}_\Delta^{-1}\left(\frac{1}{s^\alpha - \lambda}\right)(r).$$

By setting $\beta = \alpha$ in Eq. (2.2), we obtain

$$\mathcal{L}_\Delta\left\{{}_\Delta F_{\alpha,\alpha}(\lambda, r, r_0)\right\}(s, r_0) = \frac{1}{s^\alpha - \lambda}, \quad |\lambda| < |s|^\alpha.$$

Therefore

$$G_\alpha(r) = {}_\Delta F_{\alpha,\alpha}(\lambda, r, r_0).$$

As a result, Eq. (3.3), with $G_{\alpha_1,\alpha_2,\ldots,\alpha_\omega}(r) = G_\alpha(r)$, and Theorem 2.6 yield Eq. (3.17).

Theorem 3.3. *Let $f : \mathbb{T} \to \mathbb{R}$, $\eta - 1 < \alpha \le \eta\, (\eta \in \mathbb{N})$, $\lambda, \mu \in \mathbb{R}$, and $\alpha > \beta > 0$ with $\mu \ne 0$. The following equation:*

$${}^C D^\alpha_{\Delta,r_0} x(r) - \lambda {}^C D^\beta_{\Delta,r_0} x(r) - \mu x(r) = f(r), \tag{3.18}$$

has the general solution as follows:

$$x(r) = \left(G_{\alpha,\beta} * f\right)(r) + \sum_{\kappa=0}^{\eta-1} b_\kappa x_\kappa(r), \tag{3.19}$$

where

$$G_{\alpha,\beta}(r) = \sum_{\eta=0}^{\infty} \frac{\mu^\eta}{\eta!} \frac{\partial^\eta}{\partial\lambda^\eta}\, {}_\Delta F_{\alpha-\beta,\alpha+\beta\eta}(\lambda, r, r_0),$$

and $x_\kappa(r)$ are given by (3.5) and (3.6), and b_κ are arbitrary real constants.

Proof. Eq. (3.18) is Eq. (3.1) with $\omega = 2$, $\alpha_2 = \alpha$, $\alpha_1 = \beta$, $A_2 = 1$, $A_1 = -\lambda$, $A_0 = -\mu$ and using Eq. (3.2), yields

$$G_{\alpha,\beta}(r) = \mathcal{L}_\Delta^{-1}\left(\frac{1}{s^\alpha - \lambda s^\beta - \mu}\right)(r).$$

According to (3.8) for $s \in \mathbb{C}$ with $\left|\frac{\mu s^{-\beta}}{s^{\alpha-\beta} - \lambda}\right| < 1$, we have

$$G_{\alpha,\beta}(r) = \mathcal{L}_\Delta^{-1}\left(\sum_{m=0}^{\infty} \frac{\mu^\eta s^{-(\eta+1)\beta}}{(s^{\alpha-\beta} - \lambda)^{\eta+1}}\right)(r).$$

Now, use Theorem 2.5, with $|\lambda s^{\beta-\alpha}| < 1$, where α is changed by $\alpha - \beta$ and β by $\alpha + \beta n$, we obtain

$$\begin{aligned}\frac{s^{-(\eta+1)\beta}}{(s^{\alpha-\beta} - \lambda)^{\eta+1}} &= \frac{s^{(\alpha-\beta)-(\alpha+\beta\eta)}}{(s^{\alpha-\beta} - \lambda)^{\eta+1}} \\ &= \frac{1}{\eta!}\mathcal{L}_\Delta\left(\frac{\partial^\eta}{\partial\lambda^\eta}\, {}_\Delta F_{\alpha-\beta,\alpha+\beta\eta}(\lambda, r, r_0)\right)(s).\end{aligned}$$

Consequently, we can conclude

$$G_{\alpha,\beta}(r) = \sum_{\eta=0}^{\infty} \frac{\mu^\eta}{\eta!} \frac{\partial^\eta}{\partial\lambda^\eta}\, {}_\Delta F_{\alpha-\beta,\alpha+\beta\eta}(\lambda, r, r_0).$$

As a result, Eq. (3.3) with $G_{\alpha_1,\alpha_2,\ldots,\alpha_\omega}(r) = G_{\alpha,\beta}(r)$, and Theorem 3.1 result in Eq. (3.18).

