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Application the Quadratic Non-Polynomial Spline Method to the Two-Dimensional Volterra Integral Equation of the Second Kind

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Abstract. In this paper, the quadratic non-polynomial spline's method is proposed for numerical solution of two-dimensional (2D) Volterra integral equation of the second kind. The tensor product are used for extending one- dimensional quadratic non-polynomial spline ψ_1 to a two-dimensional spline $\psi_1 \otimes \psi_2$ to solve 2D Volterra integral equation. Also, we compared the absolute error for proposed method with other numerical approximations methods and the exact solution. On the other hand, a comparative study between the numerical and analytical solution is explained through the three numerical examples presented in this work.

Keywords: Two dimensional Volterra integral equation, non-polynomial spline, tensor product.

INTRODUCTION

An integral equation is used to solve many issues in applied mathematics, engineering, and physics. The 2D Volterra integral equations, in particular, are an important tool for modelling a variety of problems. These equations appear in electromagnetic and electrodynamic, elasticity and dynamic contact, heat and mass transfer, fluid mechanics, acoustic, chemical and electrochemical processes, molecular physics, population, medicine, and a variety of other fields [1].

It is considered 2D Volterra integral equation of the second kind of the form [2]

$$\eta(x,t) = g(x,t) + \int_c^t \int_a^x k\big(y,x,z,t,\eta(y,z)\big)dydz, \quad (x,t) \in \Gamma = [a,b] \times [c,d]. \tag{1}$$

Where $\eta(x,t)$ is an unknown function, $g(x,t)$ is a continuous function defined on Γ and $k\big(y,x,z,t,\eta(y,z)\big)$ and continuous function.

Many works have been published on developing and analysing numerical methods for solving two-dimensional integrals, such as the sinc method for two dimensions [1], rationalised Haar functions [2], Legendre wavelets and convergence [3], existence of solution for some nonlinear two-dimensional Volterra integral equations via measures of noncompactness [4], non-polynomial spline method for the solution of two-dimensional linear wave equations with a nonlinear source term [5], Haar wavelets [6] and convergence

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TENSOR PRODUCT OF QUADRATIC NON-POLYNOMIAL SPLINE FUNCTION

When Volterra integral equation of the second kind of one- dimensional is solved, one dimensional the quadratic non-polynomial spline function is used

$$\psi_i(x) = a_i \cos\big(p(x-x_i)\big) + b_i \sin\big(p(x-x_i)\big) + c_i(x-x_i) + d_i(x-x_i)^2 + e_i. \tag{2}$$

Let,

$$\omega_j(t) = a_i \cos\Big(p\big(t-t_j\big)\Big) + b_j \sin\Big(p\big(t-t_j\big)\Big) + c_j\big(t-t_j\big) + d_j\big(t-t_j\big)^2 + e_j, \tag{3}$$

where, $a_i, b_i, c_i, d_i, e_i, a_j, b_{j,} c_j, d_j$ and e_j are constants, $i = 0, \dots, n, j = 0, \dots, m\ and\ p$ is the frequency of the trigonometric functions which would be used to raise the method.

To treatment of 2D Volterra integral equation of the second kind, can be solved by the concept tensor product approximation. Let Ω be a region such that $\Omega = \{(x,t)|a \le x \le b, c \le t \le d\}$. The method in this work has two dimensional functions $\varphi(x,t)$ which is used in a tensor product function $\psi \otimes \omega$, such that ψ and ω are two spline functions, say $\psi = span\{\phi_0, \phi_1, \phi_2, \phi_3, \phi_4\}$ and $\omega = span\{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4\}$ appears in $\varphi(x,t) \in (\psi \otimes \omega)$ where [7-11]

$$\varphi_{ij}(x,t) = \sum_{k=0}^{4} \sum_{l=0}^{4} C_{klij}\, \phi_{ki}(x)\theta_{lj}(t), \tag{4}$$

$\phi_k(x), k = 0,1,2,3,4$ are basis of the quadratic non-polynomial spline function $\psi_i(x)$ and $\theta_l(t), l = 0,1,2,3,4$ are basis of the quadratic non-polynomial spline function $\omega_j(t)$.

