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A numerical method for solving quadratic fractional optimal control problems

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ABSTRACT

The objective of this article is to present a novel algorithm that can efficiently address fractional quadratic optimal control problems (FQOCPs) through the application of the generalized differential transform method, in conjunction with a Vandermonde matrix. The algorithm's performance, in terms of solution accuracy, reliability, and efficiency, is exemplified by a range of illustrative examples. This paper introduces an innovative methodology for the numerical resolution of FQOCPs, demonstrating its inherent capabilities and efficacy.

1. Introduction

The theory of optimal control is a mathematical branch that focuses on minimizing or maximizing a given cost function in specific dynamical systems. It has gained considerable attention from scientists due to its effectiveness in designing and analyzing real-life models, such as spacecraft [1], engineering [2], physical devices [3–5], biological systems [6–9], economics [10], and others. On the other hand, fractional calculus is employed to model real-world systems, offering enhanced accuracy, efficiency, and precision in capturing their dynamic behavior. This motivated O. P. Agarwal to apply classical control theory within a fractional framework, leading to the development of optimal fractional control theory [11,12]. Consequently, numerous researchers have focused on finding optimal solutions for dynamical systems described by fractional derivatives [13–32].

Fractional derivatives can be defined in various ways, including Caputo, Riemann–Liouville, Grünwald–Letnikov, and others. Consequently, many studies have been conducted on fractional optimal control systems described by Caputo or Riemann–Liouville fractional derivatives. As the demand for the application of fractional optimal control problems (FOCPs) grows, the need for numerical methods to solve the resulting equations has emerged as a rapidly expanding area of research. Two main approaches, direct and indirect methods, are commonly employed in numerical schemes for this purpose. Indirect methods involve solving the Pontryagin's system using suitable numerical techniques, which is complex due to the involvement of both left and right fractional derivatives in the Pontryagin's equations [11,12,33–39]. In contrast, direct methods approximate the FOCP without considering the necessary optimality conditions [40–47]. Inspired by Taylor series expansion, Zhou [48] introduced the differential transform method (DTM), which is a powerful semi-numerical technique. DTM is an appealing option for researchers to solve both linear and nonlinear problems due to its lack of necessity for linearization or domain discretization. In fact, DTM distinguishes itself from the Taylor series method by offering a simplified approach while yielding equivalent outcomes. Notably, DTM exhibits reduced computational time for higher orders, making it advantageous for handling complex calculations. In recent times, there has been an increasing scholarly focus on addressing fractional differential equations (FDEs), resulting in the emergence of dedicated methodologies tailored to this objective. Arikoglu and Ozkol [49], Odibat et al. [50], and Khdair et al. [51] introduced the fractional differential transform method (FDTM), generalized differential transform method (GDTM), and restricted differential transform

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method (RDTM), respectively. These modifications of the differential transform method (DTM) were developed to address the solution of fractional differential equations (FDEs). The FDTM, GDTM, and RDTM methods are not good for solving these problems because they involve two different Caputo fractional derivatives in the fractional two boundary value problem that has to do with FOCPs. Because of this, the study's goal is to get around this problem by using the Vandermonde matrix and the Generalized Dynamic Time Warping Method (GDTM) on the time boundaries.

2. Caputo fractional derivative(CFD)

Let $x(t) : [a, b] \to R$ be a function and $\alpha > 0, \quad \alpha \in R$ such that $m = [\alpha] + 1$ where $[\cdot]$ is the greatest integer function. The left and right α Riemann–Liouville fractional integral (RLFI) are defined as follows, respectively:

$$ {}_aI_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)ds, \quad t > a \tag{1} $$

$$ {}_tI_b{}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} x(s)ds, \qquad t < b \tag{2} $$

for $\alpha, \beta > 0, \quad 0 < \alpha \leq 1,$ and $0 \leq a < b$, we have

$$ I_a^\alpha I_a^\beta x(t) = I_a^\beta I_a^\alpha x(t) = I_a^{\alpha+\beta} x(t) \tag{3} $$

$$ I_b^\alpha I_b^\beta x(t) = I_b^\beta I_b^\alpha x(t) = I_b^{\alpha+\beta} x(t) \tag{4} $$

Based on the left α RLFI in Eq. (1), the left α CFD of $x(t)$, when it exists, is defined as

$$ {}_a^CD_t{}^\alpha x(t) = {}_aI_t{}^{m-\alpha} D^m x(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} x^{(m)}(s)ds \tag{5} $$

Also, by using the right α RLFI in Eq. (2) the right α CFD can be define as follows

$$ {}_t^CD_b^\alpha x(t) = (-1)^m \, {}_tI_b{}^{m-\alpha} D^m x(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_t^b (s-t)^{m-\alpha-1} x^{(m)}(s)ds \tag{6} $$

where $x^{(m)}(t)$ and $D^m x(t)$ are the usual mth derivative of $x(t)$.

Now, it is easy to verify the following relationships:

$$ {}_a^CD_t^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}, \quad t > a, \;\; \gamma > m \tag{7} $$

$$ {}_a^CD_t^\alpha (t-a)^k = 0, \quad t > a, \;\; k = 0, 1, \dots, m-1 \tag{8} $$

$$ {}_t^CD_b^\alpha (b-t)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (b-t)^{\gamma-\alpha}, \quad t < b, \;\; \gamma > m \tag{9} $$

$$ {}_t^CD_b^\alpha (b-t)^k = 0, \qquad b > t, \;\; k = 0, 1, \dots, m-1 \tag{10} $$

$$ {}_aI_t^\alpha \, {}_a^CD_t^\alpha x(t) = x(t) - \sum_{s=0}^{m-1} \frac{x^{(s)}(a)}{s!} (t-a)^s \tag{11} $$

$$ {}_tI_b^\alpha \, {}_t^CD_b^\alpha x(t) = x(t) - \sum_{s=0}^{m-1} \frac{(-1)^m x^{(k)}(b)}{s!} (b-t)^s \tag{12} $$

For $0 < \alpha < 1$, Eqs. (11) and (12) become

$$ {}_tI_b^\alpha [{}_a^CD_t^\alpha x(t)] = x(t) - x(a) \tag{13} $$

$$ {}_tI_b^\alpha [{}_t^CD_b^\alpha x(t)] = x(t) - x(b) \tag{14} $$

3. Problem formulation

Numerous real-world applications involve solving a system of differential equations, commonly known as state space equations, to optimize a certain performance functional, often referred to as a cost functional. These optimization problems are known as optimal control problems (OCPs). The nature of OCPs can vary significantly depending on the types of state space equations, cost functions, and admissible sets of control variables involved. In recent years, a particular focus has been given to the study of fractional dynamic systems.