Theorem 3.4. *Let $f : \mathbb{T} \to \mathbb{R}$, $\alpha > \beta > 0$, and $\lambda \in \mathbb{R}$. The FDE*

$${}^C D^\alpha_{\Delta,r_0} x(r) - \lambda {}^C D^\beta_{\Delta,r_0} x(r) = f(r), \tag{3.20}$$

has the general solution

$$x(r) = \left(G_{\alpha,\beta} * f\right)(r) + \sum_{\kappa=0}^{\eta-1} b_\kappa x_\kappa(r), \tag{3.21}$$

where

$$G_{\alpha,\beta}(r) = {}_{\Delta}F_{\alpha-\beta,\alpha}(\lambda, r, r_0),$$

and $x_\kappa(r)$ *are given by* (3.14) *and* (3.15), *and* b_κ *are arbitrary real constants.*

Proof. Eq. (3.20) is Eq. (3.1) with $\omega = 1$, $\alpha_1 = \alpha$, $A_0 = 0$, $A_1 = 1$, $A_2 = -\lambda$ and using Eq. (3.2), we get

$$G_{\alpha,\beta}(r) = \mathcal{L}_{\Delta}^{-1}\left(\frac{1}{s^{\alpha} - \lambda s^{\beta}}\right)(r).$$

Furthermore, we have

$$\frac{1}{s^{\alpha} - \lambda s^{\beta}} = \frac{s^{-\beta}}{s^{\alpha-\beta} - \lambda}.$$

Now, using Theorem 2.5 with changing α by $\alpha - \beta$ and β by α, we have

$$\mathcal{L}_{\Delta}\left\{{}_{\Delta}F_{\alpha-\beta,\alpha}(\lambda, r, r_0)\right\}(s, r_0) = \frac{s^{-\beta}}{s^{\alpha-\beta} - \lambda}.$$

By applying the Laplace inverse transform together with the above result, we get

$$G_{\alpha,\beta}(r) = {}_{\Delta}F_{\alpha-\beta,\alpha}(\lambda, r, r_0).$$

As a result, the result in (3.20) follows from (3.3) and Corollary 3.1.

3.2. Hyers-Ulam stability of FDEs on time scale

This section will demonstrate the Ulam stability of linear FDEs with Caputo Δ−derivative on time scale.

Definition 3.1. For $r \in \mathbb{T}$, the fractional dynamic equation

$$\Xi(f, x, {}^{C}D^{\alpha_1}_{\Delta,r_0}x, \ldots, {}^{C}D^{\alpha_\eta}_{\Delta,r_0}x) = 0, \tag{3.22}$$

has Hyers–Ulam stability if there exists a constant $\mathcal{K} > 0$ such that for a given $\varepsilon > 0$ and for each function $x : \mathbb{T} \to \mathbb{R}$ such that

$$\left|\Xi(f, x, {}^{C}D^{\alpha_1}_{\Delta,r_0}x, \ldots, {}^{C}D^{\alpha_\eta}_{\Delta,r_0}x)\right| \le \varepsilon,$$

then there exists a solution $x_a : \mathbb{T} \to \mathbb{R}$ of Eq. (3.22) such that

$$\left|x(r) - x_a(r)\right| \le \mathcal{K}\varepsilon.$$

If this statement is also true when we replace constants ε and $\mathcal{K}\varepsilon$ with the functions $\Theta(r)$ and $C(r)$, where these functions do not depend on x and $x_a(r)$ explicitly, then we say that the Eq. (3.22) has the generalized Hyers–Ulam stability.

Lemma 3.1. *Let a function* $\psi : \mathbb{T} \to \mathbb{R}$. *The convolution* ${}_{\Delta}F_{\alpha,\beta} * \psi$ *is*

$$({}_{\Delta}F_{\alpha,\beta} * \psi)(r) = \sum_{\kappa=0}^{\infty} \lambda^{\kappa} \int_{r_0}^{r} h_{\alpha\kappa+\beta-1}(r, \sigma(\upsilon))\psi(\upsilon)\Delta\upsilon.$$

Proof. Since

$${}_{\Delta}F_{\alpha,\beta}(\lambda, r, \upsilon) = \sum_{\kappa=0}^{\infty} \lambda^{\kappa} h_{\alpha\kappa+\beta-1}(r, \upsilon).$$

In addition, we have

$${}_{\Delta}\hat{F}_{\alpha,\beta}(\lambda, r, \sigma(\upsilon)) = \sum_{\kappa=0}^{\infty} \lambda^{\kappa} h_{\alpha\kappa+\beta-1}(r, \sigma(\upsilon)).$$

Consequently, we can conclude that

$$\begin{aligned}
({}_{\Delta}F_{\alpha,\beta} * \psi)(r) &= \int_{r_0}^{r} {}_{\Delta}\hat{F}_{\alpha,\beta}(\lambda, r, \sigma(\upsilon))\psi(\upsilon)\Delta\upsilon \\
&= \int_{r_0}^{r} \sum_{\kappa=0}^{\infty} \lambda^{\kappa} h_{\alpha\kappa+\beta-1}(r, \sigma(\upsilon))\psi(\upsilon)\Delta\upsilon \\
&= \sum_{\kappa=0}^{\infty} \lambda^{\kappa} \int_{r_0}^{r} h_{\alpha\kappa+\beta-1}(r, \sigma(\upsilon))\psi(\upsilon)\Delta\upsilon.
\end{aligned}$$

The following theorem directly leads to the determination of the Hyers–Ulam stability of Eq. (3.16).