The coefficients C_{klij} that the paper finds by differentiate equation (4) with respect to x and t, then we get:

$$\frac{\partial \varphi_{ij}(x,t)}{\partial x} = -C_{00ij}\sin(x-x_i)\cos\big(t-t_j\big) + C_{10ij}\cos(x-x_i)\cos\big(t-t_j\big) + C_{20ij}\cos\big(t-t_j\big) + 2C_{30ij}(x-x_i)\cos\big(t-t_j\big) - C_{01ij}\sin(x-x_i)\sin\big(t-t_j\big) + C_{11ij}\cos(x-x_i)\sin\big(t-t_j\big) + C_{21ij}\sin\big(t-t_j\big) + 2C_{31ij}(x-x_i)\sin\big(t-t_j\big) - C_{02ij}\sin(x-x_i)\big(t-t_j\big) + C_{12ij}\cos(x-x_i)\big(t-t_j\big) + C_{22ij}\big(t-t_j\big) + 2C_{32ij}(x-x_i)\big(t-t_j\big) - C_{03ij}sin(x-x_i)\big(t-t_j\big)^2 + C_{13ij}cos(x-x_i)\big(t-t_j\big)^2 + C_{23ij}\big(t-t_j\big)^2 + 2C_{33ij}(x-x_i)\big(t-t_j\big)^2 - C_{04ij}sin(x-x_i) + C_{14ij}cos(x-x_i) + C_{24ij} + 2C_{34ij}(x-x_i), \tag{5}$$

$$\frac{\partial^2 \varphi_{ij}(x,t)}{\partial x^2} = -C_{00ij}\cos(x-x_i)\cos\big(t-t_j\big) - C_{10ij}\sin(x-x_i)\cos\big(t-t_j\big) + 2C_{30ij}\cos\big(t-t_j\big) - C_{01ij}\cos(x-x_i)\sin\big(t-t_j\big) - C_{11ij}\sin(x-x_i)\sin\big(t-t_j\big) + 2\,C_{31ij}sin\big(t-t_j\big) - C_{02ij}cos(x-x_i)\big(t-t_j\big) - C_{12ij}sin(x-x_i)\big(t-t_j\big) + 2C_{32ij}\big(t-t_j\big) - C_{03ij}cos(x-x_i)\big(t-t_j\big)^2 - C_{13ij}sin(x-x_i)\big(t-t_j\big)^2 + 2C_{33ij}\big(t-t_j\big)^2 - C_{04ij}cos(x-x_i) - C_{14ij}sin(x-x_i) + 2C_{34ij}, \tag{6}$$

in the same way, we find the other values;

$$\frac{\partial^3 \varphi_{ij}(x,t)}{\partial x^3}, \frac{\partial \varphi_{ij}(x,t)}{\partial t}, \frac{\partial^2 \varphi_{ij}(x,t)}{\partial x \partial t}, \frac{\partial^3 \varphi_{ij}(x,t)}{\partial x^2 \partial t}, \frac{\partial^3 \varphi_{ij}(x,t)}{\partial x^3 \partial t}, \frac{\partial^2 \varphi_{ij}(x,t)}{\partial t^2}, \frac{\partial^3 \varphi_{ij}(x,t)}{\partial x \partial t^2}, \frac{\partial^4 \varphi_{ij}(x,t)}{\partial x^2 \partial t^2},$$

$$\frac{\partial^5 \varphi_{ij}(x,t)}{\partial x^3 \partial t^2}, \frac{\partial^3 \varphi_{ij}(x,t)}{\partial t^3}, \frac{\partial^4 \varphi_{ij}(x,t)}{\partial x \partial t^3}, \frac{\partial^5 \varphi_{ij}(x,t)}{\partial x^2 \partial t^3} \frac{\partial^6 \varphi_{ij}(x,t)}{\partial x^3 \partial t^3}, \frac{\partial^6 \varphi_{ij}(x,t)}{\partial x^2 \partial t^4}, \frac{\partial^7 \varphi_{ij}(x,t)}{\partial x^3 \partial t^4}, \frac{\partial^4 \varphi_{ij}(x,t)}{\partial t^4},$$

$$\frac{\partial^5\varphi_{ij}(x,t)}{\partial x\partial t^4},\frac{\partial^4\varphi_{ij}(x,t)}{\partial x^4},\frac{\partial^5\varphi_{ij}(x,t)}{\partial x^4\partial t},\frac{\partial^6\varphi_{ij}(x,t)}{\partial x^4\partial t^2},\frac{\partial^7\varphi_{ij}(x,t)}{\partial x^4\partial t^3},\frac{\partial^8\varphi_{ij}(x,t)}{\partial x^4\partial t^4}.$$