$$ {}_a^CD_t^\alpha x(t) = G_1(t, x(t)) + G_2(t, x(t))u(t), \quad x(a) = x_a \tag{15} $$

Here, the state variable is denoted as $x(t) \in R^n$, and it satisfies Eq. (15) for a given control input $u(t) \in R^m$. The objective is to minimize the performance functional, which is defined as follows:

$$\text{Min } \Im[x(t), u(t)] = \frac{1}{2} x^T(b) M_{n\times n}, x(b) + \frac{1}{2} \int_a^b \left[x^T(t) Q_{n\times n}(t) x(t) + u^T(t) R_{m\times m}(t) u(t) \right] dt \tag{16}$$

Here, M, Q, and R are positive definite $n \times n$ matrices, and a and b denote the initial and terminal times, respectively.

To find the optimal control $u^*(t) \in R^m$, we will use optimal control theory [52]. The first step is to construct the Hamiltonian function:

$$\mathbf{H}(t, x, u, \lambda) = \frac{1}{2} [x^T Q, x + u^T R, u,] + \lambda^T \left[G_1(t, x) + G_2(t, x) u \right] \tag{17}$$

Here, $\lambda(t) \in R^n$ is the co-state variable.

The second step involves applying Pontryagin's minimum principle [52]:

$$ {}_a^C D_t^\alpha x(t) = \mathbf{H}_{\lambda(t)}(t, x(t), u(t), \lambda^*(t)) \tag{18}$$

$$ {}_t^C D_b^\alpha \lambda(t) = \mathbf{H}_{x(t)}(t, x(t), u(t), \lambda^*(t)) \tag{19}$$

$$\mathbf{H}_{u(t)}(t, x(t), u(t), \lambda(t)) = 0 \tag{20}$$

$$\mathbf{H}(t, x(t), \lambda(t), u(t)) \geq \mathbf{H}(t, x(t), u(t), \lambda^*(t)) \tag{21}$$

$$\lambda(b) = \mathrm{M}, x(b) \tag{22}$$

The optimal $u(t) \in R^m$ must satisfy Eqs. (20) and (21). By finding an expression for $u(t)$ in terms of $x(t)$ and/or $\lambda(t)$ and substituting it into Eqs. (18) and (19), the quasi-fixed-point optimal control problem (QFOCP) is equivalent to solving Eqs. (18) and (19) with initial conditions $x(a) = x_a$ and $\lambda(b) = \mathrm{M}, x(b)$.

In the following sections, we will delve into the detailed solution methodology for the QFOCP and provide insights into the optimal control theory.

4. Generalized differential transform method

This section is devoted the review of GDTM. First, we state the generalized Taylor's formula (GTF) [53], which given by the following theorem

Theorem 4.1 (*[53]*). *If* ${}_a^C D_t^{s\alpha} x(t) \in C(a, b]$, $\forall s = 0, 1, \ldots, N+1$, *where* $0 < \alpha \leq 1$. *If* $t \in [a, b]$, *then*

$$x(t) = \sum_{s=0}^{N} \frac{(t-a)^{s\alpha}}{\Gamma(s\alpha+1)} {}_a^C D_t^{s\alpha} x(a) + \frac{{}_a^C D_t^{s\alpha} x(\zeta)}{\Gamma((N+1)\alpha+1)} (t-a)^{(N+1)\alpha}, \ \exists \zeta \in [a, t], \ \forall t \in (a, b] \tag{23}$$

Depend on the GTF Eq. (23), the GDTM of the sth Caputo fractional derivative of $x(t)$ is define as

$$X[s] = \frac{1}{\Gamma(\alpha s+1)} {}_a^C D_t^{s\alpha} x(t) \mid_{t=a}, \quad \forall s = 0, 1, \ldots \tag{24}$$

Where ${}_a^C D_t^{s\alpha} = {}_a^C D_t^\alpha \cdot {}_a^C D_t^\alpha \underbrace{\ldots}_{s \text{ times}} {}_a^C D_t^\alpha$, $\quad 0 < \alpha \leq 1$, and the differential inverse transform of $X[s]$ is defined as

$$x(t) = \sum_{s=0}^{\infty} X[s](t-a)^{s\alpha} \tag{25}$$

By substituting Eq. (24) into Eq. (25) and using the generalized Taylor's formula Eq. (23), one can have

$$\sum_{s=0}^{\infty} X[s](t-a)^{s\alpha} = \sum_{s=0}^{\infty} \frac{(t-a)^{s\alpha}}{\Gamma(\alpha s+1)} {}_a^C D_t^{s\alpha} x(t) = x(t) \tag{26}$$

So, Eq. (25) is the inverse of GDTM is given by Eq. (24).

$$x(t) = \sum_{s=0}^{N} X[s](t-a)^{s\alpha} \tag{27}$$

Now, let $x(t) = \sum_{s=0}^{N} X[s](t-a)^{s\alpha}$, $x_1(t) = \sum_{s=0}^{N} X[s](t-a)^{s\alpha}$ and $x_2(t) = \sum_{s=0}^{N} X_2[s](t-a)^{s\alpha}$ then the following theorems are hold:

Theorem 4.2 (*[50]*). *If* $x(t) = x_1(t) \pm x_2(t)$, *then* $X[s] = X_1[s] \pm X_2[s], \quad \forall s = 0, 1, \ldots$

Theorem 4.3 (*[50]*). *If* $x(t) = a\, x_1(t)$, *then* $X[s] = aX_1[s], \quad \forall s = 0, 1, \ldots$, *where a constant*

Theorem 4.4 ([50]). *If* $x(t) = X_1(t)\,X_2(t)$, *then* $X[s] = \sum_{r=0}^{s} X_1[r]X_2[s-r], \quad \forall s = 0, 1, \ldots$

Theorem 4.5 ([50]). *If* $x(t) = {}_a^C D_t^{\alpha} X_1(t)$, *then* $X[s] = \frac{\Gamma(\alpha(s+1)+1)}{\Gamma(\alpha s+1)} X_1[s+1], \quad \forall s = 0, 1, \ldots$

Theorem 4.6 ([50]). *If* $x(t) = (t-a)^{n\alpha}$ *then* $X[s] = \delta(s-n)$ *where* $\delta(s) = \begin{cases} 1 & if \;\; s = 0 \\ 0 & if \;\; s \neq 0 \end{cases}$

5. Main result

Since the co-state Eq. (19) represented by the right α CFD, we derive generalized Taylor's formula by using the right α CFD. The following theorem is Generalized mean value theorem in term of the right α CFD.

