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Solving nonlinear stochastic differential equations via fourth-degree hat functions

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ABSTRACT

The article focuses on introducing novel fourth-degree hat functions (FDHFs) for constructing new operations matrices used to find numerical solutions for nonlinear stochastic differential equations (NSDEs). The technique's effort is summarized by utilizing FDHFs to construct operations matrices that transform the given problem into nonlinear algebraic equations. The advantage of this technique lies in its simplicity for calculating the unknown coefficients of the function's approximation without requiring any integration. As a result, the proposed approach incurs low computational expenses. Additionally, an error analysis of this approach was conducted, demonstrating a convergence rate of $O(h^5)$. Several examples were implemented to support and illustrate the effectiveness and capability of the proposed technique.

1. Introduction

The history of stochastic differential equations (SDEs) is related to the exciting invention of the microscope, which led to great developments in most modern natural sciences. In fact, Robert Brown (1773–1858) used the microscope to examine liquids and observed rapid oscillatory motion for molecules in the fluid. This motion was called Brownian motion (BM) [1]. Unfortunately, BM remained unexplained for centuries because, at that time, most scientists were unaware of the existence of atoms and molecules.

In 1900, Louis Bachelier (1870–1946) became the first person to study BM mathematically by valuing stock options in his PhD thesis [2]. In 1905, Albert Einstein (1879–1955) provided a physical explanation for BM and formulated it in a mathematical equation called the diffusion equation. Three years later, Paul Langevin (1872–1946) used Newton's second law and a statistical approach to introduce a simpler and more mechanically compatible physical description of BM compared to Einstein's explanation [3]. Langevin's approach resulted in an ordinary differential equation for the first two moments of the distribution, while Einstein's approach yielded a partial differential equation governing the entire probability distribution [4].

The existence of BM was rigorously proved by Norbert Wiener (1894–1964), who constructed a mathematical formulation known as the Wiener process [5]. Prior to 1933, probability theory was not considered related to mathematics until Andrey Kolmogorov's work in his book "Foundations of the Theory of Probability", where he used measure theory to establish a firm connection between probability theory and mathematics. In 1937, Maurice Fréchet (1878–1973) extended the concept of convergence of real sequences to different types of convergence for sequences of random variables [6].

The ideas of mean square (m.s.) continuity, m.s. differentiability, and m.s. integrability of stochastic processes were introduced by Evgenievich Slutsky (1880–1948). Kiyosi Itô (1915–2008) recognized the difficulty of studying differential equations with a noise

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term and proved that the mean integral of the noise term does not converge in the m.s. sense to a unique limit [7]. To overcome this difficulty, Itô devised a new approach, creating a formula called the Itô integral, which forms the basis of Itô calculus (stochastic calculus). Itô is considered the pioneer of SDEs and was awarded the "Carl Friedrich Gauss Prize for Applications of Mathematics" in 2006 [2].

However, SDEs are primarily deterministic differential equations with an additional white noise term. The deterministic term describes the behavior of the phenomenon, while the stochastic term represents the random perturbation or "noise" that influences the phenomenon. Stochastic behavior naturally arises in various phenomena affected by random perturbations, such as biology [8], population dynamics [9], the movement of ions in materials [10], problems of reactor dynamics [11], optimal option pricing in finance [12], and various engineering problems [13].

When discussing the importance of mathematical modeling in real-life systems [14–17], fractional differential equations and stochastic differential equations (SDEs) are often utilized. These equations find applications in various fields such as mechanics, physics, medicine, and social sciences. They are particularly useful in studying stochastic dynamical systems with memory, such as economics, including general stock markets, insurance, portfolio management, and financial markets [18–28]. However, obtaining analytical solutions to SDEs is frequently challenging or even impossible. Hence, the development of numerical methods for solving this type of equation is inevitable. As a result, a number of authors have proposed numerical methods for solving these equations, such as stochastic collocation [29–32], spline interpolation methods [33–36], wavelet methods [37,38], wavelet-based numerical schemes [39], variational iteration method [40], Adomian decomposition method [41–43], Ito–Taylor expansions [44], stochastic Runge–Kutta methods [45], the Euler–Maruyama Method [46,47], and Petrov–Galerkin methods [48,49]. Recently, many orthogonal basic functions and polynomials have been implemented for solving SDEs, including block pulse functions [50–52], hat functions [53–63], delta functions [64], Legendre polynomials [65–67], triangular functions [68], Euler polynomials [69], polynomial chaos method [70], and Bernstein polynomials [71]. In this article, the effort is concentrated on the construction of novel FDHFs and studying their properties to build a new operations matrix technique for solving SDEs.

The remainder of the article is structured as follows: Section 2 provides fundamental motivations and novelties of this paper. Section 3 provides fundamental definitions and characteristics of stochastic calculus. In Section 4, we build FDHFs and discuss their properties while also deriving operational matrices of integration and stochastic integration. Building upon the results from the previous sections, Section 5 introduces a new algorithm for solving the NSDE, while Section 6 focuses on the error analysis of the proposed method. In Section 7, we provide illustrated examples to support and demonstrate the capability and efficacy of the proposed method.

2. Motivations

The novelty of solving SDEs using FDHFs lies in the application of a non-traditional numerical method to address a complex mathematical problem. In fact, SDEs involve random fluctuations and are notoriously challenging to solve analytically. Traditional numerical methods for SDEs, such as the Euler–Maruyama method or the Milstein method, are based on discretization schemes that can suffer from issues such as numerical instability and poor accuracy. These methods often require small time steps to achieve reasonable accuracy, which can be computationally expensive and limit their practical applicability. However, the present technique can provide accurate approximations of the solution to SDEs using fewer basis functions compared to other methods, reducing computational complexity. Also, the piecewise linear nature of FDHFs enables the capturing of discontinuities and jumps present in certain stochastic processes. In addition, the flexibility of hat functions allows for adaptive mesh refinement, concentrating computational resources in regions of interest and achieving higher accuracy where needed.

3. Preliminary

This section introduces the reader to fundamental concepts in continuous-time stochastic processes, along with key concepts and tools in stochastic calculus that will be utilized throughout the paper. Stochastic processes have been extensively studied and are well-documented in the literature. For further information, we recommend referring to the following references: [72–75].

3.1. Brownian motion

Brownian motion is a continuous-time stochastic process having stationary and independent Gaussian distributed increments, and continuous paths. To begin, we need to review the definition of BM, which is a basic example of a stochastic process.

Definition 3.1 (*[76]*)**.** The stochastic process $B(t)$, $t \geq 0$ is called a standard Brownian motion or Wiener process if the properties listed below are met:

- There are independent increments of the process. This means that the random variables $B(t_n) - B(t_{n-1})$, $B(t_{n-1}) - B(t_{n-2})$, … , $B(t_2) - B(t_1)$ are independent increments for all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq T$.
- There are stationary increments for the process. This means the increment $B(t+h) - B(t)$ has a distribution that is independent of t.
- $B(t)$, $t \geq 0$ is a continuous function of time t.
- For all $t \geq 0$, $B(t) \sim N(0,1)$.
- The stochastic process $B(t)$ has continuous paths.

3.2. Itô integral

The goal of this subsection is to define the Itô stochastic integral in order to understand stochastic differential equations. This is significant because we will define SDEs in terms of this integral in this paper.

Definition 3.2 ([7]). Let $y(t)$ be measurable stochastic process on the filtration $\{\mathfrak{I}_t\}$ for any $t \geq 0$. If the following mean square limit exist

$$\int_p^q y(t)dB(t) = \lim_{\ell\to\infty} \sum_{i=0}^{\ell-1} y(t_i)\left(B(t_{i+1}) - B(t_i)\right). \tag{1}$$

Then the above integration is called Itô integral of $y(t)$, where $t_i = p + \frac{q-p}{\ell} i$, $i = 0, 1, \ldots, \ell$ and $B(t)$ is a Brownian motion. In other words,

$$\mathrm{E}\left[\left|\int_p^q y(t)dB(t) - \sum_{i=0}^{\ell-1} y(t_i)\left(B(t_{i+1}) - B(t_i)\right)\right|^2\right] \to 0, \quad as\ \ell \to \infty.$$

3.3. Stochastic differential equations

Consider the general form of the SDE.

$$\begin{aligned} dy(t) &= f(t, y(t))dt + g(t, y(t))dB(t), \qquad t \in [0, T], \\ y(t_0) &= y_0, \end{aligned} \tag{2}$$

where $B(t)$ is the Brownian motion, f is the drift coefficient, and g is the diffusion coefficient. The SDE is linear if f and g are linear, and nonlinear if they are not. Eq. (2) has the following integral form:

$$\begin{aligned} y(t) &= y(t_0) + \int_0^t f(s, y(s))ds + \int_0^t g(s, y(s))dB(s), \\ y(t_0) &= y_0, \end{aligned} \tag{3}$$

where the first integral is on the right-hand side of Eq. (3) is an ordinary Riemann–Stieltjes integral, and the second integral is the Itô stochastic integral. We say that a stochastic process $y(t)$ solves the stochastic differential equation if it satisfies this equation. This subsection's main goal is to find conditions on the coefficients f and g that ensure the existence and uniqueness of solutions. Now we make the following assumptions about the problem's data:

Assumption 1. Assume that the functions f and g satisfy the Lipschitz condition and that for every $t \geq 0$, there exists a constant H_1 such that

$$\|f(t, y) - f(t, x)\| + \|g(t, y) - g(t, x)\| \leq H_1 \|y - x\|. \tag{4}$$

Assumption 2. Assume that the functions f and g satisfy the Lipschitz condition and that for every $t \geq 0$, there exists a constant H_2 such that

$$\|f(t, y)\| + \|g(t, y)\| \leq H_2(1 + \|y\|). \tag{5}$$

Now consider the NSDE in its general form for the unknown function $y(t)$:

$$\begin{aligned} dy(t) &= L_1(t, s)\sigma(y(t))dt + L_2(t, s)\phi(y(t))dB(t), \qquad t \in [0, T], \\ y(t_0) &= g(t). \end{aligned} \tag{6}$$

Eq. (6) has the following integral form:

$$\begin{aligned} y(t) &= g(t) + \int_0^t L_1(t, s)\sigma(y(s))ds + \int_0^t L_2(t, s)\phi(y(s))dB(s), \qquad t \in [0, T], \\ y(t_0) &= g(t), \end{aligned} \tag{7}$$

where $g(t)$, $L_1(t, s)$, and $L_2(t, s)$ for $s, t \in [0, T]$ are known stochastic processes, while $y(t)$ is an unknown stochastic process that must be calculated. The functions $\sigma(y(t))$ and $\phi(y(t))$ are two well-known analytic functions on $\mathbb{R}$, $B(t)$ is the standard Wiener process defined in the same probability space $(\Omega, \mathfrak{I}, P)$, and $\int_0^t L_2(t, s)\phi(y(s))dB(s)$ is the Itô integral. In fact, Eq. (6) covered a wide range of important SDEs in financial mathematics and physical systems, such as the pricing of options, and Black–Scholes equations, the differential equations with thermal fluctuations, Heston stochastic volatility equation, Ornstein–Uhlenbeck equation, and so on [77].