Let $\eta(x,t)$ be the exact solution of equation (1), $\varphi_{ij}(x_i,t_j)$be the approximate solution to $\eta_i=\eta(x_i,t_j)$, from the above equations we have:

$$C_{00ij}=\frac{\partial^8\varphi_{ij}(x_i,t_j)}{\partial x^4\partial t^4}, \tag{7}$$

$$C_{01ij}=-\frac{\partial^7\varphi_{ij}(x_i,t_j)}{\partial x^4\partial t^3}, \tag{8}$$

$$C_{02ij}=\frac{\partial^5\varphi_{ij}(x_i,t_j)}{\partial x^4\partial t}+\frac{\partial^7\varphi_{ij}(x_i,t_j)}{\partial x^4\partial t^3}, \tag{9}$$

$$C_{03ij}=\frac{1}{2}\left(\frac{\partial^6\varphi_{ij}(x_i,t_j)}{\partial x^4\partial t^2}+\frac{\partial^8\varphi_{ij}(x_i,t_j)}{\partial x^4\partial t^4}\right), \tag{10}$$

and so we calculate the rest;

$C_{04ij},C_{10ij},C_{20ij},C_{30ij},C_{40ij},C_{11ij},C_{21ij},C_{31ij},C_{41ij},C_{12ij},C_{22ij},C_{32ij},C_{42ij},C_{13ij},C_{23ij},C_{33ij},C_{43ij}$

$,C_{14ij},C_{24ij},C_{34ij},C_{44ij}.$

SOLVING METHOD OF TENSOR PRODUCT OF QUADRATIC NON-POLYNOMIAL SPLINE FUNCTION FOR 2D

Volterra integral equations

To find the tensor product of quadratic non-polynomial spline function (TPQNPS) of equation (1), by differentiating it with respect to x,t:

$$\frac{\partial\eta}{\partial x}=\frac{\partial}{\partial x}g(x,t)+\int_0^t\int_0^x\frac{\partial}{\partial x}k\big(y,x,z,t,\eta(y,z)\big)dydz+\int_0^t k\big(x,x,z,t,\eta(x,z)\big)dz, \tag{11}$$

$$\frac{\partial^2\eta}{\partial x^2}=\frac{\partial^2}{\partial x^2}g(x,t)+\int_0^t\int_0^x\frac{\partial^2}{\partial x^2}k\big(y,x,z,t,\eta(y,z)\big)dydz+\int_0^t(\frac{\partial}{\partial x}k\big(y,x,z,t,\eta(y,z)\big))_{y=x}dz+\int_0^t\frac{\partial}{\partial x}k\big(x,x,z,t,\eta(x,z)\big)dz, \tag{12}$$

$$\frac{\partial^3\eta}{\partial x^3}=\frac{\partial^3}{\partial x^3}g(x,t)+\int_0^t\int_0^x\frac{\partial^3}{\partial x^3}k\big(y,x,z,t,\eta(y,z)\big)dydz+\int_0^t(\frac{\partial^2}{\partial x^2}k\big(y,x,z,t,\eta(y,z)\big))_{y=x}dz\ +\int_0^t\frac{\partial}{\partial x}(\frac{\partial}{\partial x}k\big(y,x,z,t,\eta(y,z)\big))_{y=x}dz\ +\int_0^t\frac{\partial^2}{\partial x^2}k\big(x,x,z,t,\eta(x,z)\big)dz, \tag{13}$$

$$\frac{\partial\eta}{\partial t}=\frac{\partial}{\partial t}g(x,t)+\int_0^t\int_0^x\frac{\partial}{\partial t}k\big(y,x,z,t,\eta(y,z)\big)dydz+\int_0^x k\big(y,x,t,t,\eta(y,t)\big)dy, \tag{14}$$