Theorem 5.1. *For* $0 < \alpha < 1$, *if* $x(t) \in C[a,b]$ *and* ${}_t^C D_b^{\alpha} x(t) \in C(a,b]$, *then there exist* $\xi \in [t,b]$ *such that*

$$x(t) = x(b) + \frac{1}{\Gamma(\alpha+1)} {}_t^C D_b^{\alpha} x(\xi) \,.\, (b-t)^{\alpha}, \quad \forall t \in (a,b] \tag{28}$$

Proof. For $0 < \alpha < 1$, we have

$${}_t I_b^{\alpha} [{}_t^C D_b^{\alpha} x(t)] = x(t) - x(b) \tag{29}$$

Now, let $g(t) = {}_t^C D_b^{\alpha} x(t) \in C(a,b]$, Eq. (29)

$${}_t I_b^{\alpha} g(t) = x(t) - x(b) \tag{30}$$

By using the definition of the right α CFD of $g(t)$, we have

$$\frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} g(s)\, ds = x(t) - x(b) \quad b > t \tag{31}$$

By using the integral mean value theorem, there is $\xi \in [t,b]$ such that

$$\frac{1}{\Gamma(\alpha)} g(\xi) \int_t^b (s-t)^{\alpha-1} ds \;= x(t) - x(b) \quad b > t \tag{32}$$

$$\frac{1}{\Gamma(\alpha)} g(\xi) \frac{(b-t)^{\alpha}}{\alpha} = x(t) - x(b) \quad b > t\;, \quad \forall t \in (a,b] \tag{33}$$

So, one can get

$$x(t) = x(b) + \frac{1}{\Gamma(\alpha+1)} {}_t^C D_b^{\alpha} x(\xi) \,.\, (b-t)^{\alpha} \tag{34}$$

Theorem 5.2. *For* $0 < \alpha \leq 1$, *if* ${}_t^C D_b^{n\alpha} x(t)$, ${}_t^C D_b^{(n+1)\alpha} x(t) \in C(a,b]$, *then we have*

$$I_b^{n\alpha} {}_t^C D_b^{n\alpha} x(t) - I_b^{(n+1)\alpha} {}_t^C D_b^{(n+1)\alpha} x(t) = \frac{(b-t)^{n\alpha}}{\Gamma(n\alpha+1)} {}_t^C D_b^{n\alpha} x(b) \tag{35}$$

where ${}_t^C D_b^{n\alpha}$ *is the sequential right* α *CFD which is define by*

$${}_t^C D_b^{n\alpha} = {}_t^C D_b^{\alpha} . {}_t^C D_b^{\alpha} \underbrace{\ldots}_{n-times} {}_t^C D_b^{\alpha} \tag{36}$$

Proof. We have, using Eq. (35)

$$\begin{aligned} I_b^{n\alpha} {}_t^C D_b^{n\alpha} x(t) - I_b^{(n+1)\alpha} {}_t^C D_b^{(n+1)\alpha} x(t) &= I_b^{n\alpha} [{}_t^C D_b^{n\alpha} x(t) - I_b^{\alpha} {}_t^C D_b^{(n+1)\alpha} x(t)] \\ &= I_b^{n\alpha} [{}_t^C D_b^{n\alpha} x(t) - (I_b^{\alpha} {}_t^C D_b^{\alpha}) {}_t^C D_b^{n\alpha} x(t)] \\ &= I_b^{n\alpha} [{}_t^C D_b^{n\alpha} x(t) - {}_t^C D_b^{n\alpha} x(t) - {}_t^C D_b^{n\alpha} x(b)] \\ &= I_b^{n\alpha} {}_t^C D_b^{n\alpha} x(b) \\ &= \frac{(b-t)^{n\alpha}}{\Gamma(n\alpha+1)} {}_t^C D_b^{n\alpha} x(b) \end{aligned}$$

The next theorem gives a Generalized Taylor's formula in term of the right α CFD.

Theorem 5.3. *For* $0 < \alpha \leq 1$, *if* $({}_t^C D_b^{\alpha})^k x(t) \in C(a,b]$ *for* $k = 0, 1, 2, \ldots, n+1$, *then there exist* $\xi \in [t,b]$ *such that*

$$x(t) = \sum_{i=0}^{n} \frac{(b-t)^{i\alpha}}{\Gamma(i\alpha+1)} ({}_t^C D_b^{\alpha})^i x(b) + \frac{({}_t^C D_b^{\alpha})^{n+1} x(\xi)}{\Gamma((n+1)\alpha+1)} \,.\, (b-t)^{(n+1)\alpha}, \;\; \forall t \in (a,b] \tag{37}$$

Proof. By using the result in Theorem 5.2, we have

$$\sum_{i=0}^{n}[I_b^{i\alpha}({}_t^C D_b^{\alpha})^i x(t) - I_b^{(i+1)\alpha}({}_t^C D_b^{\alpha})^{(i+1)} x(t)] = \sum_{i=0}^{n} \frac{(b-t)^{i\alpha}}{\Gamma(i\alpha+1)}({}_t^C D_b^{\alpha})^i x(b) \tag{38}$$

that is

$$x(t) - I_b^{(n+1)\alpha}({}_t^C D_b^{\alpha})^{(n+1)} x(t) = \sum_{i=0}^{n} \frac{(b-t)^{i\alpha}}{\Gamma(i\alpha+1)}({}_t^C D_b^{\alpha})^i x(b) \tag{39}$$

$$I_b^{(n+1)\alpha}({}_t^C D_b^{\alpha})^{(n+1)} x(t) = \frac{1}{\Gamma((n+1)\alpha)} \int_t^b (s-t)^{(n+1)\alpha-1}({}_t^C D_b^{\alpha})^{n+1} x(s)ds \tag{40}$$

By applying the integral mean value theorem yields, there exist $\xi \in [t,b]$ such that

$$I_b^{(n+1)\alpha}({}_t^C D_b^{\alpha})^{(n+1)} x(t) = \frac{({}_t^C D_b^{\alpha})^{n+1} x(\xi)}{\Gamma((n+1)\alpha)} \int_t^b (s-t)^{(n+1)\alpha-1} ds \;\; with \;\; \forall t \in (a,b] \tag{41}$$

$$= \frac{({}_t^C D_b^{\alpha})^{n+1} x(\xi)}{\Gamma((n+1)\alpha+1)} (b-t)^{(n+1)\alpha} \tag{42}$$

From Eqs. (42) and (39), the GTF in term of the right α CFD is obtained.