4. Fourth-Degree hat functions and their properties

This section is devoted to constructing the FDHFs. We will first divide the interval $\Omega = [0, T]$ into equidistant subintervals, and then each of these subintervals must be divided again into four equidistant subintervals with a length equal to h, where $h = \frac{T}{4n}$ and $n \in \mathbb{N}$. The FDHFs form a set of $(4n+1)$ linearly independent functions in $L^2[0, T]$. These functions are defined as follows:

$$\xi_0(t) = \begin{cases} \frac{(t-h)(t-2h)(t-3h)(t-4h)}{24h^4}, & 0 \le t \le 4h, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

if $k = 1, 2, \dots, n$,

$$\xi_{4k-1}(t) = \begin{cases} \frac{-(t-4kh)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{6h^4}, & (4k-4)h \le t \le 4kh, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

$$\xi_{4k-2}(t) = \begin{cases} \frac{(t-4kh)(t-(4k-1)h)(t-(4k-3)h)(t-(4k-4)h)}{4h^4}, & (4k-4)h \le t \le 4kh, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

$$\xi_{4k-3}(t) = \begin{cases} \frac{(t-(4k-2)h)(t-(4k-1)h)(t-4kh)(t-(4k-4)h)}{6h^4}, & (4k-4)h \le t \le 4kh, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

if $k = 1, 2, \dots, n-1$,

$$\xi_{4k}(t) = \begin{cases} \frac{(t-(4k-1)h)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{24h^4}, & 4(k-1)h \le t \le 4kh, \\ \frac{(t-(4k+1)h)(t-(4k+2)h)(t-(4k+3)h)(t-(4k+4)h)}{24h^4}, & 4kh \le t \le 4(k+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

and

$$\xi_{4n}(t) = \begin{cases} \frac{(t-(T-h))(t-(T-2h))(t-(T-3h))(t-(T-4h))}{24h^4}, & T-4h \le t \le T, \\ 0, & \text{otherwise}. \end{cases} \tag{13}$$

To clarify the definition of FDHFs on the interval $[0, 1]$ and $n = 2$, one can see Fig. 1 which shows the 9-set of FDHFs.

The following are the basic properties of FDHFs:

1. Using the definition of FDHFs, there is a very important relationship as follows:

$$\xi_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases} \quad \forall i, j = 0, 1, 2, \dots, 4n. \tag{14}$$

2. The total sum of FDHFs is one, which means:

$$\sum_{i=0}^{4n} \xi_i(t) = 1. \tag{15}$$

3. The functions $\xi_0(t), \xi_1(t), \dots, \xi_{4n}(t)$ are linearly independent for all $[0, T]$.
4. The FDHFs can be used to approximate any arbitrary function $y(t) \in L^2([0, T])$, as shown below.

$$y(t) \simeq y_{4n}(t) = \sum_{\kappa=0}^{4n} y_\kappa \xi_\kappa(t) = Y^{\mathrm{T}}\, \Xi(t) = \Xi^{\mathrm{T}}(t) Y, \tag{16}$$

where

$$\Xi(t) = [\xi_0(t),\ \xi_1(t),\ \xi_2(t), \dots,\ \xi_{4n}(t)]^{\mathrm{T}}, \tag{17}$$

and

$$Y = [y_0, y_1, y_2, \dots, y_{4n}]^{\mathrm{T}}. \tag{18}$$

The coefficients in Eq. (16) are given by

$$y_\kappa = y(\kappa h), \qquad \kappa = 0, 1, \dots, 4n. \tag{19}$$

5. In a similar way, on $[0\,, T] \times [0\,, T]$, the function $L(t, s)$ can be expanded by the FDHFs as follows:

$$L(t, s) \simeq L_{4n}(t, s) = \sum_{r=0}^{4n} \sum_{\kappa=0}^{4n} L_{\kappa r} \xi(t) \xi(s) = \Xi^{\mathrm{T}}(t) Y\, \Xi(s) = \Xi^{\mathrm{T}}(s) Y^{\mathrm{T}}\, \Xi(t), \tag{20}$$

where $Y = \left(L_{\kappa r}\right)_{(4n+1)\times(4n+1)}$, and $L_{\kappa r}(t, s) = L(\kappa h, rh)$, $\forall \kappa, r = 0, 1, 2, \dots, 4n$.

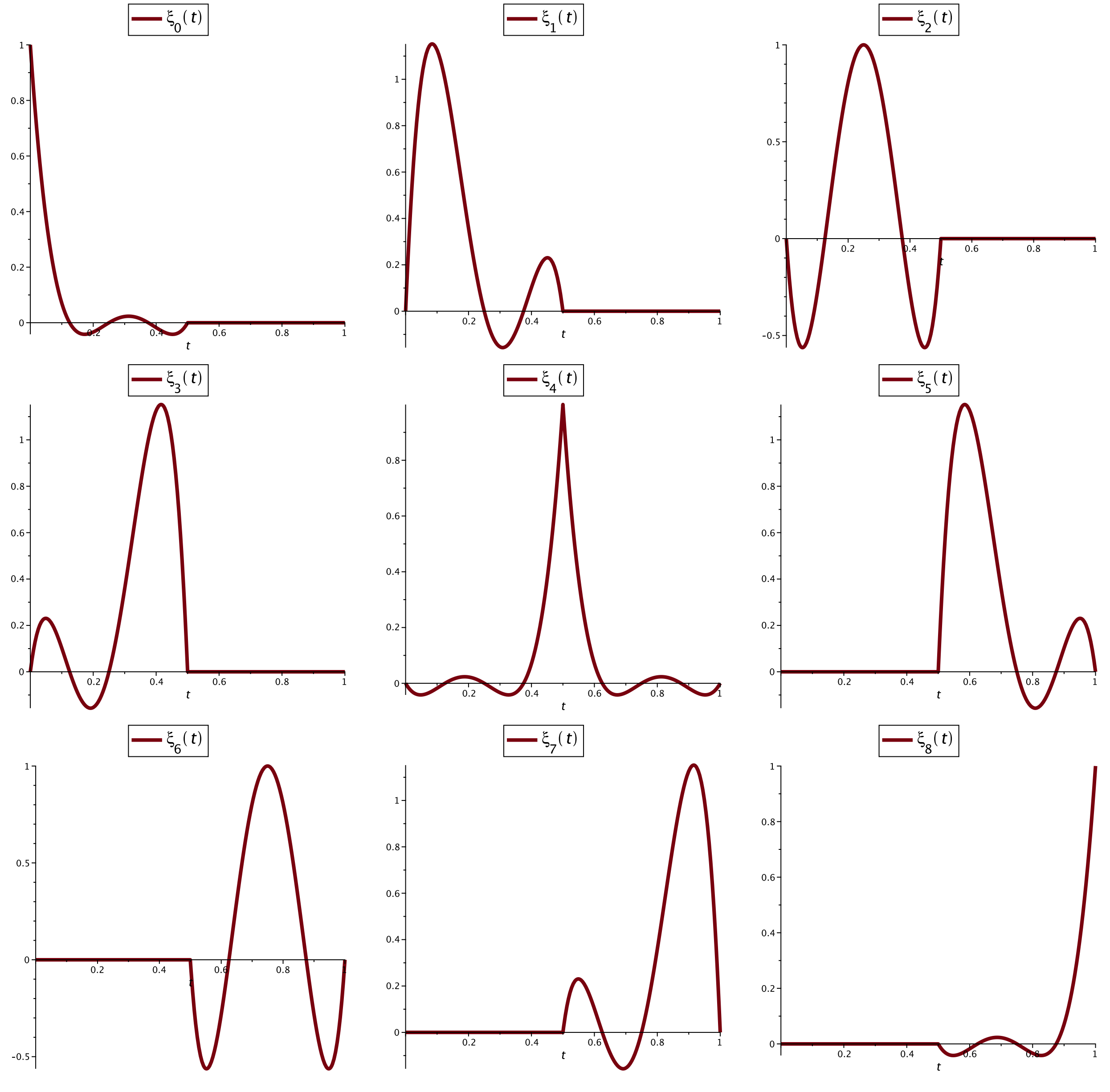

Fig. 1. Plot of FDHFs with $T = 1$ and $n = 2$.

4.1. Operational matrices of integration

In this subsection, we derive both the operational matrix for integrating the vector $\Xi(t)$, which is symbolized by P, as well as Itô's stochastic operational matrix for integrating the vector $\Xi(t)$, which is symbolized by P_S, for FDHFs in Theorems 4.1 and 4.2, respectively, that will be used in our proposed method.

To construct an operational matrix P that satisfies

$$\int_0^t \Xi(s)ds \simeq \mathrm{P}\Xi(t), \tag{21}$$

where $\Xi(t)$ is the vector defined in relation (17). Now we are attempting to write $\int_0^t \xi_\kappa(s)ds$ as a linear combination of the functions $\xi_0(t), \xi_1(t), \ldots, \xi_{4n}(t)$ as follows:

$$\int_0^t \xi_\kappa(s)ds \simeq \sum_{r=0}^{4n} \mathrm{P}_{\kappa,r}\xi_r(t), \qquad \forall \kappa = 0, 1, 2, \ldots, 4n. \tag{22}$$

The coefficients $\mathrm{P}_{\kappa,r}$ can be calculated as follows:

$$\mathrm{P}_{\kappa,r} = \int_0^{rh} \xi_\kappa(s)ds, \qquad \forall\, r,\kappa = 0,1,2,\ldots,4n. \tag{23}$$

As a direct consequence of this, we can state the following theorem:

Theorem 4.1. *Suppose $\Xi(t)$ is a vector defined by (17). Then, we can express the integration of $\Xi(t)$ as:*

$$\int_0^t \Xi(s)ds \simeq \mathrm{P}\,\Xi(t), \tag{24}$$

where P is the $(4n+1)\times(4n+1)$ operational integration matrix for the FDHFs, which is defined as:

$$\mathrm{P} = \frac{h}{720}\begin{pmatrix} 0 & Q_1 & Q_2 & Q_2 & Q_2 & Q_2 & \cdots & Q_2 & Q_2 \\ Q_3 & Q_4 & Q_5 & Q_6 & Q_6 & Q_6 & \cdots & Q_6 & Q_6 \\ Q_3 & Q_7 & Q_4 & Q_5 & Q_6 & Q_6 & \cdots & Q_6 & Q_6 \\ Q_3 & Q_7 & Q_7 & Q_4 & Q_5 & Q_6 & \cdots & Q_6 & Q_6 \\ Q_3 & Q_7 & Q_7 & Q_7 & Q_4 & Q_5 & \cdots & Q_6 & Q_6 \\ Q_3 & Q_7 & Q_7 & Q_7 & Q_7 & Q_4 & \cdots & Q_6 & Q_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ Q_3 & Q_7 & Q_7 & Q_7 & Q_7 & Q_7 & \cdots & Q_4 & Q_5 \\ Q_3 & Q_7 & Q_7 & Q_7 & Q_7 & Q_7 & \cdots & Q_7 & Q_4 \end{pmatrix}, \tag{25}$$

where $Q_1 = (251,232,243,224)$, $Q_2 = (224,224,224,224)$, $Q_3 = (0,0,0,0)^{\mathrm{T}}$, $Q_4 = \begin{pmatrix} 646 & 992 & 918 & 1024 \\ -264 & 192 & 648 & 384 \\ 106 & 32 & 378 & 1024 \\ -19 & -8 & -27 & 224 \end{pmatrix}$,
$Q_5 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 475 & 456 & 467 & 448 \end{pmatrix}$, $Q_6 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 448 & 448 & 448 & 448 \end{pmatrix}$, $Q_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Now, to construct an operational matrix P_S, we give the following theorem:

Theorem 4.2. *Suppose $\Xi(t)$ is a vector defined by (17). Then, we can express the Itô integration of $\Xi(t)$ as:*

$$\int_0^t \Xi(s)dB(s) \simeq \mathrm{P}_S\,\Xi(t), \tag{26}$$

where P_S is the $(4n+1)\times(4n+1)$ stochastic operational integration matrix for the FDHFs that are given as:

$$\mathrm{P}_S = \begin{pmatrix} 0 & \beta_{0,1}(h) & \beta_{0,2}(h) & \beta_{0,3}(h) & \beta_{0,4}(h) & \beta_{0,4}(h) & \cdots & \beta_{0,4}(h) \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_4 & \eta_4 & \cdots & \eta_4 \\ \eta_1 & \eta_5 & \eta_2 & \eta_3 & \eta_4 & \eta_4 & \cdots & \eta_4 \\ \eta_1 & \eta_5 & \eta_5 & \eta_2 & \eta_3 & \eta_4 & \cdots & \eta_4 \\ \eta_1 & \eta_5 & \eta_5 & \eta_5 & \eta_2 & \eta_3 & \cdots & \eta_4 \\ \eta_1 & \eta_5 & \eta_5 & \eta_5 & \eta_5 & \eta_2 & \cdots & \eta_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_1 & \eta_5 & \eta_5 & \eta_5 & \eta_5 & \eta_5 & \cdots & \eta_2 \end{pmatrix}_{(4n+1)\times(4n+1)}, \tag{27}$$

where

$$\eta_2 = \begin{pmatrix} B(4k-3)+\beta_{4k-3,4k-3} & \beta_{4k-3,4k-2} & \beta_{4k-3,4k-1} & \beta_{4k-3,4k} \\ \varsigma_{4k-2,4k-3} & B(4k-2)+\varsigma_{4k-2,4k-2} & \varsigma_{4k-2,4k-1} & \varsigma_{4k-2,4k} \\ \varphi_{4k-1,4k-3} & \varphi_{4k-1,4k-2} & B(4k-1)+\varphi_{4k-1,4k-1} & \varphi_{4k-1,4k} \\ \gamma_{4k,4k-3} & \gamma_{4k,4k-2} & \gamma_{4k,4k-1} & B(4k)+\gamma_{4k,4k} \end{pmatrix},$$

$\eta_3 = \begin{pmatrix} \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} \\ \varsigma_{4k-2,4k} & \varsigma_{4k-2,4k} & \varsigma_{4k-2,4k} & \varsigma_{4k-2,4k} \\ \varphi_{4k-1,4k} & \varphi_{4k-1,4k} & \varphi_{4k-1,4k} & \varphi_{4k-1,4k} \\ \gamma_{4k,4k+1} & \gamma_{4k,4k+2} & \gamma_{4k,4k+3} & \gamma_{4k,4k+4} \end{pmatrix}$, $\eta_4 = \begin{pmatrix} \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} & \beta_{4k-3,4k} \\ \varsigma_{4k-2,4k} & \varsigma_{4k-2,4k} & \varsigma_{4k-2,4k} & \varsigma_{4k-2,4k} \\ \varphi_{4k-1,4k} & \varphi_{4k-1,4k} & \varphi_{4k-1,4k} & \varphi_{4k-1,4k} \\ \gamma_{4k,4k+4} & \gamma_{4k,4k+4} & \gamma_{4k,4k+4} & \gamma_{4k,4k+4} \end{pmatrix}$, $\eta_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\eta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$, and $\beta_{0,j} = \frac{-1}{12h^4}\int_0^{jh}\left(2s^3 - 15hs^2 + 35h^2 s - 25h^3\right)B(s)ds, \quad j = 1,2,3,4,$ if $k = 1,2,\ldots,n$

$$\beta_{4k-3,j} = \frac{-1}{6h^4}\int_{(4k-4)h}^{jh}\left(4s^3 - 3(16k-7)hs^2 + 2(96k^2-84k+14)h^2 s\right.$$
$$\left. -(256k^3 - 336k^2 + 112k - 8)h^3\right)B(s)ds, \quad j = 4k-3, 4k-2, 4k-1, 4k,$$

$$\varsigma_{4k-2,j} = \frac{-1}{2h^4}\int_{(4k-4)h}^{jh} \left(2s^3 - 12(2k-1)hs^2 + (96k^2-96k+19)h^2 s \right.$$
$$\left. -(128k^3-192k^2+76k-6)h^3\right) B(s)ds, \quad j = 4k-3, 4k-2, 4k-1, 4k,$$

$$\varphi_{4k-1,j} = \frac{1}{6h^4}\int_{(4k-4)h}^{jh} \left(4s^3 - 3(16k-9)hs^2 + 2(96k^2-108k+26)h^2 s \right.$$
$$\left. -(256k^3-432k^2+208k-24)h^3\right) B(s)ds, \quad j = 4k-3, 4k-2, 4k-1, 4k,$$

if $k = 1, 2, \dots, n-1$,

$$\gamma_{4k,j} = \frac{-1}{12h^4}\int_{4(k-1)h}^{jh} \left(2s^3 - 3(8k-5)hs^2 + (96k^2-120k+35)h^2 s \right.$$
$$\left. -(128k^3-240k^2+140k-25)h^3\right) B(s)ds, \quad j = 4k-3, 4k-2, 4k-1, 4k,$$

and

$$\gamma_{4k,j} = \frac{-1}{12h^4}\left(\left(\int_{4(k-1)h}^{4kh} (2s^3 - 3(8k-5)hs^2 + (96k^2-120k+35)h^2 s \right.\right.$$
$$- (128k^3-240k^2+140k-25)h^3) + \int_{4kh}^{jh} (2s^3 - 3(8k+5)hs^2 + (96k^2+120k+35)h^2 s$$
$$\left.\left. -(128k^3+240k^2+140k+25)h^3)\right) B(s)ds, \quad j = 4k+1, 4k+2, 4k+3, 4k+4.\right.$$

Proof. Using integration by parts and the definitions of $\xi_i(t)$, $i = 0, 1, \dots, 4n$, we can obtain the following results:

$$\int_0^t \xi_0(s)\, dB(s) = \xi_0(t)B(t) - \int_0^t \xi'_0(s)B(s)ds, \tag{28}$$

where

$$\xi'_0(t) = \begin{cases} \frac{1}{12h^4}\left(2t^3 - 15ht^2 + 35h^2 t - 25h^3\right), & 0 < t < 4h, \\ 0, & \text{otherwise.} \end{cases}$$

When we expand relation (28) in terms of FDHFs, we get:

$$\int_0^t \xi_0(s)\, dB(s) \simeq \sum_{j=0}^{4n} \varpi_{0j}\, \xi_j(t),$$

where

$$\varpi_{0j} = \int_0^{jh} \xi_0(s)dB(s)$$
$$= \xi_0(jh)B(jh) - \frac{1}{12h^4}\int_0^{jh}\left(2s^3 - 15hs^2 + 35h^2 s - 25h^3\right) B(s)ds. \tag{29}$$

Thence, from the relations (14) and (29), we obtain:

$$\varpi_{0j} = \begin{cases} 0, & j = 0, \\ \frac{-1}{12h^4}\int_0^{h}\left(2s^3 - 15hs^2 + 35h^2 s - 25h^3\right) B(s)ds, & j = 1, \\ \frac{-1}{12h^4}\int_0^{2h}\left(2s^3 - 15hs^2 + 35h^2 s - 25h^3\right) B(s)ds, & j = 2, \\ \frac{-1}{12h^4}\int_0^{3h}\left(2s^3 - 15hs^2 + 35h^2 s - 25h^3\right) B(s)ds, & j = 3, \\ \frac{-1}{12h^4}\int_0^{4h}\left(2s^3 - 15hs^2 + 35h^2 s - 25h^3\right) B(s)ds, & j \geq 4, \end{cases}$$

if $k = 1, 2, \dots, n$. So

$$\int_0^t \xi_{4k-1}(s)dB(s) = \int_{(4k-4)h}^t \xi_{4k-1}(s)dB(s)$$
$$= \xi_{4k-1}(t)B(t) - \int_{(4k-4)h}^t \xi'_{4k-1}(s)B(s)ds, \tag{30}$$

where

$$\xi'_{4k-1}(t) = \begin{cases} \frac{-(4t^3 - 3(16k-9)ht^2 + 2(96k^2-108k+26)h^2 t - (256k^3-432k^2+208k-24)h^3)}{6h^4}, & (4k-4)h < t < 4kh, \\ 0, & \text{otherwise.} \end{cases}$$

When we expand relation (30) in terms of FDHFs, we get:

$$\int_0^t \xi_{4k-1}(s)\,dB(s) \simeq \sum_{j=0}^{4n} \varpi_{(4k-1)j}\,\xi_j(t),$$

where

$$\begin{aligned}\varpi_{(4k-1)j} &= \int_{(4k-4)h}^{jh} \xi_{(4k-1)}(s)dB(s)\\ &= \xi_{(4k-1)}(jh)B(jh) + \frac{1}{6h^4}\int_{(4k-4)h}^{jh} \big(4s^3 - 3(16k-9)hs^2 + 2(96k^2-108k+26)h^2 s\\ &\qquad -(256k^3-432k^2+208k-24)h^3\big)\,B(s)ds.\end{aligned} \tag{31}$$

Thence, from the relations (14) and (31), we obtain:

$$\varpi_{(4k-1)j} = \begin{cases} 0, & j \le 4k-4,\\ \frac{1}{6h^4}\int_{(4k-4)h}^{(4k-3)h}\big(4s^3-3(16k-9)hs^2+2(96k^2-108k+26)h^2s\\ \quad -(256k^3-432k^2+208k-24)h^3\big)\,B(s)ds, & j=4k-3,\\ \frac{1}{6h^4}\int_{(4k-4)h}^{(4k-2)h}\big(4s^3-3(16k-9)hs^2+2(96k^2-108k+26)h^2s\\ \quad -(256k^3-432k^2+208k-24)h^3\big)\,B(s)ds, & j=4k-2,\\ B((4k-1)h)+\frac{1}{6h^4}\int_{(4k-4)h}^{(4k-1)h}\big(4s^3-3(16k-9)hs^2+2(96k^2-108k+26)h^2s\\ \quad -(256k^3-432k^2+208k-24)h^3\big)\,B(s)ds, & j=4k-1,\\ \frac{1}{6h^4}\int_{(4k-4)h}^{4kh}\big(4s^3-3(16k-9)hs^2+2(96k^2-108k+26)h^2s\\ \quad -(256k^3-432k^2+208k-24)h^3\big)\,B(s)ds, & j\ge 4k.\end{cases}$$