$$\frac{\partial\eta}{\partial x\partial t}=\frac{\partial^2}{\partial x\partial t}g(x,t)+\int_0^t\int_0^x\frac{\partial^2}{\partial x\partial t}k\big(y,x,z,t,\eta(y,z)\big)dydz+\int_0^t(\frac{\partial}{\partial t}k\big(y,x,z,t,\eta(y,z)\big))_{y=x}dz+\int_0^x\frac{\partial}{\partial x}k\big(y,x,t,t,\eta(y,t)\big)dy+k\big(x,x,t,t,\eta(x,t)\big)\ , \tag{15}$$

$$\frac{\partial^3\eta}{\partial x^2\partial t}=\frac{\partial^3}{\partial x^2\partial t}g(x,t)+\int_0^t\int_0^x\frac{\partial^3}{\partial x^2\partial t}k\big(y,x,z,t,\eta(y,z)\big)dydz+\int_0^t(\frac{\partial^2}{\partial x\partial t}k\big(y,x,z,t,\eta(y,z)\big))_{y=x}dz+\int_0^x\frac{\partial^2}{\partial x^2}k\big(y,x,t,t,\eta(y,t)\big)dy+(\frac{\partial}{\partial x}k\big(y,x,t,t,\eta(y,t)\big))_{y=x}\int_0^t\frac{\partial}{\partial x}(\frac{\partial}{\partial t}k\big(y,x,z,t,\eta(y,z)\big))_{y=x}dz+\frac{\partial}{\partial x}k\big(x,x,t,t,\eta(x,t)\big), \tag{16}$$

Similarly, we find other derivatives;

$$\frac{\partial^4\eta}{\partial x^3\partial t},\frac{\partial^2\eta}{\partial t^2},\frac{\partial^3\eta}{\partial x\partial t^2},\frac{\partial^4\eta}{\partial x^2\partial t^2},\frac{\partial^5\eta}{\partial x^3\partial t^2},\frac{\partial^3\eta}{\partial t^3},\frac{\partial^4\eta}{\partial x\partial t^3},\frac{\partial^5\eta}{\partial x^2\partial t^3},\frac{\partial^6\eta}{\partial x^3\partial t^3},\frac{\partial^4\eta}{\partial t^4},\frac{\partial^5\eta}{\partial x\partial t^4},\frac{\partial^6\eta}{\partial x^2\partial t^4},$$
$$\frac{\partial^7\eta}{\partial x^3\partial t^4},\frac{\partial^4\eta}{\partial x^4},\frac{\partial^5\eta}{\partial x^4\partial t},\frac{\partial^6\eta}{\partial x^4\partial t^2},\frac{\partial^7\eta}{\partial x^4\partial t^3},\frac{\partial^8\eta}{\partial x^4\partial t^4},$$

at the same way.

Now substitute $x=a\ and\ t=c$ in equation above, we get:

$$\frac{\partial}{\partial x}\eta(a,c)=\frac{\partial}{\partial x}g(a,c), \tag{17}$$

$$\frac{\partial^2}{\partial x^2}\eta(a,c)=\frac{\partial^2}{\partial x^2}g(a,c), \tag{18}$$

$\frac{\partial^3}{\partial x^3}\eta(a,c) = \frac{\partial^3}{\partial x^3} g(a,c),$ (19)
$\frac{\partial}{\partial t}\eta(a,c) = \frac{\partial}{\partial t} g(a,c),$ (20)
$\frac{\partial}{\partial x \partial t}\eta(a,c) = \frac{\partial^2}{\partial x \partial t} g(a,c) \; + k\big(a,a,c,c,\eta(a,c)\big),$ (21)
$\frac{\partial^3}{\partial x^2 \partial t}\eta(a,c) = \frac{\partial^3}{\partial x^2 \partial t} g(a,c) + ((\frac{\partial}{\partial x} k\big(y,x,t,t,\eta(y,t)\big))_{y=x})_{x=a,t=c} + (\frac{\partial}{\partial x} k\big(x,x,t,t,\eta(x,t)\big))_{x=a,t=c},$
(22)
Similarly, we find other
derivatives $\frac{\partial^4}{\partial x^3 \partial t}, \frac{\partial^2}{\partial t^2}, \frac{\partial^3}{\partial x \partial t^2}, \frac{\partial^5}{\partial x^3 \partial t^2}, \frac{\partial^4}{\partial x^2 \partial t^2}, \frac{\partial^3\eta}{\partial t^3}, \frac{\partial^4\eta}{\partial x \partial t^3}, \frac{\partial^5\eta}{\partial x^2 \partial t^3}, \frac{\partial^6\eta}{\partial x^3 \partial t^3}, \frac{\partial^4\eta}{\partial t^4}, \frac{\partial^5\eta}{\partial x \partial t^4}, \frac{\partial^6\eta}{\partial x^2 \partial t^4}, \frac{\partial^7\eta}{\partial x^3 \partial t^4},$
$\frac{\partial^4\eta}{\partial x^4}, \frac{\partial^5\eta}{\partial x^4 \partial t}, \frac{\partial^6\eta}{\partial x^4 \partial t^2}, \frac{\partial^7\eta}{\partial x^4 \partial t^3}, \frac{\partial^8\eta}{\partial x^4 \partial t^4}$ at substitute $x = a, t = c$.