Now, by using the result in Theorem 5.3, we define the GDTM about $t=b$ of $x(t)$ as follows:

$$X[s] = \frac{1}{\Gamma(\alpha s+1)} {}_t^C D_b^{s\alpha} x(t)\,|_{t=b}\,, \quad 0<\alpha\le 1, \quad \forall s=0,1,\dots \tag{43}$$

Also, the differential inverse transform of $X[s]$ is defined as

$$x(t) = \sum_{s=0}^{\infty} X[s](b-t)^{s\alpha} \tag{44}$$

It is easy verify this claim by substituting Eq. (43) into Eq. (44) and Theorem 5.3, one can have

$$\sum_{s=0}^{\infty} X[s](b-t)^{s\alpha} = \sum_{s=0}^{\infty} \frac{(b-t)^{s\alpha}}{\Gamma(\alpha s+1)} {}_t^C D_b^{s\alpha} x(t) = x(t) \tag{45}$$

So, Eq. (44) is the inverse of the GDTM about $t=b$ Eq. (43).

$$x(t) = \sum_{s=0}^{N} X[s](t-a)^{s\alpha} \tag{46}$$

Now, let $x(t) = \sum_{s=0}^{N} X[s](b-t)^{s\alpha}$, $x_1(t) = \sum_{s=0}^{N} X[s](b-t)^{s\alpha}$ and $x_2(t) = \sum_{s=0}^{N} X_2[s](b-t)^{s\alpha}$ then the following theorems are hold:

Theorem 5.4. *If* $x(t) = x_1(t) \pm x_2(t)$*, then* $X[s] = X_1[s] \pm X_2[s], \quad \forall s = 0,1,\dots$

Proof. Since $x_1(t) = \sum_{s=0}^{N} \chi_1[s](b-t)^{s\alpha}$ and $x_2(t) = \sum_{s=0}^{N} \chi_2[s](b-t)^{s\alpha}$, then

$$\begin{aligned} x(t) &= \sum_{s=0}^{N} \chi[s](b-t)^{s\alpha} \\ &= a\,x_1(t) \pm\, b\,x_2(t) \\ &= a \sum_{s=0}^{N} \chi_1[s](b-t)^{s\alpha} \pm\, b \sum_{s=0}^{N} \chi_2[s](b-t)^{s\alpha} \\ &= \sum_{s=0}^{N} (a\chi_1[s] \pm\, b\,\chi_2[s])(b-t)^{s\alpha} \end{aligned}$$

By compare the coefficients of $(b-t)^{s\alpha}$ for all $s = 0,1,2,\dots$, one can get

$$\chi[s] = a\,\chi_1[s] \pm b\,\chi_2[s], \quad \forall s=0,1,\dots$$

Theorem 5.5. *If* $x(t) = x_1(t)\,x_2(t)$*, then* $X[s] = \sum_{r=0}^{s} X_1[r]X_2[s-r], \quad \forall s=0,1,\dots$

Proof. Since $x_1(t) = \sum_{s=0}^{N} \chi_1[s](b-t)^{s\alpha}$ and $x_2(t) = \sum_{s=0}^{N} \chi_2[s](b-t)^{s\alpha}$, then

$$\begin{aligned} x(t) &= \sum_{s=0}^{N} \chi[s](b-t)^{s\alpha} \\ &= x_1(t)\,x_2(t) \end{aligned}$$

$$= (\sum_{s=0}^{N} \chi_1[s](b-t)^{s\alpha})(\sum_{s=0}^{N} \chi_2[s](b-t)^{s\alpha})$$
$$= \sum_{s=0}^{N}\sum_{r=0}^{s} \chi_1[r]\chi_2[s-r](b-t)^{s\alpha}$$

By compare the coefficients of $(b-t)^{s\alpha}$ for all $s = 0, 1, 2, \ldots$, one can get
$X[s] = \sum_{r=0}^{s} \chi_1[r]\chi_2[s-r], \quad \forall s = 0, 1, \ldots$

Theorem 5.6. *If* $x(t) = {}_t^C D_b^{\alpha} X_1(t)$, *then* $X[s] = \frac{\Gamma(\alpha(s+1)+1)}{\Gamma(\alpha s+1)} X_1[s+1], \quad \forall s = 0, 1, \ldots$

Proof. Since $x_1(t) = \sum_{s=0}^{N} \chi_1[s](b-t)^{s\alpha}$, then

$$x(t) = \sum_{s=0}^{N} \chi[s](b-t)^{s\alpha}$$
$$= {}_t^C D_b^{\alpha} x_1(t)$$
$$= \sum_{s=1}^{N} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\alpha+1)} \chi_1[s](b-t)^{s\alpha-\alpha}$$
$$= \sum_{s=0}^{N} \frac{\Gamma(s\alpha+\alpha+1)}{\Gamma(s\alpha+1)} \chi_1[s+1](b-t)^{s\alpha}$$

By compare the coefficients of $(b-t)^{s\alpha}$ for all $s = 0, 1, 2, \ldots$, one can get

$$\chi[s] = \frac{\Gamma(\alpha(s+1)+1)}{\Gamma(\alpha s+1)} \chi_1[s+1], \quad \forall s = 0, 1, \ldots$$

Theorem 5.7. *If* $x(t) = (b-t)^{n\alpha}$, *then then* $X[s] = \delta(s-n)$ *where* $\delta(s) = \begin{cases} 1 & if \ \ s = 0 \\ 0 & if \ \ s \neq 0 \end{cases}$