Theorem 3.5. *On the time scale $\mathbb{T}$, let $f : \mathbb{T} \to \mathbb{R}$. If a function $x : \mathbb{T} \to \mathbb{R}$ fulfills*

$$\left|{}^{C}D^{\alpha}_{\Delta,r_0}x(r) - \lambda x(r) - f(r)\right| \le \varepsilon, \ \ \forall r \in \mathbb{T}, \tag{3.23}$$

where $\eta - 1 < \alpha \le \eta$, $\eta \in \mathbb{N}$, $\eta = -\overline{[-\alpha]}$ and $\lambda \in \mathbb{R}$, and for some $\varepsilon > 0$, then there exists a solution $x_a(r) : \mathbb{T} \to \mathbb{R}$ of Eq. (3.16) such that

$$\left|x(r) - x_a(r)\right| \le \varepsilon \sum_{\vartheta=0}^{\infty} |\lambda|^{\vartheta} \int_{r_0}^{r} \left|h_{(\vartheta+1)\alpha-1}(r,\sigma(v))\right| \Delta v, \ \ \ \forall\, r \in \mathbb{T}.$$

Proof. Define

$$\psi(r) = {}^{C}D^{\alpha}_{\Delta,r_0}x(r) - \lambda x(r) - f(r), \ \ \forall\, r \in \mathbb{T}. \tag{3.24}$$

Using Laplace transform to $\psi(r)$, we get

$$\begin{aligned}\mathcal{L}_{\Delta}\left\{\psi(r)\right\}(s) &= s^{\alpha}\mathcal{L}_{\Delta}\left\{x(r)\right\}(s) - \sum_{\vartheta=0}^{\eta-1} b_{\vartheta}s^{\alpha-\vartheta-1} \\ &\quad - \lambda\mathcal{L}_{\Delta}\left\{x(r)\right\}(s) - \mathcal{L}_{\Delta}\left\{f(r)\right\}(s).\end{aligned} \tag{3.25}$$

where $b_{\vartheta} = x^{\Delta^{\vartheta}}(r_0)$. Then,

$$\mathcal{L}_{\Delta}\left\{x(r)\right\}(s) = \sum_{\vartheta=0}^{\eta-1} b_{\vartheta}\frac{s^{\alpha-\vartheta-1}}{s^{\alpha}-\lambda} + \frac{\mathcal{L}_{\Delta}\left\{f(r)\right\}(s)}{s^{\alpha}-\lambda} + \frac{\mathcal{L}_{\Delta}\left\{\psi(r)\right\}(s)}{s^{\alpha}-\lambda}. \tag{3.26}$$

Define

$$x_a(r) = \sum_{\vartheta=0}^{\eta-1} b_{\vartheta}x_{\vartheta}(r) + (G_{\alpha} * f)(r),$$

where

$$\begin{aligned} x_{\vartheta}(r) &= {}_{\Delta}F_{\alpha,\vartheta+1}(\lambda, r, r_0), \\ G_{\alpha}(r) &= {}_{\Delta}F_{\alpha,\alpha}(\lambda, r, r_0), \\ (G_{\alpha} * f)(r) &= \sum_{\vartheta=0}^{\infty} \lambda^{\vartheta} \int_{r_0}^{r} h_{(\vartheta+1)\alpha-1}(r,\sigma(v))f(v)\Delta v. \end{aligned}$$

By implementing Theorem 3.3 with changing β by $\kappa + 1$, one can have

$$\mathcal{L}_{\Delta}\left\{x_{\kappa}(r)\right\}(s) = \mathcal{L}_{\Delta}\left\{{}_{\Delta}F_{\alpha,\vartheta+1}(\lambda, r, r_0)\right\}(s) = \frac{s^{\alpha-\vartheta-1}}{s^{\alpha}-\lambda}, \quad |\lambda| < |s|^{\alpha}. \tag{3.27}$$

Again, we replace β by α, and we get

$$\mathcal{L}_{\Delta}\left\{G_{\alpha}(r)\right\}(s) = \mathcal{L}_{\Delta}\left\{{}_{\Delta}F_{\alpha,\alpha}(\lambda, r, r_0)\right\}(s) = \frac{1}{s^{\alpha}-\lambda}, \quad |\lambda| < |s|^{\alpha}. \tag{3.28}$$