Similarly, for the computation of $\int_0^t \xi_{4k-2}(s)dB(s)$, we obtain:

$$\begin{aligned}\int_0^t \xi_{4k-2}(s)dB(s) &= \int_{(4k-4)h}^t \xi_{4k-2}(s)dB(s)\\ &= \xi_{4k-2}(t)B(t) - \int_{(4k-4)h}^t {\xi'}_{4k-2}(s)B(s)ds,\end{aligned} \tag{32}$$

where

$${\xi'}_{4k-2}(t) = \begin{cases} \frac{(2t^3-12(2k-1)ht^2+(96k^2-96k+19)h^2t-(128k^3-192k^2+76k-6)h^3)}{2h^4}, & (4k-4)h < t < 4kh,\\ 0, & \text{otherwise}.\end{cases}$$

When we expand relation (32) in terms of FDHFs, we get:

$$\int_0^t \xi_{4k-2}(s)\,dB(s) \simeq \sum_{j=0}^{4n} \varpi_{(4k-2)j}\,\xi_j(t),$$

where

$$\begin{aligned}\varpi_{(4k-2)j} &= \int_{(4k-4)h}^{jh} \xi_{(4k-2)}(s)dB(s)\\ &= \xi_{(4k-2)}(jh)B(jh) - \frac{1}{2h^4}\int_{(4k-4)h}^{jh}\big(2s^3-12(2k-1)hs^2+(96k^2-96k+19)h^2s\\ &\qquad -(128k^3-192k^2+76k-6)h^3\big)\,B(s)ds.\end{aligned} \tag{33}$$

Based on the relations (14) and (33), we obtain:

$$\varpi_{(4k-2)j} = \begin{cases} 0, & j \le 4k-4,\\ \frac{-1}{2h^4}\int_{(4k-4)h}^{(4k-3)h}\big(2s^3-12(2k-1)hs^2+(96k^2-96k+19)h^2s\\ \quad -(128k^3-192k^2+76k-6)h^3\big)\,B(s)ds, & j=4k-3,\\ B((4k-2)h)-\frac{1}{2h^4}\int_{(4k-4)h}^{(4k-2)h}\big(2s^3-12(2k-1)hs^2+(96k^2-96k+19)h^2s\\ \quad -(128k^3-192k^2+76k-6)h^3\big)\,B(s)ds, & j=4k-2,\\ \frac{-1}{2h^4}\int_{(4k-4)h}^{(4k-1)h}\big(2s^3-12(2k-1)hs^2+(96k^2-96k+19)h^2s\\ \quad -(128k^3-192k^2+76k-6)h^3\big)\,B(s)ds, & j=4k-1,\\ \frac{-1}{2h^4}\int_{(4k-4)h}^{4kh}\big(2s^3-12(2k-1)hs^2+(96k^2-96k+19)h^2s\\ \quad -(128k^3-192k^2+76k-6)h^3\big)\,B(s)ds, & j\ge 4k.\end{cases}$$

Also, for the computation of $\int_0^t \xi_{4k-3}(s)dB(s)$, we get:

$$\int_0^t \xi_{4k-3}(s)dB(s) = \int_{(4k-4)h}^t \xi_{4k-3}(s)dB(s)$$
$$= \xi_{4k-3}(t)B(t) - \int_{(4k-4)h}^t \xi'_{4k-3}(s)B(s)ds, \tag{34}$$

where

$$\xi'_{4k-3}(t) = \begin{cases} \frac{\left(4t^3-3(16k-7)ht^2+2(96k^2-84k+14)h^2t-(256k^3-336k^2+112k-8)h^3\right)}{6h^4}, & (4k-4)h < t < 4kh, \\ 0, & \text{otherwise.} \end{cases}$$

When we expand relation (34) in terms of FDHFs, we get:

$$\int_0^t \xi_{4k-3}(s)dB(s) \simeq \sum_{j=0}^{N} \varpi_{(4k-3)j}\, \xi_j(t),$$

where

$$\varpi_{(4k-3)j} = \int_{(4k-4)h}^{jh} \xi_{(4k-3)}(s)dB(s)$$
$$= \xi_{(4k-3)}(jh)B(jh) - \frac{1}{6h^4}\int_{(4k-4)h}^{jh} \left(4s^3 - 3(16k-7)hs^2 + 2(96k^2-84k+14)h^2 s \right.$$
$$\left. -(256k^3-336k^2+112k-8)h^3\right) B(s)ds. \tag{35}$$

Thence, from the relations (14) and (35), we obtain:

$$\varpi_{(4k-3)j} = \begin{cases} 0, & j \le 4k-4, \\ B((4k-3)h) - \frac{1}{6h^4}\int_{(4k-4)h}^{(4k-3)h}\left(4s^3 - 3(16k-7)hs^2 + 2(96k^2-84k+14)h^2 s \right. \\ \quad \left. -(256k^3-336k^2+112k-8)h^3\right) B(s)ds, & j = 4k-3, \\ \frac{-1}{6h^4}\int_{(4k-4)h}^{(4k-2)h}\left(4s^3 - 3(16k-7)hs^2 + 2(96k^2-84k+14)h^2 s \right. \\ \quad \left. -(256k^3-336k^2+112k-8)h^3\right) B(s)ds, & j = 4k-2, \\ \frac{-1}{6h^4}\int_{(4k-4)h}^{(4k-1)h}\left(4s^3 - 3(16k-7)hs^2 + 2(96k^2-84k+14)h^2 s \right. \\ \quad \left. -(256k^3-336k^2+112k-8)h^3\right) B(s)ds, & j = 4k-1, \\ \frac{-1}{6h^4}\int_{(4k-4)h}^{4kh}\left(4s^3 - 3(16k-7)hs^2 + 2(96k^2-84k+14)h^2 s \right. \\ \quad \left. -(256k^3-336k^2+112k-8)h^3\right) B(s)ds, & j \ge 4k. \end{cases}$$

Now, if $k = 1, 2, \ldots, n-1$. So

$$\int_0^t \xi_{4k}(s)dB(s) = \int_{4(k-1)h}^t \xi_{4k}(s)dB(s)$$
$$= \xi_{4k}(t)B(t) - \int_{4(k-1)h}^t \xi'_{4k}(s)B(s)ds, \tag{36}$$

where

$$\xi'_{4k}(t) = \begin{cases} \frac{2t^3-3(8k-5)ht^2+(96k^2-120k+35)h^2t-(128k^3-240k^2+140k-25)h^3}{12h^4}, & 4(k-1)h < t < 4kh, \\ \frac{2t^3-3(8k+5)ht^2+(96k^2+120k+35)h^2t-(128k^3+240k^2+140k+25)h^3}{12h^4}, & 4kh < t < 4(k+1)h, \\ 0, & \text{otherwise.} \end{cases}$$

When we expand relation (36) in terms of FDHFs, we get:

$$\int_0^t \xi_{4k}(s)\, dB(s) \simeq \sum_{j=0}^{4n} \varpi_{4kj}\, \xi_j(t),$$

where

$$\varpi_{4kj} = \int_{4(k-1)h}^{jh} \xi_{4k}(s)dB(s)$$
$$= \xi_{4k}(jh)B(jh) - \frac{1}{12h^4}\int_{4(k-1)h}^{jh}\left(2s^3 - 3(8k-5)hs^2 + (96k^2-120k+35)h^2 s\right.$$
$$-(128k^3-240k^2+140k-25)h^3 + 2s^3 - 3(8k+5)hs^2$$
$$\left.+(96k^2+120k+35)h^2 s - (128k^3+240k^2+140k+25)h^3\right) B(s)ds. \tag{37}$$

Thence, from the relations (14) and (37), we obtain:

$$\varpi_{4kj}=\begin{cases}0, & j\le 4k-4,\\ \frac{-1}{12h^4}\int_{4(k-1)h}^{(4k-3)h}\big(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3\big)\,B(s)ds, & j=4k-3,\\ \frac{-1}{12h^4}\int_{4(k-1)h}^{(4k-2)h}\big(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3\big)\,B(s)ds, & j=4k-2,\\ \frac{-1}{12h^4}\int_{4(k-1)h}^{(4k-1)h}\big(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3\big)\,B(s)ds, & j=4k-1,\\ B(4kh)-\frac{1}{12h^4}\int_{4(k-1)h}^{4kh}\big(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3\big)\,B(s)ds, & j=4k,\\ \frac{-1}{12h^4}\Big(\int_{4(k-1)h}^{4kh}(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3)+\int_{4kh}^{(4k+1)h}(2s^3-3(8k+5)hs^2\\ \quad+(96k^2+120k+35)h^2s-(128k^3+240k^2+140k+25)h^3)\Big)\,B(s)ds, & j=4k+1,\\ \frac{-1}{12h^4}\Big(\int_{4(k-1)h}^{4kh}(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3)+\int_{4kh}^{(4k+2)h}(2s^3-3(8k+5)hs^2\\ \quad+(96k^2+120k+35)h^2s-(128k^3+240k^2+140k+25)h^3)\Big)\,B(s)ds, & j=4k+2,\\ \frac{-1}{12h^4}\Big(\int_{4(k-1)h}^{4kh}(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3)+\int_{4kh}^{(4k+3)h}(2s^3-3(8k+5)hs^2\\ \quad+(96k^2+120k+35)h^2s-(128k^3+240k^2+140k+25)h^3)\Big)\,B(s)ds, & j=4k+3,\\ \frac{-1}{12h^4}\Big(\int_{4(k-1)h}^{4kh}(2s^3-3(8k-5)hs^2+(96k^2-120k+35)h^2s\\ \quad-(128k^3-240k^2+140k-25)h^3)+\int_{4kh}^{4(k+1)h}(2s^3-3(8k+5)hs^2\\ \quad+(96k^2+120k+35)h^2s-(128k^3+240k^2+140k+25)h^3)\Big)\,B(s)ds, & j\ge 4(k+1).\end{cases}$$

Finally, for the computation of $\int_0^t \xi_{4n}(s)\,dB(s)$, we get:

$$\begin{aligned}\int_0^t \xi_{4n}(s)dB(s)&=\int_{T-4h}^{t}\xi_{4n}(s)\,dB(s)\\ &=\xi_{4n}(t)B(t)-\int_{T-4h}^{t}\xi'_{4n}(s)B(s)ds,\end{aligned}\tag{38}$$

where

$$\xi'_{4n}(t)=\begin{cases}\frac{(2t^3-3(2T-5h)t^2+(6T^2-30Th+35h^2)t-(2T^3-15T^2h+35Th^2-25h^3))}{12h^4}, & T-4h<t<T,\\ 0, & \text{otherwise.}\end{cases}$$

When we expand relation (38) in terms of FDHFs, we get:

$$\int_0^t \xi_{4n}(s)\,dB(s)\simeq\sum_{j=0}^{4n}\varpi_{4nj}\,\xi_j(t),$$

where

$$\begin{aligned}\varpi_{4nj}&=\int_{T-4h}^{jh}\xi_{4n}(s)dB(s)\\ &=\xi_{4n}(jh)B(jh)-\frac{1}{12h^4}\int_{T-4h}^{jh}(2s^3-3(2T-5h)s^2+(6T^2-30Th+35h^2))s\\ &\qquad-(2T^3-15T^2h+35Th^2-25h^3)B(s)ds.\end{aligned}\tag{39}$$

Thence, from the relations (14) and (39), we obtain:

$$\varpi_{4nj} = \begin{cases} 0, & j \le 4n-4, \\ \frac{-1}{12h^4}\int_{T-4h}^{T-3h} (2s^3 - 3(2T-5h)s^2 + (6T^2 - 30Th + 35h^2))s \\ \qquad -(2T^3 - 15T^2h + 35Th^2 - 25h^3)B(s)ds, & j = 4n-3, \\ \frac{-1}{12h^4}\int_{T-4h}^{T-2h} (2s^3 - 3(2T-5h)s^2 + (6T^2 - 30Th + 35h^2))s \\ \qquad -(2T^3 - 15T^2h + 35Th^2 - 25h^3)B(s)ds, & j = 4n-2, \\ \frac{-1}{12h^4}\int_{T-4h}^{T-h} (2s^3 - 3(2T-5h)s^2 + (6T^2 - 30Th + 35h^2))s \\ \qquad -(2T^3 - 15T^2h + 35Th^2 - 25h^3)B(s)ds, & j = 4n-1, \\ B(T) - \frac{1}{12h^4}\int_{T-4h}^{T} (2s^3 - 3(2T-5h)s^2 + (6T^2 - 30Th + 35h^2))s \\ \qquad -(2T^3 - 15T^2h + 35Th^2 - 25h^3)B(s)ds, & j = 4n. \end{cases}$$

As a result, the proof has been completed.