Proof. Since $x(t) = \sum_{s=0}^{N} \chi[s](b-t)^{s\alpha}$, then

$$x(t) = \sum_{s=0}^{N} \chi[s](b-t)^{s\alpha}$$
$$= (b-t)^{k\alpha}$$

By compare the coefficients of $(b-t)^{s\alpha}$ for all $s = 0, 1, 2, \ldots$, one can get

$\chi[s] = 0, \ \forall s = 0, 1, \ldots, k-1, k+1, \ldots$ and $\chi[k] = 1$, that is
$\chi[s] = \delta(s-k)$

Definition 5.1 ([54,55]). For any real values $\xi_1, \xi_2, \ldots, \xi_n$ the following $n \times n$ is called a Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{n-1} & \xi_n \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \cdots & \xi_{n-1}^2 & \xi_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_1^{n-2} & \xi_2^{n-2} & \xi_3^{n-2} & \cdots & \xi_{n-1}^{n-2} & \xi_n^{n-2} \\ \xi_1^{n-1} & \xi_2^{n-1} & \xi_3^{n-1} & \cdots & \xi_{n-1}^{n-1} & \xi_n^{n-1} \end{pmatrix} \tag{47}$$

Lemma 5.1 ([54]). *The determent of the Vandermonde matrix in Eq. (47) is*

$$|V| = \prod_{1<s<r<n} (\xi_r - \xi_s) \tag{48}$$

By using Lemma 5.1, it can easily deduce that the Vandermonde matrix is invertible if all $\xi_1, \xi_2, \ldots, \xi_n$ *are different nonzero real values.*

Theorem 5.8. *Let* $\chi[s], \ \forall s = 0, 1, 2, \ldots$ *is the GDTM about* $t = b$ *of* $x(t)$ *and* $\chi[s], \forall s = 0, 1, 2, \ldots$ *is the GDTM about* $t = a$ *of* $x(t)$. *Then it is always possible to write* $\chi[s], \ \forall s = 0, 1, 2, \ldots$ *in term of* $\chi[s], \forall s = 0, 1, 2, \ldots$ *and vice versa.*

Proof. Since $\chi[s], \ \forall s = 0, 1, 2, \ldots$ is the GDTM about $t = b$ of $x(t)$, we have

$$x(t) = \sum_{s=0}^{N} \chi[s](b-t)^{s\alpha} \tag{49}$$

Also, since $\chi[s]$, $\forall s=0,1,2,\ldots$ is the GDTM about $t=a$ of $x(t)$, we get

$$x(t)=\sum_{s=0}^{N}\chi[s](t-a)^{s\alpha} \tag{50}$$

By using Eqs. (49) and (50), we have

$$\sum_{s=0}^{N}\chi[s](b-t)^{s\alpha}=\sum_{s=0}^{N}\chi[s](t-a)^{s\alpha} \tag{51}$$

Substitute $t=a+\frac{b-a}{N+1}(s-1), \forall s=1,2,\ldots,N+1$ in (51), we have the following linear system

$$A_{(N+1)\times(N+1)}\chi_{(N+1)\times 1}=B_{(N+1)\times(N+1)}\chi_{(N+1)\times 1} \tag{52}$$

where $A_{1,j}=B_{1,j}=1$, $\forall j=1,2,\ldots,N+1$, $A_{i,j}=[b-a-\frac{b-a}{N+1}(j-1)]^{\alpha(i-1)}$ and $B_{i,j}=[\frac{b-a}{N+1}(j-1)]^{\alpha(i-1)}$ for all $i=2,\ldots,N+1$ and for all $j=1,2,\ldots,N+1$

Clearly, $A=V(\xi_1,\xi_2,\ldots,\xi_{N+1})$ is $N+1$ Vandermonde matrix with $\xi_j=(b-a-\frac{b-a}{N+1}(j-1))^{\alpha}$, for all $j=1,2,\ldots,N+1$ where ξ_j are different real values. Also, $B=V(\zeta_1,\zeta_2,\ldots,\zeta_{N+1})$ is $N+1$ Vandermonde matrix with $\zeta_1=0$, $\zeta_j=(b-a-\frac{b-a}{N+1}(j-1))^{\alpha}$, for all $j=2,\ldots,N+1$, where ζ_j are different real values.

Now, by using Lemma 5.1, we find that A and B are invertible matrices. Therefore, we use Eq. (52) to have

$$\chi=A^{-1}B\chi \tag{53}$$

and

$$\chi=B^{-1}A\chi \tag{54}$$

To this end, we will establish an algorithm to find a numerical solution of quadratic time varying fractional optimal control problems. Therefore, assume $\chi[s]$, $\forall s=0,1,2,\ldots$ is the GDTM about $t=b$ of the state $x(t)$, $\chi[s]$, $\forall s=0,1,2,\ldots$ is the GDTM about $t=a$ of the state $x(t)$, $Y[s]$, $\forall s=0,1,2,\ldots$ is the GDTM about $t=b$ of the co-state $\lambda(t)$ and $\Lambda[s]$, $\forall s=0,1,2,\ldots$ is the GDTM about $t=a$ of the co-state$\lambda(t)$.

Algorithm 5.1.

1. Choose a value for N.
2. Apply the discretization in time method (DTM) about $t=a$ to the state equation and the DTM about $t=b$ to the co-state equations.
3. Apply the DTM about $t=a$ for the initial condition $x(a)=x_a$ and the DTM about $t=b$ for the final condition $\lambda(b)=\lambda_b$. This yields $X[0]=x_a$ and $Y[0]=\lambda_b$, respectively.
4. Apply theorem (5.2) and substitute the result from step (1) into the equations. This results in a system of $2N$ linear equations with the unknown variables $X[s]$ for $s=1,2,\ldots,N$ and $Y[s]$ for $s=1,2,\ldots,N$.
5. Solve the linear system obtained in step (3) to find the unknown variables $X[s]$ for $s=1,2,\ldots,N$ and $\Lambda[s]$ for $s=0,1,2,\ldots,N$.
6. To obtain $x(t)$ and $\lambda(t)$, utilize the results from step (5) by approximating $x(t)$ as $\sum_{s=0}^{N}Xs^{\alpha s}$ and $\lambda(t)$ as $\sum_{s=0}^{N}Y^{\alpha s}$.

6. Illustrated examples

This section is devoted to illustrating some examples to show the importance, accuracy, effectiveness and efficiency of this algorithm for solving the proposed method for solving a problem.