From Eqs. (3.27) and (3.28), we can conclude that

$$\begin{aligned}\mathcal{L}_{\Delta}\left\{x_a(r)\right\}(s) &= \sum_{\vartheta=0}^{\eta-1} b_{\vartheta}\mathcal{L}_{\Delta}\left\{x_{\vartheta}(r)\right\}(s) + \mathcal{L}_{\Delta}\left\{(G_{\alpha} * f)(r)\right\}(s) \\ &= \sum_{\vartheta=0}^{\eta-1} b_{\vartheta}\frac{s^{\alpha-\vartheta-1}}{s^{\alpha}-\lambda} + \frac{\mathcal{L}_{\Delta}\left\{f(r)\right\}(s)}{s^{\alpha}-\lambda}.\end{aligned} \tag{3.29}$$

Using Theorem 2.4, Eq. (3.29) and a simple computation, one can get

$$\begin{aligned}\mathcal{L}_{\Delta}\left\{{}^{C}D^{\alpha}_{\Delta,r_0}x_a(r) - \lambda x_a(r)\right\}(s) &= s^{\alpha}\mathcal{L}_{\Delta}\left\{x_a(r)\right\}(s) - \sum_{\vartheta=0}^{m-1} s^{\alpha-\vartheta-1}b_{\vartheta} - \lambda\mathcal{L}_{\Delta}\left\{x_a(r)\right\}(s) \\ &= \mathcal{L}_{\Delta}\left\{f(r)\right\}(s).\end{aligned} \tag{3.30}$$

So $x_a(r)$ is the solution of Eq. (3.16). Furthermore, it results from Eqs. (3.26) and (3.29) that

$$\mathcal{L}_{\Delta}\left\{x(r)\right\}(s) - \mathcal{L}_{\Delta}\left\{x_a(r)\right\}(s) = \frac{\mathcal{L}_{\Delta}\left\{\psi(r)\right\}(s)}{s^{\alpha}-\lambda} = \mathcal{L}_{\Delta}\left\{(G_{\alpha} * \psi)(r)\right\}(s). \tag{3.31}$$

Using the inverse Laplace transform of Eq. (3.31), we get

$$x(r) - x_a(r) = (G_{\alpha} * \psi)(r), \quad \forall\, r \in \mathbb{T}.$$

From the inequality (3.23), we know that $|\psi(r)| \le \varepsilon$, we can acquire

$$\left|x(r) - x_a(r)\right| = \left|(G_{\alpha} * \psi)(r)\right|$$

$$
\begin{aligned}
&= \left| \sum_{\vartheta=0}^{\infty} \lambda^{\vartheta} \int_{r_0}^{r} h_{(\vartheta+1)\alpha-1}(r,\sigma(v))\psi(v)\Delta v \right| \\
&\le \sum_{\vartheta=0}^{\infty} \left|\lambda^{\vartheta}\right| \int_{r_0}^{r} \left|h_{(\vartheta+1)\alpha-1}(r,\sigma(v))\psi(v)\right| \Delta v \\
&\le \varepsilon \sum_{\vartheta=0}^{\infty} \left|\lambda^{\vartheta}\right| \int_{r_0}^{r} \left|h_{(\vartheta+1)\alpha-1}(r,\sigma(v))\right| \Delta v, \quad \forall\, r \in \mathbb{T}.
\end{aligned}
$$

Similarly, we can prove that the Eq. (3.16) is generalized Hyers–Ulam stable.

Corollary 3.2. *Let $\eta - 1 < \alpha \le \eta$, $\eta \in \mathbb{N}$, $\eta = -\overline{[-\alpha]}$ and $\lambda \in \mathbb{R}$. If the function $x : \mathbb{T} \to \mathbb{R}$ fulfills*

$$
\left|{}^{C}D^{\alpha}_{\Delta,r_0} x(r) - \lambda x(r) - f(r)\right| \le \Theta(r), \quad \forall\, r \in \mathbb{T},
$$

then there is a solution $x_a : \mathbb{T} \to \mathbb{R}$ of Eq. (3.16) such that

$$
\left|x(r) - x_a(r)\right| \le C(r),
$$

where

$$
C(r) = \sum_{\vartheta=0}^{\infty} |\lambda|^{\vartheta} \int_{r_0}^{r} \left|h_{(\vartheta+1)\alpha-1}(r,\sigma(v))\right| \Theta(v)\Delta v.
$$