5. Description of the proposed computational method

This section focuses on designing a computational method based on FDHFs to solve NSDEs. This method converts the NSDE (7) problem into a nonlinear algebraic system, which is then numerically solved.

First, the functions $g(t)$, $L_1(t,s)\sigma(y(s))$, and $L_2(t,s)\phi(y(s))$ must be approximated as follows:

$$g(t) \simeq g_{4n}(t) = \sum_{\kappa=0}^{4n} g_\kappa \xi_\kappa(t), \tag{40}$$

$$L_1(t,s)\sigma(y(s)) \simeq L_{1(4n)}(t,s)\sigma(y_{4n}(s)) = \sum_{\tau=0}^{4n}\sum_{\kappa=0}^{4n} \psi_{\kappa,\tau}\xi_\kappa(t)\xi_\tau(s), \tag{41}$$

and

$$L_2(t,s)\phi(y(s)) \simeq L_{2(4n)}(t,s)\phi(y_{4n}(s)) = \sum_{\tau=0}^{4n}\sum_{\kappa=0}^{4n} \omega_{\kappa,\tau}\xi_\kappa(t)\xi_\tau(s), \tag{42}$$

where

$$\psi_{i,j} = L_1(ih, jh)\sigma(y(jh)), \qquad \forall i,j = 0,1,\dots,4n, \tag{43}$$

$$\omega_{i,j} = L_2(ih, jh)\phi(y(jh)), \qquad \forall i,j = 0,1,\dots,4n. \tag{44}$$

Second, we assume the solution of Eq. (7) has the following form:

$$y(t) = \sum_{\kappa=0}^{4n} y_\kappa \xi_\kappa(t). \tag{45}$$

Hence, the NSDE (7) is transformed into

$$\sum_{\kappa=0}^{4n} y_\kappa\xi_\kappa(t) = \sum_{\kappa=0}^{4n} g_\kappa\xi_\kappa(t) + \sum_{\tau=0}^{4n}\sum_{\kappa=0}^{4n} L_1(\kappa h, \tau h)\,\sigma(y_\tau)\,\xi_\kappa(t)\int_0^t \xi_\tau(s)\,ds$$
$$+ \sum_{\tau=0}^{4n}\sum_{\kappa=0}^{4n} L_2(\kappa h, \tau h)\,\phi(y_\tau)\,\xi_\kappa(t)\int_0^t \xi_\tau(s)dB(s). \tag{46}$$

Then at the ith node point (ih), Eq. (46) becomes:

$$\sum_{\kappa=0}^{4n} y_\kappa\xi_\kappa(ih) = \sum_{\kappa=0}^{4n} g_\kappa\xi_\kappa(ih) + \sum_{\tau=0}^{4n}\sum_{\kappa=0}^{4n} L_1(\kappa h, \tau h)\,\sigma(y_\tau)\,\xi_\kappa(ih)\int_0^{ih} \xi_\tau(s)ds$$
$$+ \sum_{\tau=0}^{4n}\sum_{\kappa=0}^{4n} L_2(\kappa h, \tau h)\,\phi(y_\tau)\xi_\kappa(ih)\int_0^{ih} \xi_\tau(s)dB(s), \tag{47}$$

where $i = 0,1,2,\dots,4n$.

By employing Eq. (14) and simplifying the system given in Eq. (47), we obtain:

$$y_i = g_i + \sum_{\tau=0}^{4n} L_1(ih,\tau h)\sigma(y_\tau)\int_0^{ih} \xi_\tau(s)ds + \sum_{\tau=0}^{4n} L_2(ih,\tau h)\phi(y_\tau)\int_0^{ih} \xi_\tau(s)dB(s). \tag{48}$$

By using Eqs. (24) and (26) together, we have reached the following:

$$y_i = g_i + \sum_{\tau=0}^{4n} \left(L_1(ih,\tau h)\sigma(y_\tau)\right)\mathrm{P}_{i,\tau} + \sum_{\tau=0}^{4n} \left(L_2(ih,\tau h)\phi(y_\tau)\right)\mathrm{P}^S_{i,\tau}, \quad i = 0,1,2,\dots,4n, \tag{49}$$

where P and P_S are the operational matrix of integration and stochastic operational matrix of integration given in Eqs. (25) and (27), respectively.

Finally, the approximate solution of Eq. (7) can be obtained using Eq. (45) after solving the nonlinear system (49) using one of the numerical methods and finding the unknown coefficients, $y_\kappa, \forall \kappa = 0, 1, 2, \ldots, 4n$.

6. Error analysis

This section focuses on the error analysis of the proposed approach for solving NSDE. First, we begin by defining

$$\|y\| = \sup_{t \in \Omega} |y(t)| . \tag{50}$$

Theorem 6.1. *Suppose that $t_\kappa = \kappa h$, $\kappa = 0, 1, \ldots, 4n$, $g(t) \in C^5(\Omega)$ and $g_{4n}(t) = \sum_{\kappa=0}^{4n} g(t_\kappa)\xi_\kappa(t)$ be the FDHFs expanded of $g(t)$. As well, suppose that $e(t) = g(t) - g_{4n}(t)$, $t \in \Omega$. Next, we have*

$$\|g(t) - g_{4n}(t)\| \leq \lambda h^5, \tag{51}$$

where λ is a constant number, and therefore $\|e(t)\| \simeq O(h^5)$.

Proof. Suppose that

$$e_i(t) = \begin{cases} g(t) - g_{4n}(t), & t \in V_i, \\ 0, & t \in \Omega - V_i, \end{cases}$$

where $V_i = \{t | ih \leq t \leq (i+4)h, \quad h = \frac{T}{4n}\}$, $i = 0, 4, 8, \ldots, 4n-4$. Then, we get:

$$e_i(t) = g(t) - g_{4n}(t) = g(t) - \sum_{\kappa=0}^{4n} g(\kappa h)\xi_\kappa(t),$$

$$e_i(t) = g(t) - [g(ih)\xi_i(t) + g((i+1)h)\xi_{i+1}(t) + g((i+2)h)\xi_{i+2}(t) + g((i+3)h)\xi_{i+3}(t) + g((i+4)h)\xi_{i+4}(t)].$$

When the fourth-degree interpolation error is used, then we have [78–80]

$$e_i(t) = \frac{(t-ih)(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)(t-(i+4)h)}{120} \cdot \frac{d^5 g(\chi_i)}{dt^5},$$

where $\chi_i \in (ih, (i+4)h)$.

Suppose that $\varphi(t) = (t-ih)(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)(t-(i+4)h)$. Because V_i is compacted and $\varphi(t)$ is a continuous function, we have:

$$\sup_{t \in V_i} |\varphi(t)| = \max_{t \in V_i} |\varphi(t)| = 3.6314h^5.$$

Therefore, we have

$$|e_i(t)| \leq \frac{1}{120} |\varphi(t)| \left| \frac{d^5 g(\chi_i)}{dt^5} \right| .$$

As a result, we have

$$\|e(t)\| = \max_{i=0,4,\ldots,4n-4} \sup_{t \in V_i} |e_i(t)| \leq \max_{i=0,4,\ldots,4n-4} 0.03026h^5 \left| \frac{d^5 g(\chi_i)}{dt^5} \right| .$$

After that, there is $\varepsilon \in \{0, 4, \ldots, 4n-4\}$, we get:

$$\|e(t)\| \leq \max_{i=0,4,\ldots,4n-4} 0.03026h^5 \left| \frac{d^5 g(\chi_i)}{dt^5} \right| = 0.03026h^5 \left| \frac{d^5 g(\chi_\varepsilon)}{dt^5} \right| .$$

Lastly, by using the relation (50), we obtain

$$\|e(t)\| \leq 0.03026h^5 \left| \frac{d^5 g(\chi_\varepsilon)}{dt^5} \right| \leq 0.03026h^5 \left\| \frac{d^5 g(t)}{dt^5} \right\| \leq \lambda h^5. \tag{52}$$

According to the relation (52), we get

$$\|e(t)\| \simeq O(h^5).$$

Eventually, the proof was completed.

Theorem 6.2. *Assume that $L_\tau(t,s) \in C^5(\Omega\times\Omega)$, and $e(t,s) = L_\tau(t,s) - L_{\tau(4n)}(t,s)$, $(t,s)\in D = (\Omega\times\Omega)$ be the truncation error where $L_{\tau(4n)}(t,s) = \sum_{i=0}^{4n}\sum_{j=0}^{4n} L_\tau(ih,jh)\xi_i(t)\xi_j(s)$ is the FDHFs approximate of $L_\tau(t,s)$. Following that, we get*

$$\left\| L_\tau(t,s) - L_{\tau(4n)}(t,s)\right\| \le \lambda_\tau h^5, \tag{53}$$

where λ_τ, $\tau = 1,2$ are constant numbers, and therefore $\|e(t,s)\| \simeq O(h^5)$.