Example 6.1 (*[56]*)**.** For the following problem

$$\min J[u(t),x(t)]=\frac{1}{2}\int_0^1 (u^2(t)+x^2(t))dt \tag{55}$$

subject to

$${}_0^C D_t^{\alpha}x(t)=\frac{1}{4}u(t)-\frac{1}{4}x(t)+t^{\alpha},\quad x(0)=1,\quad 0<\alpha\le 1 \tag{56}$$

The exact optimal state space solution when $\alpha=1$ is

$$\begin{aligned}x^*(t)=&\frac{(-2+(9\sqrt{2}+9)e^{\frac{1}{4}\sqrt{2}})e^{-\frac{1}{4}\sqrt{2}t}+(2+(9\sqrt{2}-9)e^{-\frac{1}{4}\sqrt{2}})e^{\frac{1}{4}\sqrt{2}t}}{(\sqrt{2}-1)e^{-\frac{1}{4}\sqrt{2}}+e^{\frac{1}{4}\sqrt{2}}(1+\sqrt{2})}\\&+\frac{2(t-4)((\sqrt{2}-1)e^{-\frac{1}{4}\sqrt{2}}+e^{\frac{1}{4}\sqrt{2}}(1+\sqrt{2}))}{(\sqrt{2}-1)e^{-\frac{1}{4}\sqrt{2}}+e^{\frac{1}{4}\sqrt{2}}(1+\sqrt{2})}\end{aligned} \tag{57}$$

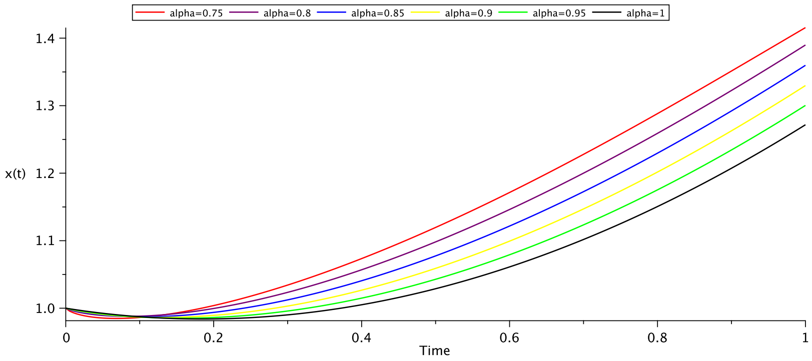

Fig. 1. The state variable $x(t)$ for Example 6.1 with various choices of α .

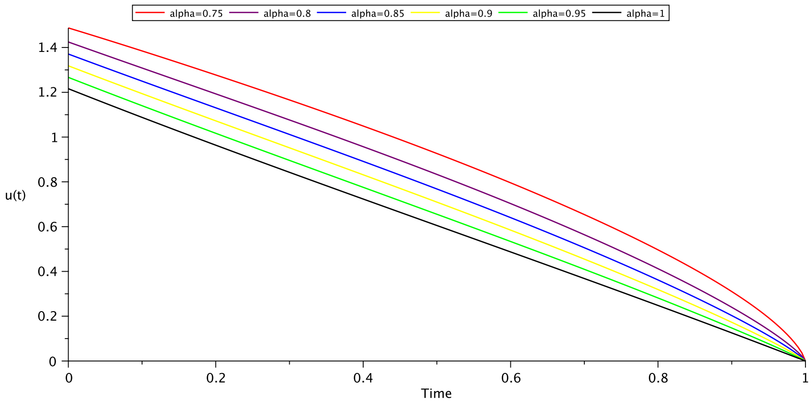

Fig. 2. The control variable $u(t)$ for Example 6.1 with various choices of α .

while the exact optimal control is

$$u^*(t) = -\frac{1}{4}\Big[\frac{(-8\sqrt{2}+36e^{\frac{1}{4}\sqrt{2}}+8)e^{-\frac{1}{4}\sqrt{2}t}}{(\sqrt{2}-1)e^{-\frac{1}{4}\sqrt{2}}+e^{\frac{1}{4}\sqrt{2}}(1+\sqrt{2})} + \frac{(-8\sqrt{2}-36e^{-\frac{1}{4}\sqrt{2}}-8)e^{\frac{1}{4}\sqrt{2}t}}{(\sqrt{2}-1)e^{-\frac{1}{4}\sqrt{2}}+e^{\frac{1}{4}\sqrt{2}}(1+\sqrt{2})} + \frac{8((\sqrt{2}-1)e^{-\frac{1}{4}\sqrt{2}}+e^{\frac{1}{4}\sqrt{2}}(1+\sqrt{2}))t}{(\sqrt{2}-1)e^{-\frac{1}{4}\sqrt{2}}+e^{\frac{1}{4}\sqrt{2}}(1+\sqrt{2})}\Big] \tag{58}$$

The relevant TBFDEs for this issue are

$${}_0^C D_t^\alpha x^*(t) = -\frac{1}{4}x^*(t) - \frac{1}{16}\lambda^*(t) + t, \quad x^*(0) = 1 \tag{59}$$

$${}_t^C D_1^\alpha \lambda^*(t) = x^*(t) - \frac{1}{4}\lambda^*(t), \quad \lambda^*(1) = 0 \tag{60}$$

where the optimal control is

$$u^*(t) = -\frac{1}{4}\lambda^*(t) \tag{61}$$

Now, we consider the GDT at $t = 0$ for Eq. (59), and then, we employ the GDT at $t = 1$ for Eq. (60).

$$\frac{\Gamma(s\alpha+\alpha+1)}{\Gamma(s\alpha+1)}\mathrm{X}[s+1] = -\frac{1}{4}\mathrm{X}[s] - \frac{1}{16}\Lambda[s] + \delta(1), \quad \forall s = 0, 1, 2, \dots, N \tag{62}$$

$$\frac{\Gamma(s\alpha+\alpha+1)}{\Gamma(s\alpha+1)}Y[s+1] = \chi[s] - \frac{1}{4}Y[s], \quad \forall s = 0, 1, 2, \dots, N \tag{63}$$

where $X[0] = 1$ and $Y[0] = 0$.