Theorem 3.6. *On the time scale $\mathbb{T}$, let $\lambda, \mu \in \mathbb{R}$ with $\mu \ne 0$, $\eta - 1 < \alpha \le \eta$, $\alpha > \beta > 0$, and $\eta = -\overline{[-\alpha]}$. Let $f(r)$ be a function defined on time scale. If a function $x : \mathbb{T} \to \mathbb{R}$ fulfills*

$$
\left|{}^{C}D^{\alpha}_{\Delta,r_0} x(r) - \lambda\, {}^{C}D^{\beta}_{\Delta,r_0} x(r) - \mu x(r) - f(r)\right| \le \varepsilon, \tag{3.32}
$$

at all $r \in \mathbb{T}$ and some $\varepsilon > 0$, then there exists a solution $x_a : \mathbb{T} \to \mathbb{R}$ of Eq. (3.4) such that

$$
\left|x(r) - x_a(r)\right| \le \varepsilon \int_{r_0}^{r} \left|\widehat{G_{\alpha,\beta}}(r,\sigma(v))\right| \Delta v, \quad \forall\, r \in \mathbb{T}.
$$

Proof. Define

$$
\psi(r) = {}^{C}D^{\alpha}_{\Delta,r_0} x(r) - \lambda\, {}^{C}D^{\beta}_{\Delta,r_0} x(r) - \mu x(r) - f(r), \quad \forall\, r \in \mathbb{T}.
$$

Let $m - 1 < \beta \le m$ and $m \in \mathbb{N}$. We can clearly see $m \le \eta$ as a result of $0 < \beta < \alpha$. Using Lemma 3.1 and Laplace transform, one can have

$$
\begin{aligned}
\mathcal{L}_{\Delta}\{\psi(r)\}(s) &= s^{\alpha}\mathcal{L}_{\Delta}\{x(r)\}(s) - \sum_{\vartheta=0}^{\eta-1} b_{\vartheta} s^{\alpha-\vartheta-1} - \lambda s^{\beta}\mathcal{L}_{\Delta}\{x(r)\}(s) \\
&\quad + \lambda \sum_{\vartheta=0}^{m-1} b_{\vartheta} s^{\beta-\vartheta-1} - \mu\mathcal{L}_{\Delta}\{x(r)\}(s) - \mathcal{L}_{\Delta}\{f(r)\}(s).
\end{aligned} \tag{3.33}
$$

By (3.33), it follows that

$$
\begin{aligned}
\mathcal{L}_{\Delta}\{x(r)\}(s) &= \frac{\sum_{\vartheta=0}^{\eta-1} b_{\vartheta} s^{\alpha-\vartheta-1} - \lambda \sum_{\vartheta=0}^{m-1} b_{\vartheta} s^{\beta-\vartheta-1} + \mathcal{L}_{\Delta}\{f(r)\}(s)}{s^{\alpha} - \lambda s^{\beta} - \mu} \\
&\quad + \frac{\mathcal{L}_{\Delta}\{\psi(r)\}(s)}{s^{\alpha} - \lambda s^{\beta} - \mu}.
\end{aligned} \tag{3.34}
$$

Now, we set

$$
x_a(r) = \sum_{\vartheta=0}^{\eta-1} b_{\vartheta} x_{\vartheta}(r) + (G_{\alpha,\beta} * f)(r), \tag{3.35}
$$

where

$$
\begin{aligned}
x_{\vartheta}(r) &= \sum_{\ell=0}^{\infty} \frac{\mu^{\ell}}{\ell!} \frac{\partial^{\ell}}{\partial\lambda^{\ell}}\, {}_{\Delta}F_{\alpha-\beta,\beta\ell+\vartheta+1}(\lambda,r,r_0) \\
&\quad - \lambda \sum_{\ell=0}^{\infty} \frac{\mu^{\ell}}{\ell!} \frac{\partial^{\ell}}{\partial\lambda^{\ell}}\, {}_{\Delta}F_{\alpha-\beta,\beta\ell+\vartheta+1+\alpha-\beta}(\lambda,r,r_0), \quad \forall\, \vartheta = 0,\dots,m-1,
\end{aligned} \tag{3.36}
$$

and

$$
x_{\vartheta}(r) = \sum_{\ell=0}^{\infty} \frac{\mu^{\ell}}{\ell!} \frac{\partial^{\ell}}{\partial\lambda^{\ell}}\, {}_{\Delta}F_{\alpha-\beta,\beta\ell+\vartheta+1}(\lambda,r,r_0), \quad \forall\, \vartheta = m,\dots,\eta-1. \tag{3.37}
$$