Proof. Suppose that

$$e_{qr}(t,s) = \begin{cases} L_\tau(t,s) - L_{\tau(4n)}(t,s), & (t,s)\in V_{qr},\\ 0, & (t,s)\in D - V_{qr},\end{cases}$$

where $V_{qr} = \{(t,s)|qh \le t \le (q+4)h,\ rh\le s\le (r+4)h,\ h = \frac{T}{4n}\}$, $q,r = 0,4,8,\dots,4n-4$. Then, we get:

$$e_{qr}(t,s) = L_\tau(t,s) - L_{\tau(4n)}(t,s) = L_\tau(t,s) - \sum_{i=0}^{4n}\sum_{j=0}^{4n} L_\tau(ih,jh)\xi_i(t)\xi_j(s),$$

$$\begin{aligned} e_{qr}(t,s) = L_\tau(t,s) - [&L_\tau(qh,rh)\xi_q(t)\xi_r(s) + L_\tau(qh,(r+1)h)\xi_q(t)\xi_{r+1}(s) + \cdots\\ &+ L_\tau(qh,(r+4)h)\xi_q(t)\xi_{r+4}(s) + \cdots\\ &+ L_\tau((q+4)h,(r+4)h)\xi_{q+4}(t)\xi_{r+4}(s)].\end{aligned}$$

When the fourth-degree interpolation error is used, then we have [81]:

$$\begin{aligned} e_{qr}(t,s) = &\frac{(t-qh)(t-(q+1)h)(t-(q+2)h)(t-(q+3)h)(t-(q+4)h)}{120}\cdot\frac{\partial^5 L_\tau(\chi_q,s)}{\partial t^5}\\ &+\frac{(s-rh)(s-(r+1)h)(s-(r+2)h)(s-(r+3)h)(s-(r+4)h)}{120}\cdot\frac{\partial^5 L_\tau(t,\eta_r)}{\partial s^5}\\ &-\frac{(t-qh)(t-(q+1)h)\dots(t-(q+4)h)(s-rh)\dots(s-(r+4)h)}{14400}\cdot\frac{\partial^{10}L_\tau(\bar\chi_q,\bar\eta_r)}{\partial t^5\partial s^5},\end{aligned}$$

where $\chi_q,\bar\chi_q \in (qh,(q+4)h)$ and $\eta_r,\bar\eta_r\in(rh,(r+4)h)$.

Suppose that $u(t) = (t-qh)(t-(q+1)h)(t-(q+2)h)(t-(q+3)h)(t-(q+4)h)$ as well as $v(s) = (s-rh)(s-(r+1)h)(s-(r+2)h)(s-(r+3)h)(s-(r+4)h)$. Thus, we get:

$$\begin{aligned}\left|e_{qr}(t,s)\right| \le &\frac{1}{120}|u(t)|\left|\frac{\partial^5 L_\tau(\chi_q,s)}{\partial t^5}\right| + \frac{1}{120}|v(s)|\left|\frac{\partial^5 L_\tau(t,\eta_r)}{\partial s^5}\right|\\ &+\frac{1}{14400}|u(t)|\,|v(s)|\left|\frac{\partial^{10}L_\tau(\bar\chi_q,\bar\eta_r)}{\partial t^5\partial s^5}\right|.\end{aligned}$$

Since $\sup_{t\in(qh,(q+4)h)}|u(t)| = 3.6314h^5$, and $\sup_{s\in(rh,(r+4)h)}|v(s)| = 3.6314h^5$, we obtain

$$\begin{aligned}\|e(t,s)\| &= \max_{\substack{q=0,4,\dots,4n-1\\ r=0,4,\dots,4n-1}}\sup_{(t,s)\in V_{qr}}\left|e_{qr}(t,s)\right|\\ &\le 0.03026h^5 \max_{\substack{q=0,4,\dots,4n-1\\ r=0,4,\dots,4n-1}}\sup_{(t,s)\in V_{qr}}\left(\left|\frac{\partial^5 L_\tau(\chi_q,s)}{\partial t^5}\right| + \left|\frac{\partial^5 L_\tau(t,\eta_r)}{\partial s^5}\right|\right.\\ &\qquad\left.+0.03026h^5\left|\frac{\partial^{10}L_\tau(\bar\chi_q,\bar\eta_r)}{\partial t^5\partial s^5}\right|\right).\end{aligned}$$

Then there are $\alpha,\iota\in\{0,4,\dots,4n-4\}$, where

$$\|e(t,s)\| \le 0.03026h^5\sup_{(t,s)\in V_{qr}}\left(\left|\frac{\partial^5 L_\tau(\chi_\alpha,s)}{\partial t^5}\right| + \left|\frac{\partial^5 L_\tau(t,\eta_\iota)}{\partial s^5}\right| + 0.03026h^5\left|\frac{\partial^{10}L_\tau(\bar\chi_\alpha,\bar\eta_\iota)}{\partial t^5\partial s^5}\right|\right).$$

Lastly, by using the relation (50), we obtain

$$\begin{aligned}\|e(t,s)\| &\le 0.03026h^5\left(\left\|\frac{\partial^5 L_\tau(t,s)}{\partial t^5}\right\| + \left\|\frac{\partial^5 L_\tau(t,s)}{\partial s^5}\right\| + 0.03026h^5\left\|\frac{\partial^{10}L_\tau(t,s)}{\partial t^5\partial s^5}\right\|\right)\\ &\le \lambda_\tau h^5.\end{aligned}\tag{54}$$

According to relation (54), we get:

$$\|e(t,s)\| \simeq O(h^5).$$

Finally, the proof is completed.

Theorem 6.3. *Suppose the exact solution is $y(t)$ and $y_{4n}(t)$ is the fourth-degree of the hat series approximate solution of Eq. (7). Furthermore, suppose*

(i) $\|\sigma(y(t))\| \leq Z_1, \qquad t \in \Omega,$

(ii) $\|\phi(y(t))\| \leq Z_2, \qquad t \in \Omega,$

(iii) $\|L_\tau(t,s)\| \leq \nu_\tau, \quad \tau = 1,2, \quad (t,s) \in \Omega \times \Omega,$

(iv) $1 - \nu_1 M_1 T - M_1 \lambda_1 T h^5 - \nu_2 M_2 Y - M_2 \lambda_2 Y h^5 > 0,$

where Z_1, Z_2, ν_1, ν_2, M_1, and M_2 are all positive constants, and

$$Y = \sup\{B(t); t \in \Omega\}.$$

Furthermore, suppose that the nonlinear terms $\sigma(y(t))$ and $\phi(y(t))$ fulfill Lipschitz's condition, i.e.,

$$\|\sigma(y(t)) - \sigma(y_{4n}(t))\| \leq M_1 \|y(t) - y_{4n}(t)\|, \tag{55}$$

$$\|\phi(y(t)) - \phi(y_{4n}(t))\| \leq M_2 \|y(t) - y_{4n}(t)\|. \tag{56}$$

Then

$$\|y(t) - y_{4n}(t)\| \leq \frac{(\lambda + Z_1\lambda_1 T + Z_2 \lambda_2 Y)h^5}{1 - \nu_1 M_1 T - M_1 \lambda_1 T h^5 - \nu_2 M_2 Y - M_2 \lambda_2 Y h^5}, \tag{57}$$

and $\|y(t) - y_{4n}(t)\| \simeq O(h^5)$.

Proof. Now, using fourth-degree hat functions to approximate Eq. (7), we obtain:

$$y_{4n}(t) = g_{4n}(t) + \int_0^t L_{1(4n)}(t,s)\sigma(y_{4n}(s))ds + \int_0^t L_{2(4n)}(t,s)\phi(y_{4n}(s))dB(s), \quad t \in \Omega. \tag{58}$$

Using norm properties and Eqs. (7) and (58), we have:

$$\begin{aligned}\|y(t) - y_{4n}(t)\| \leq \|g(t) - g_{4n}(t)\| &+ \int_0^t \left\|L_1(t,s)\sigma(y(s)) - L_{1(4n)}(t,s)\sigma(y_{4n}(s))\right\| ds \\ &+ \int_0^t \left\|L_2(t,s)\phi(y(s)) - L_{2(4n)}(t,s)\phi(y_{4n}(s))\right\| dB(s).\end{aligned}$$

Therefore,

$$\begin{aligned}\|y(t) - y_{4n}(t)\| \leq \|g(t) - g_{4n}(t)\| &+ \|t\| \left\|L_1(t,s)\sigma(y(s)) - L_{1(4n)}(t,s)\sigma(y_{4n}(s))\right\| \\ &+ \|B(t)\| \left\|L_2(t,s)\phi(y(s)) - L_{2(4n)}(t,s)\phi(y_{4n}(s))\right\|.\end{aligned}$$

It is clear that $\|t\| \leq T$, while noting that $Y = \|B\|$, we have

$$\begin{aligned}\|y(t) - y_{4n}(t)\| \leq \|g(t) - g_{4n}(t)\| &+ T \left\|L_1(t,s)\sigma(y(s)) - L_{1(4n)}(t,s)\sigma(y_{4n}(s))\right\| \\ &+ Y \left\|L_2(t,s)\phi(y(s)) - L_{2(4n)}(t,s)\phi(y_{4n}(s))\right\|.\end{aligned} \tag{59}$$

Now, according to Eqs. (53) and (55) and hypotheses (i) and (iii), we conclude that

$$\begin{aligned}&\left\|L_1(t,s)\sigma(y(s)) - L_{1(4n)}(t,s)\sigma(y_{4n}(s))\right\| \\ &\leq \|L_1(t,s)\| \|\sigma(y(t)) - \sigma(y_{4n}(t))\| + \left\|L_1(t,s) - L_{1(4n)}(t,s)\right\| \|\sigma(y(t)) - \sigma(y_{4n}(t))\| \\ &\qquad + \left\|L_1(t,s) - L_{1(4n)}(t,s)\right\| \|\sigma(y(t))\| \\ &\leq \nu_1 M_1 \|y(t) - y_{4n}(t)\| + M_1 \lambda_1 h^5 \|y(t) - y_{4n}(t)\| + Z_1 \lambda_1 h^5.\end{aligned} \tag{60}$$

By applying Eqs. (53) and (56) as well as hypotheses (ii) and (iii), we obtain

$$\begin{aligned}&\left\|L_2(t,s)\phi(y(s)) - L_{2(4n)}(t,s)\phi(y_{4n}(s))\right\| \\ &\leq \|L_2(t,s)\| \|\phi(y(t)) - \phi(y_{4n}(t))\| + \left\|L_2(t,s) - L_{2(4n)}(t,s)\right\| \|\phi(y(t)) - \phi(y_{4n}(t))\| \\ &\qquad + \left\|L_2(t,s) - L_{2(4n)}(t,s)\right\| \|\phi(y(t))\| \\ &\leq \nu_2 M_2 \|y(t) - y_{4n}(t)\| + M_2 \lambda_2 h^5 \|y(t) - y_{4n}(t)\| + Z_2 \lambda_2 h^5.\end{aligned} \tag{61}$$

Using Theorem 6.1 as well as Eqs. (59), (60), and (61), we can now obtain the following:

$$\begin{aligned}\|y(t) - y_{4n}(t)\| \leq \lambda h^5 &+ \nu_1 M_1 T \|y(t) - y_{4n}(t)\| + M_1 \lambda_1 T h^5 \|y(t) - y_{4n}(t)\| + Z_1 \lambda_1 T h^5 \\ &+ \nu_2 M_2 Y \|y(t) - y_{4n}(t)\| + M_2 \lambda_2 Y h^5 \|y(t) - y_{4n}(t)\| + Z_2 \lambda_2 Y h^5.\end{aligned} \tag{62}$$

Table 1
The absolute errors for Example 7.1.