For $N = 10$, we perform the recurrence relations Eqs. (62) and (63) with $X[0] = 1$ and $Y[0] = 0$. The results can be reported by the following figures (see Figs. 1 and 2):

In Tables 1 and 2, the absolute error of $x(t)$ and $u(t)$ have been computed respectively for various choices of N. In fact, these Tables show the ability, reliability, accuracy and efficiency of the propose algorithms to solve FQOCPs.

Table 1
The absolute errors of $x(t)$ for Example 6.1 at $\alpha = 1$ and various choices of N.

Time	$N = 5$	$N = 10$	$N = 20$	$N = 30$
0.1	6.53×10^{-8}	1.91×10^{-14}	2.62×10^{-31}	4.34×10^{-50}
0.2	1.08×10^{-8}	2.37×10^{-14}	2.94×10^{-31}	4.23×10^{-50}
0.3	1.37×10^{-7}	2.37×10^{-14}	3.01×10^{-31}	3.97×10^{-50}
0.4	1.59×10^{-7}	2.25×10^{-14}	3.05×10^{-31}	3.71×10^{-50}
0.5	1.73×10^{-7}	2.11×10^{-14}	3.12×10^{-31}	3.46×10^{-50}
0.6	1.74×10^{-7}	1.98×10^{-14}	1.62×10^{-31}	3.22×10^{-50}
0.7	1.48×10^{-7}	1.88×10^{-14}	3.14×10^{-31}	2.99×10^{-50}
0.8	6.45×10^{-8}	1.94×10^{-14}	3.14×10^{-31}	2.76×10^{-50}
0.9	1.28×10^{-7}	2.57×10^{-14}	2.82×10^{-31}	2.75×10^{-50}
1	5.17×10^{-7}	5.00×10^{-14}	4.23×10^{-31}	8.35×10^{-50}

Table 2
The absolute errors of $u(t)$ for Example 6.1 at $\alpha = 1$ and various choices of N.

Time	$N = 5$	$N = 10$	$N = 20$	$N = 30$
0	4.23×10^{-6}	4.03×10^{-14}	3.28×10^{-30}	7.66×10^{-50}
0.1	3.96×10^{-6}	1.19×10^{-13}	2.19×10^{-30}	2.56×10^{-49}
0.2	3.77×10^{-6}	1.41×10^{-13}	2.01×10^{-30}	2.61×10^{-49}
0.3	3.62×10^{-6}	1.46×10^{-13}	1.91×10^{-30}	2.58×10^{-49}
0.4	3.50×10^{-6}	1.46×10^{-13}	1.84×10^{-30}	2.56×10^{-49}
0.5	3.73×10^{-6}	1.45×10^{-13}	1.77×10^{-30}	2.53×10^{-49}
0.6	3.21×10^{-6}	1.43×10^{-13}	1.69×10^{-30}	2.50×10^{-49}
0.7	2.94×10^{-6}	1.40×10^{-13}	1.64×10^{-30}	2.47×10^{-49}
0.8	2.47×10^{-6}	1.30×10^{-13}	1.54×10^{-30}	2.43×10^{-49}
0.9	1.59×10^{-6}	9.98×10^{-14}	1.34×10^{-30}	2.32×10^{-49}

Example 6.2 ([56]). Consider the problem

$$\min J[u(t), x(t)] = \frac{1}{2}\int_0^1 [(u(t) + \frac{t^\alpha}{\Gamma(\alpha+3)} - \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)})^2 + (x(t) - t^{3\alpha})^2]dt \tag{64}$$

subject to

$${}_0^C D_t^\alpha x(t) = t^{2\alpha}u(t) + \frac{x(t)}{\Gamma(\alpha+3)} \quad , \quad x(0) = 0, \quad 0 < \alpha \le 1 \tag{65}$$

The exact optimal state space solution when $\alpha = 1$ is

$$x^*(t) = t^{3\alpha} \tag{66}$$

while the exact optimal control is

$$u^*(t) = -\frac{t^\alpha}{\Gamma(\alpha+3)} + \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} \tag{67}$$

$$\text{Let } x(t) = \sum_{s=0}^{N} \chi[s]\, t^{s\alpha} = \sum_{s=0}^{N} \chi[s]\,(1-t)^{s\alpha},$$
$$\lambda(t) = \sum_{s=0}^{N} \Lambda[s]\, t^{s\alpha} = \sum_{s=0}^{N} \Upsilon[s]\,(1-t)^{s\alpha}$$

The relevant TBFDEs for this issue are

$${}_0^C D_t^\alpha x^*(t) = \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{1}{\Gamma(\alpha+3)} t^{3\alpha} - \lambda^*(t) t^{4\alpha} + \frac{x^*(t)}{\Gamma(\alpha+3)} \quad , \quad x^*(0) = 0 \tag{68}$$

$${}_t^C D_1^\alpha \lambda^*(t) = x^*(t) - t^{3\alpha} + \frac{\lambda^*(t)}{\Gamma(\alpha+3)}, \quad \lambda^*(1) = 0 \tag{69}$$

where the optimal control is

$$u^*(t) = \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma(\alpha+3)} t^\alpha - \lambda^*(t) t^{2\alpha} \tag{70}$$

By take the GDT about $t = 0$ for the Eq. (68) and take the GDT about $t = 1$ for Eq. (69), one can get

$$\frac{\Gamma(s\alpha+\alpha+1)}{\Gamma(s\alpha+1)} \mathrm{X}[s+1] = \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} \delta(2,s) - \frac{1}{\Gamma(\alpha+3)} \delta(3,s)$$
$$- \sum_{i=0}^{s} \delta(4,i) \Lambda(s-i) + \frac{\chi[s]}{\Gamma(\alpha+3)}, \quad \forall s = 0, 1, 2, \ldots, N \tag{71}$$

Table 3
The absolute errors of $x(t)$ Example 6.2 at $\alpha = 0.6$ and various choices of N.