By Theorem 3.3, we get

$$\mathcal{L}_\Delta\left\{G_{\alpha,\beta}(r)\right\}(s)=\frac{1}{s^\alpha-\lambda s^\beta-\mu}.$$

Then we can obtain

$$\mathcal{L}_\Delta\left\{(G_{\alpha,\beta}*f)(r)\right\}(s)=\frac{\mathcal{L}_\Delta\{f(r)\}(s)}{s^\alpha-\lambda s^\beta-\mu}. \tag{3.38}$$

By Theorem 2.3, Theorem 2.5 and Eq. (3.38), we have

$$\begin{aligned}\mathcal{L}_\Delta\left\{x_a(r)\right\}(s)&=\mathcal{L}_\Delta\left\{\sum_{\vartheta=0}^{m-1}b_\vartheta x_\vartheta(r)\right\}(s)+\mathcal{L}_\Delta\left\{\sum_{\vartheta=m}^{\eta-1}b_\vartheta x_\vartheta(r)\right\}(s)\\&\quad+\mathcal{L}_\Delta\left\{(G_{\alpha,\beta}*f)(r)\right\}(s)\\&=\frac{\sum_{\vartheta=0}^{\eta-1}b_\vartheta s^{\alpha-\vartheta-1}-\lambda\sum_{\vartheta=0}^{m-1}b_\vartheta s^{\beta-\vartheta-1}+\mathcal{L}_\Delta\{f(r)\}(s)}{s^\alpha-\lambda s^\beta-\mu}.\end{aligned} \tag{3.39}$$

From Eq. (3.39), we have

$$\mathcal{L}_\Delta\left\{{}^CD^\alpha_{\Delta,r_0}x_a(r)\right\}(s)-\lambda\mathcal{L}_\Delta\left\{{}^CD^\beta_{\Delta,r_0}x_a(r)\right\}(s)-\mu\mathcal{L}_\Delta\left\{x_a(r)\right\}(s)=\mathcal{L}_\Delta\{f(r)\}(s). \tag{3.40}$$

Using Eqs. (3.34) and (3.39), we obtain

$$\begin{aligned}\mathcal{L}_\Delta\{x(r)\}(s)-\mathcal{L}_\Delta\left\{x_a(r)\right\}(s)&=\frac{\mathcal{L}_\Delta\{\psi(r)\}(s)}{s^\alpha-\lambda s^\beta-\mu}\\&=\mathcal{L}_\Delta\left\{(G_{\alpha,\beta}*\psi)(r)\right\}(s).\end{aligned} \tag{3.41}$$

Using the inverse Laplace transform to both sides of Eq. (3.41), we get

$$x(r)-x_a(r)=(G_{\alpha,\beta}*\psi)(r),\quad \forall r\in\mathbb{T}.$$

By Eq. (3.32), we have

$$|\psi(r)|\le\varepsilon,\quad \forall r\in\mathbb{T}.$$

Then, we can obtain

$$\begin{aligned}\left|x(r)-x_a(r)\right|&=\left|(G_{\alpha,\beta}*\psi)(r)\right|\\&=\left|\int_{r_0}^r\widehat{G_{\alpha,\beta}}(r,\sigma(\upsilon))\psi(\upsilon)\Delta\upsilon\right|\\&\le\int_{r_0}^r\left|\widehat{G_{\alpha,\beta}}(r,\sigma(\upsilon))\psi(\upsilon)\right|\Delta\upsilon\\&\le\varepsilon\int_{r_0}^r\left|\widehat{G_{\alpha,\beta}}(r,\sigma(\upsilon))\right|\Delta\upsilon.\end{aligned}$$

Corollary 3.3. *Let $f:\mathbb{T}\to\mathbb{R}$, and the integral $\int_{r_0}^r\left|\widehat{G_{\alpha,\beta}}(r,\sigma(\upsilon))\right|\Theta(\upsilon)\Delta\upsilon$ exists at all $r\in\mathbb{T}$. If a function $x:\mathbb{T}\to\mathbb{R}$ satisfies the following inequality*

$$\left|{}^CD^\alpha_{\Delta,r_0}x(r)-\lambda{}^CD^\beta_{\Delta,r_0}x(r)-\mu x(r)-f(r)\right|\le\Theta(r),\quad \forall r\in\mathbb{T}$$

where $\eta-1<\alpha\le\eta$, $\eta\in\mathbb{N}$, $\alpha>\beta>0$, $\eta=-\overline{[-\alpha]}$, $\lambda,\mu\in\mathbb{R}$ with $\mu\neq 0$. Then there is a solution $x:\mathbb{T}\to\mathbb{R}$ of Eq. (3.4) such that

$$\left|x(r)-x_a(r)\right|\le C(r),$$

where

$$C(r)=\int_{r_0}^r\left|\widehat{G_{\alpha,\beta}}(r,\sigma(\upsilon))\right|\Theta(\upsilon)\Delta\upsilon.$$

In order to complete Theorem 3.6, we also take into account the Ulam stability of Eq. (3.18) with the coefficient $\mu=0$.