Time	GHFs $m = 1,\ n = 24$	MHFs $m = 2,\ n = 12$	AHFs $m = 3,\ n = 8$	Present method $m = 4,\ n = 6$
0.125	2.8467999×10^{-4}	2.2180293×10^{-4}	1.7122739×10^{-4}	1.2466918×10^{-4}
0.250	3.8129877×10^{-4}	3.4086141×10^{-4}	3.4383876×10^{-4}	1.7466224×10^{-4}
0.375	1.0486267×10^{-3}	7.8732379×10^{-4}	7.1210596×10^{-4}	6.2833165×10^{-4}
0.500	1.2213487×10^{-3}	9.7799104×10^{-4}	1.1191313×10^{-3}	8.3703359×10^{-4}
0.625	1.3838426×10^{-3}	1.1955101×10^{-3}	1.1639424×10^{-3}	7.3948241×10^{-4}
0.750	1.8037148×10^{-3}	1.5201921×10^{-3}	1.6052961×10^{-3}	1.0010297×10^{-3}
0.875	1.6331990×10^{-3}	1.4177357×10^{-3}	1.4084576×10^{-3}	9.1134462×10^{-4}
1.000	1.8601470×10^{-3}	1.6951507×10^{-3}	1.6980067×10^{-3}	1.2607231×10^{-3}

According to Eq. (62) and the assumption (iv), we get:

$$\left\| y(t) - y_{4n}(t) \right\| \leq \frac{(\lambda + Z_1\lambda_1 T + Z_2\lambda_2 Y)h^5}{1 - \nu_1 M_1 T - M_1\lambda_1 T h^5 - \nu_2 M_2 Y - M_2\lambda_2 Y h^5}. \tag{63}$$

Moreover, we can deduce from Eq. (63) that:

$$\left\| y(t) - y_{4n}(t) \right\| \simeq O(h^5). \qquad \square$$

Through the above analysis, one can draw many advantages from the proposed method, as follows:

✓ Using FDHFs, problem under consideration is converted to a system of algebraic equations which can be easily solved.

✓ The proposed approach is convergent and the rate of convergence is $O(h^5)$.

✓ It is simple to calculate the unknown coefficients of the function's approximation based on this approach without integrating anything. Consequently, the proposed approach has a low computational expense.

✓ Because of the simplicity of FDHFs, this approach is a powerful mathematical tool to solve various kinds of equations with little additional works.

7. Numerical examples

Certain numerical examples have been solved to prove the accuracy, efficiency, as well as dependability of the method that is suggested and described in the preceding section. The Tables and Figures, contained below show the numerical results of this method. In comparison with the exact solution, the approximate solution demonstrates that this method's accuracy and applicability are both favorable. To prove the vantage of this method over other methods, we made a comparison between the mean error obtained from the proposed method and that obtained by the methods of hat basis functions, i.e., generalized hat basis functions (GHFs), modified hat basis functions (MHFs), and Adjustment hat basis functions (AHFs). We use the same length of subintervals (same $h = \frac{T}{mn}$, where m is the degree of polynomials that used in the definition of bases function in each method) to ensure a fair comparison between these base functions; that is, we use the same number of basis functions in each method. All computations in this paper have been performed using Maple 2020. The numerical results mentioned in the tables were obtained by running Maple software-based computer programs. The error in our suggested method was discussed and examined using the absolute error function, as shown below.

$$e(t) = \left| y(t) - y_{4n}(t) \right|,$$

where $y(t)$ and $y_{4n}(t)$ are the exact solution and the approximate solution of NSDE (7), respectively.

Example 7.1. We discuss the stochastic differential equation of option pricing, which is given by: [82]

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t). \tag{64}$$

The exact solution is $S(t) = S(0)e^{((\mu - \frac{\sigma^2}{2})t + \sigma B(t))}$, where $S(0) = 1$, $\mu = \frac{1}{10}$, and $\sigma = \frac{1}{20}$.

Table 1 shows a comparison of the absolute error obtained using the suggested method for $n = 6$ with the absolute errors of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$. As shown in this Table, the results obtained using our suggested method are better than those obtained by using the other hat functions methods. Moreover, Fig. 2 shows the exact and approximate solutions calculated using our method for $n = 6$. As can be seen, our suggested method for solving this problem is both accurate and fast and easy to implement.

Example 7.2. Take into consideration the following NSDE [83]:

$$dy(t) = -\frac{1}{900}y(t)(1 - y^2(t))dt + \frac{1}{30}(1 - y^2(t))dB(t), \qquad t \in [0, 1], \tag{65}$$

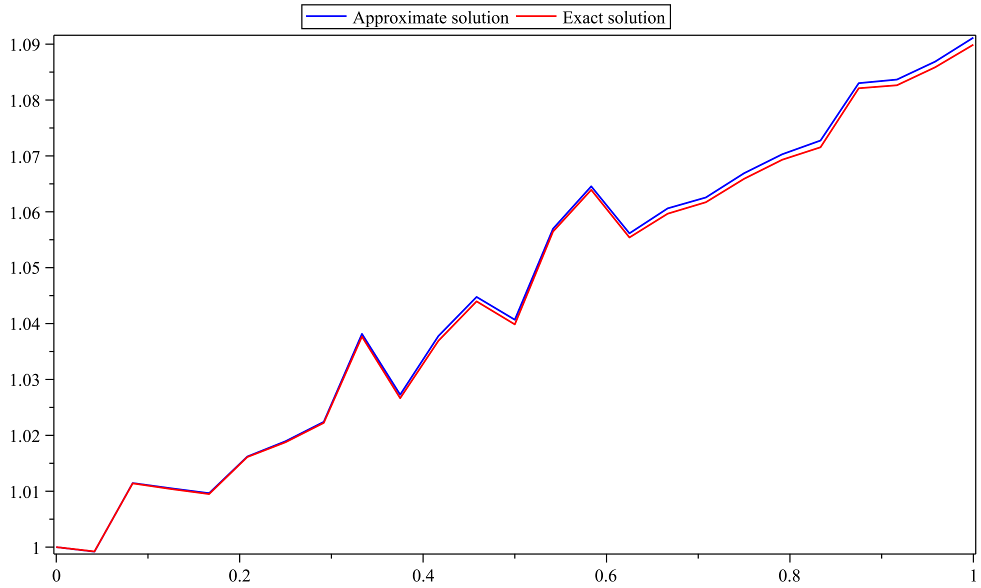

Fig. 2. The approximate and exact solution of Example 7.1 for $m = 4$ and $n = 6$.

Table 2
The absolute errors for Example 7.2.

Time	GHFs $m = 1$, $n = 24$	MHFs $m = 2$, $n = 12$	AHFs $m = 3$, $n = 8$	Present method $m = 4$, $n = 6$
0.125	2.7928130×10^{-5}	2.4814940×10^{-5}	2.0400690×10^{-5}	1.6936240×10^{-5}
0.250	3.0907780×10^{-5}	2.9044020×10^{-5}	2.7227830×10^{-5}	1.9024290×10^{-5}
0.375	9.5118200×10^{-5}	7.8680180×10^{-5}	6.8963700×10^{-5}	7.2041950×10^{-5}
0.500	9.4241180×10^{-5}	8.0186010×10^{-5}	7.7696120×10^{-5}	6.9532690×10^{-5}
0.625	1.1787564×10^{-4}	1.0837266×10^{-4}	9.4989170×10^{-5}	7.5862360×10^{-5}
0.750	1.3636557×10^{-4}	1.2117259×10^{-4}	1.0727745×10^{-4}	9.1880550×10^{-5}
0.875	1.3105202×10^{-4}	1.2087019×10^{-4}	1.0938854×10^{-4}	8.7081410×10^{-5}
1.000	1.3734810×10^{-4}	1.3061978×10^{-4}	1.1463472×10^{-4}	9.6373190×10^{-5}

where $y(0) = \frac{1}{10}$, and the exact solution to this equation is $y(t) = \tanh(\frac{1}{30}B(t) + \operatorname{arctanh}(\frac{1}{10}))$.

Table 2 shows a comparison of the absolute error obtained using the suggested method for $n = 6$ with the absolute errors of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$. As shown in this Table, the results obtained using our suggested method are better than those obtained by using the other hat functions methods. Moreover, Fig. 3 shows the exact and approximate solutions calculated using our method for $n = 6$. As can be seen, our suggested method for solving this problem is both accurate and fast and easy to implement.

Example 7.3. Take into consideration the following NSDE [84]:

$$dy(t) = -\frac{1}{16}\sin(y(t))\cos^3(y(t)) + \frac{1}{4}\cos^2(y(t))dB(t), \qquad t \in [0, 1], \tag{66}$$

For $y(0) = \frac{1}{20}$, the exact solution of this NSDE is $y(t) = \arctan(\frac{1}{4}B(t) + \tan(\frac{1}{20}))$.

Table 3 shows a comparison of the absolute error obtained using the suggested method for $n = 6$ with the absolute errors of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$. As shown in this Table, the results obtained using our suggested method are better than those obtained by using the other hat functions methods. Moreover, Fig. 4 shows the exact and approximate solutions calculated using our method for $n = 6$. As can be seen, our suggested method for solving this problem is both accurate and fast and easy to implement.

Example 7.4. Take into consideration the following NSDE [83]:

$$dy(t) = -\frac{1}{800}\tanh(y(t))sech^2(y(t)) + \frac{1}{20}sech(y(t))dB(t), \qquad t \in [0, 1], \tag{67}$$

For $y(0) = \frac{1}{10}$, the exact solution of this NSDE is $y(t) = \operatorname{arcsinh}(\frac{1}{20}B(t) + \sinh(\frac{1}{10}))$.

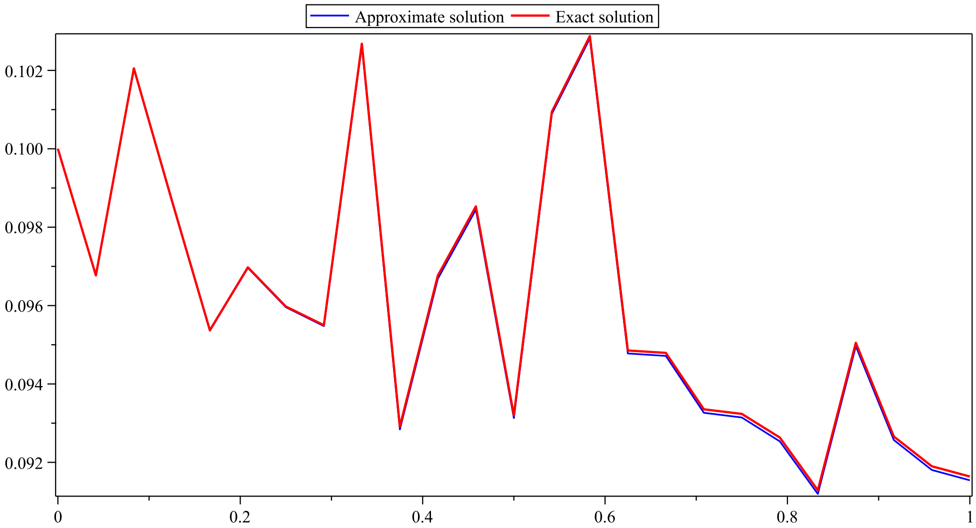

Fig. 3. The approximate and exact solution of Example 7.2 for $m = 4$ and $n = 6$.