Time	VIM [56]			Present method
	$N = 5$	$N = 10$	$N = 20$	$N \geq 3$
0.1	2.98×10^{-6}	1.04×10^{-17}	6.93×10^{-18}	0
0.2	2.34×10^{-5}	8.39×10^{-17}	2.08×10^{-18}	0
0.3	7.81×10^{-5}	4.58×10^{-14}	2.77×10^{-17}	0
0.4	1.83×10^{-4}	1.33×10^{-12}	5.55×10^{-17}	0
0.5	3.53×10^{-4}	2.68×10^{-11}	5.55×10^{-17}	0
0.6	6.66×10^{-4}	4.07×10^{-10}	0	0
0.7	9.23×10^{-4}	4.83×10^{-9}	1.11×10^{-16}	0
0.8	1.28×10^{-3}	4.60×10^{-8}	0	0
0.9	1.58×10^{-3}	3.62×10^{-7}	2.22×10^{-16}	0

$$\frac{\Gamma(s\alpha+\alpha+1)}{\Gamma(s\alpha+1)}\Upsilon[s+1] = \chi[s] - T[s] + \frac{\Upsilon[s]}{\Gamma(\alpha+3)}, \quad \forall s = 0, 1, 2, \ldots, N \tag{72}$$

where $X[0] = 0$ and $\Upsilon[0] = 0$.

If $s = 0$, we have

$$\frac{\Gamma(\alpha+1)}{\Gamma(1)}\mathrm{X}[1] = \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\delta(2,0) - \frac{1}{\Gamma(\alpha+3)}\delta(3,0) - \delta(4,0)\Lambda(0-i) + \frac{\chi[0]}{\Gamma(\alpha+3)} \tag{73}$$

then X[1] = 0.

$$\frac{\Gamma(\alpha+1)}{\Gamma(1)}\Upsilon[1] = \chi[0] - T[0] + \frac{\Upsilon[0]}{\Gamma(\alpha+3)} \tag{74}$$

then $\Upsilon[1] = 0$.

If $s = 1$, we have

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{X}[2] = \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\delta(2,1) - \frac{1}{\Gamma(\alpha+3)}\delta(3,1) - \delta(4,0)\Lambda(1) - \delta(4,1)\Lambda(0) + \frac{\chi[1]}{\Gamma(\alpha+3)} \tag{75}$$

then X[2] = 0.

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\Upsilon[2] = \chi[1] - T[1] + \frac{\Upsilon[1]}{\Gamma(\alpha+3)} \tag{76}$$

then $\Upsilon[2] = 0$.

If $s = 2$, we have

$$\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\mathrm{X}[3] = \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\delta(2,2) - \frac{1}{\Gamma(\alpha+3)}\delta(3,2) - \delta(4,0)\Lambda(2) - \delta(4,1)\Lambda(1) - \delta(4,2)\Lambda(0) + \frac{\chi[2]}{\Gamma(\alpha+3)} \tag{77}$$

then X[3] = 1.

$$\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\Upsilon[3] = \chi[2] - T[2] + \frac{\Upsilon[2]}{\Gamma(\alpha+3)} \tag{78}$$

Then $\Upsilon[3] = 0$.

Then, we get $\Upsilon[s] = 0, \ \forall s = 0, 1, \ldots$

Then $x(t) = t^{3\alpha}$ and $\lambda(t) = 0$. In fact, we have the exact solution only in three iterations.

In Table 3, the absolute error of $x(t)$ for $N = 5, 10, 20$ by using variational iteration method (VIM) [56] are compare with the absolute error obtained by the present method for $n \geq 3$. In fact, we have the exact solution of this problem only in three iterations (see Figs. 3 and 4).

7. Conclusions

The application of the FDTM, GDTM, and RDTM poses challenges due to the unique representation of the co-state equation through the utilization of the right α CFD, as well as the representation of the state equation through the use of the left α CFD. In order to address this challenge, we present an innovative methodology that integrates the Generalized Dynamic Time Warping Measure (GDTM) at the temporal boundaries with the Vandermonde matrix. By implementing this approach, we propose a novel algorithm for efficiently resolving First-Order Constraint Problems (FOCPs). By presenting two illustrated examples, we have effectively showcased

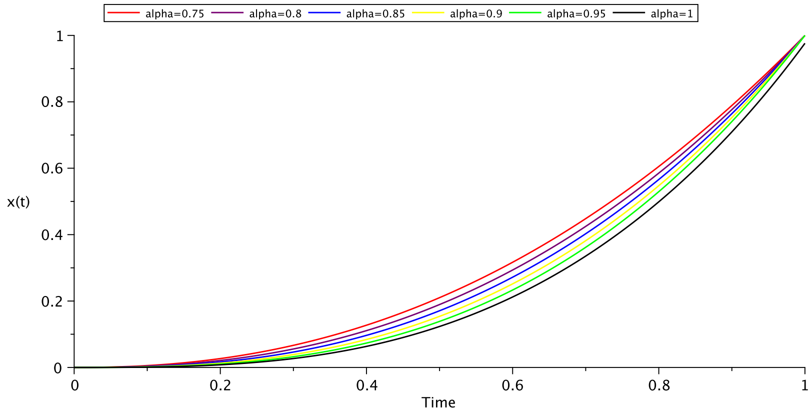

Fig. 3. The state variable $x(t)$ for Example 6.2 with various choices of α .

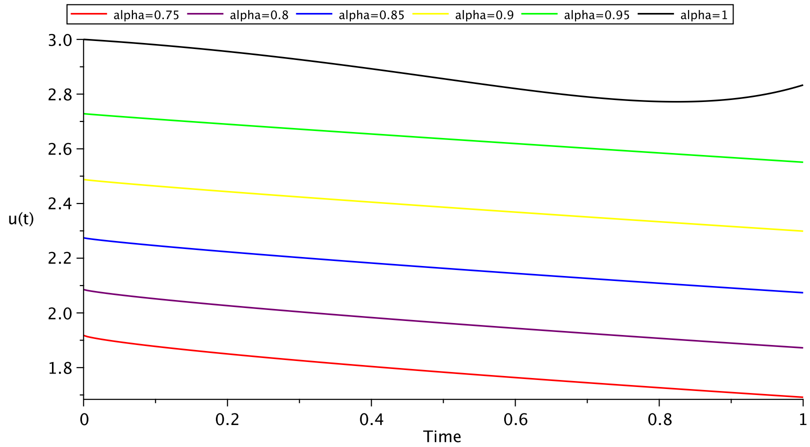

Fig. 4. The control variable $u(t)$ for Example 6.2 with various choices of α .

the remarkable capabilities, dependability, precision, and effectiveness of the proposed algorithms in addressing FOCPs. We believe that these algorithms are suitable for solving time delay optimal control problems when its dynamic equation is ordinary (fractional) differential equations. Also, these algorithms may need some modification in order to be suitable for singular optimal control problems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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