ALGORITHM

The following algorithm tensor product of quadratic non-polynomial spline function (TPQNPS) for solving 2D Volterra integral equations:

Step1: Set $h = \frac{b-a}{n}$ and $k = \frac{d-c}{m}, x_i = x_0 + ih,\; i = 0,1,\dots,n,$ where($x_0 = a$, x_n =b) $and t_j = t_0 + jk, j = 0,1,\dots,m$ where($t_0 = c$, t_m =d) and $\eta(a,c) = g(a,c)$.

Step2: Evaluate
$C_{0000}, C_{0100}, C_{0200}, C_{0300}, C_{0400}, C_{1000}, C_{2000}, C_{3000}, C_{4000}, C_{1100}, C_{2100}, C_{3100}, C_{4100}, C_{1200}, C_{2200}, C_{3200},$
$C_{4200}, C_{1300}, C_{2300}, C_{3300}, C_{4300}, C_{1400}, C_{2400}, C_{3400}, C_{4400},$ by substituting (17)-(22) and
$\frac{\partial^4\eta}{\partial x^3 \partial t}, \frac{\partial^2\eta}{\partial t^2}, \frac{\partial^3\eta}{\partial x \partial t^2}, \frac{\partial^4\eta}{\partial x^2 \partial t^2}, \frac{\partial^5}{\partial x^3 \partial t^2}, \frac{\partial^4}{\partial x^2 \partial t^2}, \frac{\partial^3\eta}{\partial t^3}, \frac{\partial^4\eta}{\partial x \partial t^3}, \frac{\partial^5\eta}{\partial x^2 \partial t^3}, \frac{\partial^6\eta}{\partial x^3 \partial t^3}, \frac{\partial^4\eta}{\partial t^4}, \frac{\partial^5\eta}{\partial x \partial t^4}, \frac{\partial^6\eta}{\partial x^2 \partial t^4}, \frac{\partial^7\eta}{\partial x^3 \partial t^4}, \frac{\partial^4\eta}{\partial x^4}, \frac{\partial^5\eta}{\partial x^4 \partial t}$
$, \frac{\partial^6\eta}{\partial x^4 \partial t^2}, \frac{\partial^7\eta}{\partial x^4 \partial t^3}, \frac{\partial^8\eta}{\partial x^4 \partial t^4}$ at substitute $x = a,\; t = c$, in equations (7)-(10)and
$C_{04ij}, C_{10ij}, C_{20ij}, C_{30ij}, C_{40ij}, C_{11ij}, C_{21ij}, C_{31ij}, C_{41ij}, C_{12ij}, C_{22ij}, C_{32ij}, C_{42ij}, C_{13ij}, C_{23ij}, C_{33ij}, C_{43ij}$
$, C_{14ij}, C_{24ij}, C_{34ij}, C_{44ij}$.

Step3: Calculate $\varphi_{00}(x,t)$ using step 2 and equation (4) for $i = 0, j = 0$.