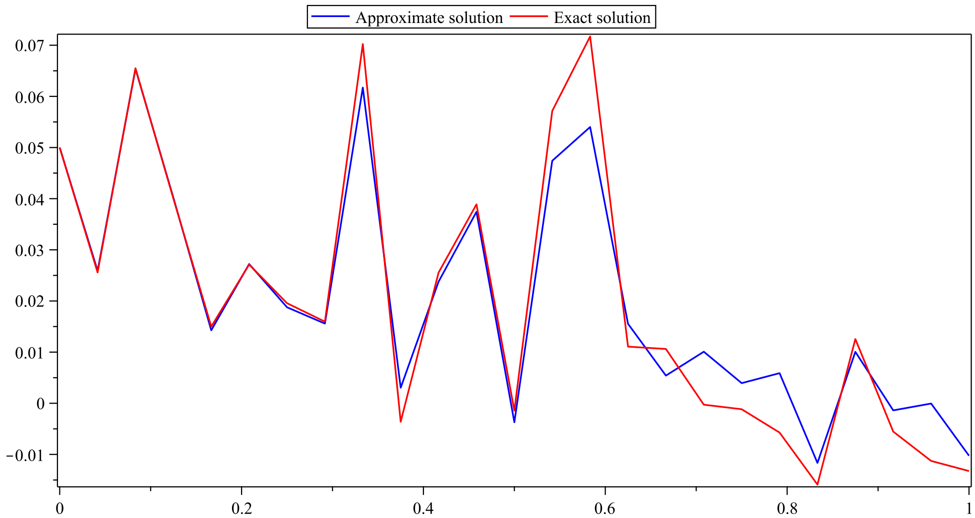

Fig. 4. The approximate and exact solution of Example 7.3 for $m = 4$ and $n = 6$.

Table 3
The absolute errors for Example 7.3.

Time	GHFs $m = 1, n = 24$	MHFs $m = 2, n = 12$	AHFs $m = 3, n = 8$	Present method $m = 4, n = 6$
0.125	8.9078580×10^{-5}	2.3702533×10^{-4}	1.9580883×10^{-4}	1.8082771×10^{-4}
0.250	1.7273269×10^{-3}	1.5964788×10^{-3}	2.2412381×10^{-3}	8.0152909×10^{-4}
0.375	3.1939696×10^{-3}	4.3901188×10^{-3}	3.7322939×10^{-3}	6.6506474×10^{-3}
0.500	2.5994360×10^{-3}	1.3179872×10^{-3}	6.6494761×10^{-3}	2.2658036×10^{-3}
0.625	1.8556922×10^{-3}	3.0877012×10^{-3}	1.5971960×10^{-5}	4.4612011×10^{-3}
0.750	2.5491245×10^{-3}	7.2882496×10^{-4}	9.3402864×10^{-3}	5.1032955×10^{-3}
0.875	6.3009202×10^{-3}	4.8949379×10^{-3}	7.8117225×10^{-3}	2.5012577×10^{-3}
1.000	3.0681918×10^{-3}	4.1396683×10^{-3}	2.0188410×10^{-3}	2.9954754×10^{-3}

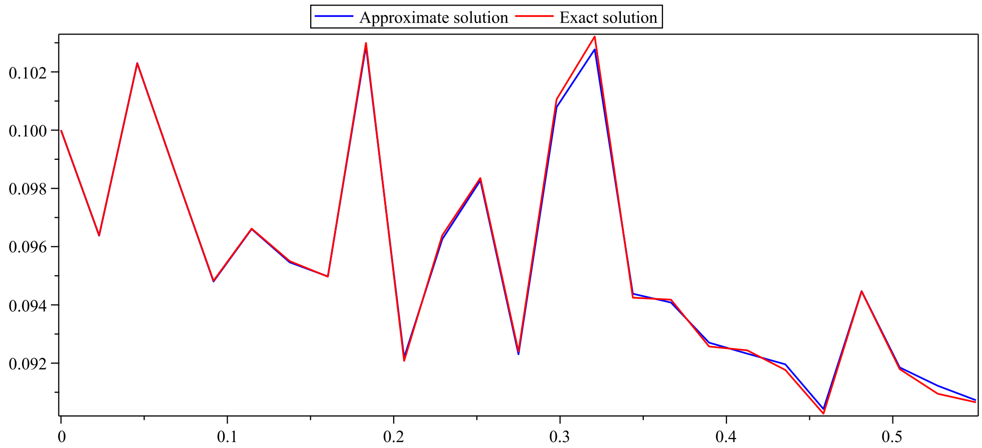

Fig. 5. The approximate and exact solution of Example 7.4 for $m = 4$ and $n = 6$.

Table 4
The absolute errors for Example 7.4.

Time	GHFs $m = 1,\ n = 24$	MHFs $m = 2,\ n = 12$	AHFs $m = 3,\ n = 8$	Present method $m = 4,\ n = 6$
0.125	3.9535570×10^{-5}	5.4358530×10^{-5}	5.0273440×10^{-5}	4.8772240×10^{-5}
0.250	4.9369540×10^{-5}	3.6166890×10^{-5}	1.0018047×10^{-4}	4.2602040×10^{-5}
0.375	5.7688676×10^{-4}	6.9481071×10^{-4}	6.3026366×10^{-4}	9.1584092×10^{-4}
0.500	1.9739477×10^{-4}	3.2351344×10^{-4}	1.9740436×10^{-4}	2.3194892×10^{-4}
0.625	8.2270372×10^{-4}	9.4418996×10^{-4}	6.3983681×10^{-4}	1.0784057×10^{-3}
0.750	5.4544166×10^{-4}	7.2560586×10^{-4}	1.1725651×10^{-4}	1.2989674×10^{-3}
0.875	3.2041392×10^{-4}	4.6260325×10^{-4}	1.8398321×10^{-4}	7.1106639×10^{-4}
1.000	1.2659096×10^{-3}	1.3799745×10^{-3}	7.5573765×10^{-4}	1.2488298×10^{-3}

Table 5
The absolute errors for Example 7.5.

Time	GHFs $m = 1,\ n = 24$	MHFs $m = 2,\ n = 12$	AHFs $m = 3,\ n = 8$	Present method $m = 4,\ n = 6$
0.125	5.0894620×10^{-7}	4.5374620×10^{-7}	3.7144620×10^{-7}	3.0964620×10^{-7}
0.250	5.6508804×10^{-7}	5.3378804×10^{-7}	5.0038804×10^{-7}	3.5068804×10^{-7}
0.375	1.7532806×10^{-6}	1.4530806×10^{-6}	1.2742806×10^{-6}	1.3310806×10^{-6}
0.500	1.7445638×10^{-6}	1.4887638×10^{-6}	1.4428638×10^{-6}	1.2845638×10^{-6}
0.625	2.1668476×10^{-6}	1.9985476×10^{-6}	1.7512476×10^{-6}	1.3930476×10^{-6}
0.750	2.5226843×10^{-6}	2.2489843×10^{-6}	1.9882843×10^{-6}	1.7057843×10^{-6}
0.875	2.4201219×10^{-6}	2.2434219×10^{-6}	2.0320219×10^{-6}	1.6117219×10^{-6}
1.000	2.5429527×10^{-6}	2.4334527×10^{-6}	2.1347527×10^{-6}	1.7932527×10^{-6}

Table 4 shows a comparison of the absolute error obtained using the suggested method for $n = 6$ with the absolute errors of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$. As shown in this Table, the results obtained using our suggested method are better than those obtained by using the other hat functions methods. Moreover, Fig. 5 shows the exact and approximate solutions calculated using our method for $n = 6$. As can be seen, our suggested method for solving this problem is both accurate and fast and easy to implement.

Example 7.5. Take into consideration the following NSDE [85]:

$$dy(t) = a^2 y(t)(1 + y^2(t))dt + a(1 + y^2(t))dB(t), \qquad t \in [0, 1], \tag{68}$$

For $y(0) = y_0$, the exact solution of this NSDE is $y(t) = \tan(aB(t) + \arctan(y_0))$.

We used the proposed method to solve the problem for $a = 0.001$, $y_0 = 1$. Table 5 shows a comparison of the absolute error obtained using the suggested method for $n = 6$ with the absolute errors of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$. As shown in this Table, the results obtained using our suggested method are better than those obtained by using the other hat functions methods. Moreover, Fig. 6 shows the exact and approximate solutions calculated using our method for $n = 6$. As can be seen, our suggested method for solving this problem is both accurate and fast and easy to implement.

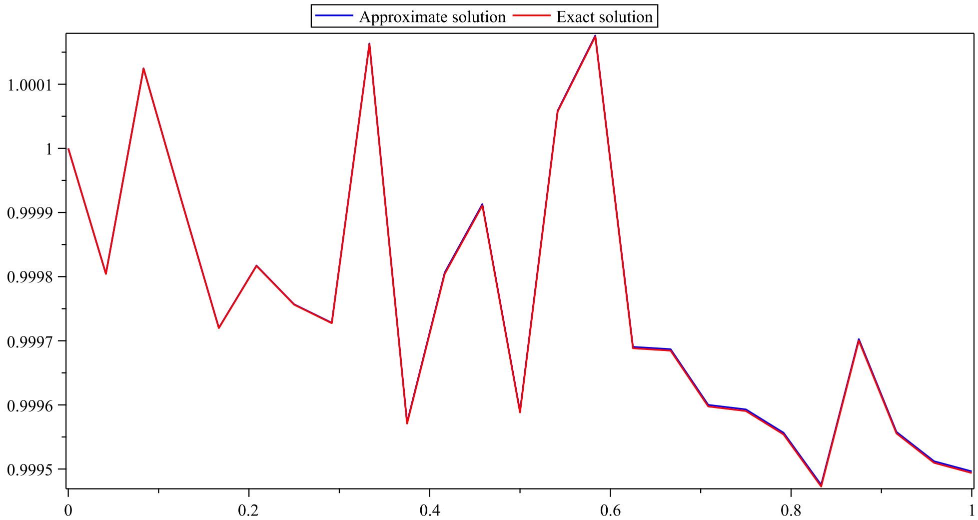

Fig. 6. The approximate and exact solution of Example 7.5 for $m = 4$ and $n = 6$.

8. Conclusion

The analytical solution of SDEs is typically difficult to obtain, necessitating the use of approximate solutions in many cases. This paper introduces a new numerical method based on FDHFs to solve these equations. This method transforms SDE into a system of nonlinear algebraic equations that can be solved using one of the well-known iterative methods. Several theorems were employed to discuss the convergence analysis of our proposed method. We have proven that our suggested method for solving the aforementioned equation has a convergence rate of $O(h^5)$. The results obtained by the present method are efficient and convenient compared to the exact solution and the results of other hat basis function methods. Therefore, we can conclude that the proposed method demonstrates favorable accuracy and applicability.

Declaration of competing interest

The authors certify that they have no financial or non-financial interests (such as honoraria, educational grants, participation in speakers bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements) in this manuscript.

Data availability

No data was used for the research described in the article.

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