Theorem 3.7. *On the time scale, Let $f:\mathbb{T}\to\mathbb{R}$ be a function, $\eta-1<\alpha\le\eta$, $\eta\in\mathbb{N}$, $\alpha>\beta>0$, $\eta=-\overline{[-\alpha]}$, and $\lambda\in\mathbb{R}$. If a function $x:\mathbb{T}\to\mathbb{R}$ satisfies*

$$\left|{}^CD^\alpha_{\Delta,r_0}x(r)-\lambda{}^CD^\beta_{\Delta,r_0}x(r)-f(r)\right|\le\varepsilon \tag{3.42}$$

at all $r\in\mathbb{T}$, and some $\varepsilon>0$, then there exists a solution $x_a:\mathbb{T}\to\mathbb{R}$ of Eq. (3.20) such that

$$\left|x(r)-x_a(r)\right|\le\varepsilon\sum_{\vartheta=0}^\infty|\lambda|^\vartheta\int_{r_0}^r\left|h_{(\vartheta+1)\alpha-\vartheta\beta-1}(r,\sigma(\upsilon))\right|\Delta\upsilon,\quad\forall r\in\mathbb{T}.$$

Proof. Define

$$\psi(r) = {}^C D^{\alpha}_{\Delta,r_0} x(r) - \lambda {}^C D^{\beta}_{\Delta,r_0} x(r) - f(r), \forall r \in \mathbb{T}.$$

Using Lemma 3.1 with Laplace transform, on can have

$$\mathcal{L}_\Delta\{\psi(r)\}(s) = s^\alpha \mathcal{L}_\Delta\{x(r)\}(s) - \sum_{\vartheta=0}^{m-1} b_\vartheta s^{\alpha-\vartheta-1} - \lambda s^\beta \mathcal{L}_\Delta\{x(r)\}(s) + \lambda \sum_{\vartheta=0}^{m-1} b_\vartheta s^{\beta-\vartheta-1} - \mathcal{L}_\Delta\{f(r)\}(s). \tag{3.43}$$

By Eq. (3.43), it follows that

$$\mathcal{L}_\Delta\{x(r)\}(s) = \frac{\sum_{\vartheta=0}^{\eta-1} b_\vartheta s^{\alpha-\vartheta-1} - \lambda \sum_{\vartheta=0}^{m-1} b_\vartheta s^{\beta-\vartheta-1} + \mathcal{L}_\Delta\{f(r)\}(s)}{s^\alpha - \lambda s^\beta} + \frac{\mathcal{L}_\Delta\{\psi(r)\}(s)}{s^\alpha - \lambda s^\beta}. \tag{3.44}$$

Now, we set

$$x_a(r) = \sum_{\vartheta=0}^{\eta-1} b_\vartheta x_\vartheta(r) + (G_{\alpha,\beta} * f)(r), \tag{3.45}$$

where

$$x_\vartheta = {}_\Delta F_{\alpha-\beta,\vartheta+1}(\lambda, r, r_0) - \lambda\, {}_\Delta F_{\alpha-\beta,\vartheta+1+\alpha-\beta}(\lambda, r, r_0), \forall \vartheta = 0, \ldots, m-1, \tag{3.46}$$

and

$$x_\vartheta = {}_\Delta F_{\alpha-\beta,\vartheta+1}(\lambda, r, r_0), \forall \vartheta = m, \ldots, \eta - 1. \tag{3.47}$$

Such that

$$G_{\alpha,\beta}(r) = {}_\Delta F_{\alpha-\beta,\alpha}(\lambda, r, r_0), \tag{3.48}$$

and

$$(G_{\alpha,\beta} * f)(r) = \sum_{\vartheta=0}^{\infty} \lambda^\vartheta \int_{r_0}^{r} h_{(\vartheta+1)\alpha-\vartheta\beta-1}(r, \sigma(v)) f(v) \Delta v.$$