Step4: Approximation $\eta_1 \approx \varphi_{00}(x_1, t_1)$

Step5: For $i = 1$ to $n - 1, j = 1$ to $m - 1$ do the following steps:

Step6: Evaluate
$C_{00ij}, C_{01ij}, C_{02ij}, C_{03ij}, C_{04ij}, C_{10ij}, C_{20ij}, C_{30ij}, C_{40ij}, C_{11ij}, C_{21ij}, C_{31ij}, C_{41ij}, C_{12ij}, C_{22ij}, C_{32ij}, C_{42ij},$
$C_{13ij}, C_{23ij}, C_{33ij}, C_{43ij}, C_{14ij}, C_{24ij}, C_{34ij}$ and C_{44ij}, by using equation (7)-(10)and
$C_{04ij}, C_{10ij}, C_{20ij}, C_{30ij}, C_{40ij}, C_{11ij}, C_{21ij}, C_{31ij}, C_{41ij}, C_{12ij}, C_{22ij}, C_{32ij}, C_{42ij}, C_{13ij}, C_{23ij}, C_{33ij}, C_{43ij}$
$, C_{14ij}, C_{24ij}, C_{34ij}, C_{44ij}$, and
replacing; $\eta(x_i, t_j), \frac{\partial\eta(x_i,t_j)}{\partial x}, \frac{\partial^2\eta(x_i,t_j)}{\partial x^2}, \frac{\partial^3\eta(x_i,t_j)}{\partial x^3}, \frac{\partial\eta(x_i,t_j)}{\partial t}, \frac{\partial\eta(x_i,t_j)}{\partial x \partial t}, \frac{\partial^3\eta(x_i,t_j)}{\partial x^2 \partial t}, \frac{\partial^4\eta(x_i,t_j)}{\partial x^3 \partial t}, \frac{\partial^2\eta(x_i,t_j)}{\partial t^2}, \frac{\partial^3\eta(x_i,t_j)}{\partial x \partial t^2},$
$, \frac{\partial^4\eta(x_i,t_j)}{\partial x^2 \partial t^2}, \frac{\partial^5\eta(x_i,t_j)}{\partial x^3 \partial t^2}, \frac{\partial^3\eta(x_i,t_j)}{\partial t^3} \frac{\partial^4\eta(x_i,t_j)}{\partial x \partial t^3}, \frac{\partial^5\eta(x_i,t_j)}{\partial x^2 \partial t^3}, \frac{\partial^6\eta(x_i,t_j)}{\partial x^3 \partial t^3}, \frac{\partial^4\eta(x_i,t_j)}{\partial t^4}, \frac{\partial^5\eta(x_i,t_j)}{\partial x \partial t^4}, \frac{\partial^6\eta(x_i,t_j)}{\partial x^2 \partial t^4},$
$\frac{\partial^5\eta(x_i,t_j)}{\partial x^4 \partial t}, \frac{\partial^7\eta(x_i,t_j)}{\partial x^4 \partial t^3}, \frac{\partial^7\eta(x_i,t_j)}{\partial x^3 \partial t^4}, \frac{\partial^4\eta(x_i,t_j)}{\partial x^4} \frac{\partial^6\eta(x_i,t_j)}{\partial x^4 \partial t^2}$, and $\frac{\partial^8\eta(x_i,t_j)}{\partial x^4 \partial t^8}$ by
$\varphi_{ij}(x_i, t_j), \frac{\partial\varphi_{ij}(x_i,t_j)}{\partial x}, \frac{\partial^2\varphi_{ij}(x_i,t_j)}{\partial x^2}, \frac{\partial^3\varphi_{ij}(x_i,t_j)}{\partial x^3}, \frac{\partial\varphi_{ij}(x_i,t_j)}{\partial t}, \frac{\partial\varphi_{ij}(x_i,t_j)}{\partial x \partial t}, \frac{\partial^3\varphi_{ij}(x_i,t_j)}{\partial x^2 \partial t}, \frac{\partial^4\varphi_{ij}(x_i,t_j)}{\partial x^3 \partial t},$
$\frac{\partial^2\varphi_{ij}(x_i,t_j)}{\partial t^2}, \frac{\partial^3\varphi_{ij}(x_i,t_j)}{\partial x \partial t^2}, \frac{\partial^4\varphi_{ij}(x_i,t_j)}{\partial x^2 \partial t^2}, \frac{\partial^3\varphi_{ij}(x_i,t_j)}{\partial t^3}, \frac{\partial^5\varphi_{ij}(x_i,t_j)}{\partial x^3 \partial t^2}, \frac{\partial^4\varphi_{ij}(x_i,t_j)}{\partial x \partial t^3}, \frac{\partial^5\varphi_{ij}(x_i,t_j)}{\partial x^2 \partial t^3}, \frac{\partial^6\varphi_{ij}(x_i,t_j)}{\partial x^3 \partial t^3}$
$, \frac{\partial^4\varphi_{ij}(x_i,t_j)}{\partial t^4}, \frac{\partial^5\varphi_{ij}(x_i,t_j)}{\partial x \partial t^4}, \frac{\partial^6\varphi_{ij}(x_i,t_j)}{\partial x^2 \partial t^4}, \frac{\partial^7\varphi_{ij}(x_i,t_j)}{\partial x^3 \partial t^4}, \frac{\partial^4\varphi_{ij}(x_i,t_j)}{\partial x^4}, \frac{\partial^5\varphi_{ij}(x_i,t_j)}{\partial x^4 \partial t}, \frac{\partial^6\varphi_{ij}(x_i,t_j)}{\partial x^4 \partial t^2}, \frac{\partial^7\varphi_{ij}(x_i,t_j)}{\partial x^4 \partial t^3}$ and $\frac{\partial^7\varphi_{ij}(x_i,t_j)}{\partial x^4 \partial t^3}$.