By Theorem 2.3 and Theorem 2.5, we have

$$\begin{aligned}\mathcal{L}_\Delta\left\{x_a(r)\right\}(s) &= \mathcal{L}_\Delta\left\{\sum_{\vartheta=0}^{m-1} b_\vartheta x_\vartheta(r)\right\}(s) + \mathcal{L}_\Delta\left\{\sum_{\vartheta=m}^{\eta-1} b_\vartheta x_\vartheta(r)\right\}(s) + \mathcal{L}_\Delta\left\{(G_{\alpha,\beta} * f)(r)\right\}(s) \\ &= \sum_{\vartheta=0}^{m-1} b_\kappa \mathcal{L}_\Delta\left\{{}_\Delta F_{\alpha-\beta,\vartheta+1}(\lambda, r, r_0) - \lambda_\Delta F_{\alpha-\beta,\vartheta+1+\alpha-\beta}(\lambda, r, r_0)\right\}(s) \\ &+ \sum_{\vartheta=m}^{\eta-1} b_\vartheta \mathcal{L}_\Delta\left\{{}_\Delta F_{\alpha-\beta,\vartheta+1}(\lambda, r, r_0)\right\}(s) + \frac{\mathcal{L}_\Delta\{f(r)\}(s)}{s^\alpha - \lambda s^\beta} \\ &= \frac{\sum_{\vartheta=0}^{\eta-1} b_\vartheta s^{\alpha-\vartheta-1} - \lambda \sum_{\vartheta=0}^{m-1} b_\vartheta s^{\beta-\vartheta-1} + \mathcal{L}_\Delta\{f(r)\}(s)}{s^\alpha - \lambda s^\beta}.\end{aligned} \tag{3.49}$$

From Eq. (3.49), one can get

$$\mathcal{L}_\Delta\left\{{}^C D^{\alpha}_{\Delta,r_0} x_a(r) - \lambda {}^C D^{\beta}_{\Delta,r_0} x_a(r)\right\}(s) = \mathcal{L}_\Delta\{f(r)\}(s), \tag{3.50}$$

so $x_a(r)$ is a solution of Eq. (3.20). Using Eqs. (3.44) and (3.49), we obtain

$$\mathcal{L}_\Delta\{x(r)\}(s) - \mathcal{L}_\Delta\left\{x_a(r)\right\}(s) = \frac{\mathcal{L}_\Delta\{\psi(r)\}(s)}{s^\alpha - \lambda s^\beta} = \mathcal{L}_\Delta\left\{(G_{\alpha,\beta} * \psi)(r)\right\}(s). \tag{3.51}$$

Using the inverse time scale Laplace transform to both sides of Eq. (3.51), we get

$$x(r) - x_a(r) = (G_{\alpha,\beta} * \psi)(r), \quad \forall r \in \mathbb{T}.$$

Similar to the above theorems' proof, we obtain

$$\begin{aligned}\left|x(r) - x_a(r)\right| &= \left|(G_{\alpha,\beta} * \psi)(r)\right| \\ &\leq \varepsilon \sum_{\vartheta=0}^{\infty} |\lambda|^\vartheta \int_{r_0}^{r} \left|h_{(\vartheta+1)\alpha-\vartheta\beta-1}(r, \sigma(v))\right| \Delta v.\end{aligned}$$

Corollary 3.4. *On the time scale, let* $f : \mathbb{T} \to \mathbb{R}$ *be a function,* $\eta - 1 < \alpha \leq \eta$, $\eta \in \mathbb{N}$, $\alpha > \beta > 0$, $\eta = -\overline{[-\alpha]}$ *and* $\lambda \in \mathbb{R}$. *If a function* $x : \mathbb{T} \to \mathbb{R}$ *satisfies the following inequality for a given* $\varepsilon > 0$

$$\left|{}^{C}D^{\alpha}_{\Delta,r_0} x(r) - \lambda {}^{C}D^{\beta}_{\Delta,r_0} x(r) - f(r)\right| \leq \Theta(r), \quad \forall\, r \in \mathbb{T},$$

then there is a solution $x_a : \mathbb{T} \to \mathbb{R}$ *of Eq.* (3.20) *such that*

$$\left|x(r) - x_a(r)\right| \leq C(r), \; \forall\, r \in \mathbb{T}$$

where

$$C(r) = \sum_{\vartheta=0}^{\infty} |\lambda|^{\vartheta} \int_{r_0}^{r} \left|h_{(\vartheta+1)\alpha-\vartheta\beta-1}(r, \sigma(v))\right| \Theta(v)\Delta v.$$

4. Conclusions

The HUS study of a class of linear FDEs with Caputo Δ–derivative on time scale is our target in this paper. For this purpose, the Laplace transform in its time scale version has been used. If the exact solution does not exist or is difficult to find, the approximate solutions for these types of equations are sufficient to study HUS. In fact, this is the main advantage of our main results in studying HUS, which is very important in various fields, including optimization, numerical analysis, economics, and biology.

Declaration of competing interest

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

Data availability

No data was used for the research described in the article.

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