Step7: Calculate $\varphi_{ij}(x,t)$ using step6 and equation(4).

Step8: Approximate $\eta_{i+1,j+1} = \varphi_{ij}(x_{i+1}, t_{j+1})$.

NUMERICAL EXAMPLES

This paper shows some of the numerical examples to explain that above methods for solving the 2D Volterra integral equation of the second kind. The exact solution is defined and used to illustrate that the numerical solution get with above method is correct, all calculation are implement by maple18 to solve all examples.

Example (1) [12, 13]: Consider the following nonlinear 2D Volterra integral equation

$$\eta(x,t) = g(x,t) + \int_0^t \int_0^x (xy^2 + \cos(z))\eta^2(y,z)\,dydz, (x,t) \in [0,1] \times [0,1]$$

where

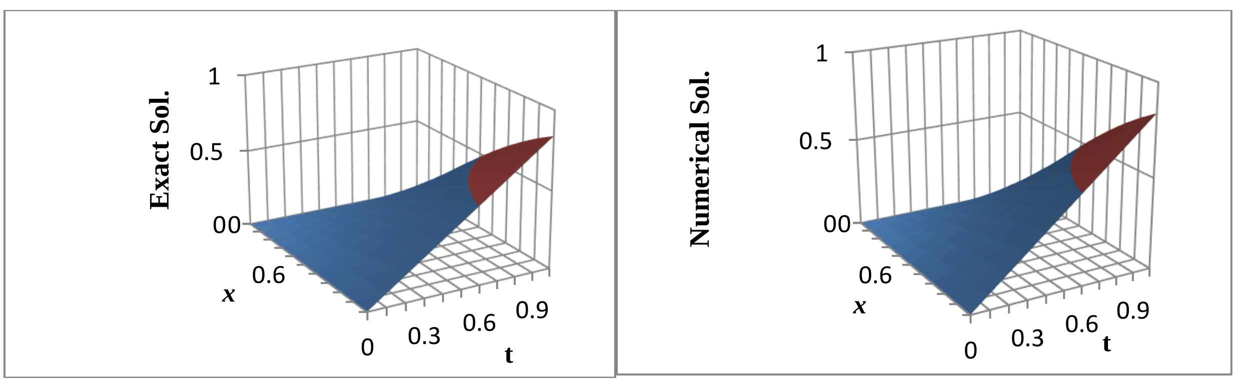
$g(x,t) = x\sin(t)\left(1 - \frac{1}{9}x^2 sin^2 t\right) + \frac{1}{10}x^6\left(\frac{1}{2}\sin(2t) - t\right).$

Exact solution is $\eta(x,t) = xsin(t)$

Table (1) shows that comparison between absolute error of (TPQNPS) and absolute error of [12, 13] for example (1) where$(x,t) = \left(\frac{1}{2^i}, \frac{1}{2^j}\right)$ $and\ i = 0,1,\ldots,6,\ j = 0,1,\ldots,6.$ In Figure (1) comparison between exact solution and numerical solution of $\eta(x,t)$ for example (1).

TABLE (1): Comparison between the errors of our method with [12, 13] for example (1) for $n = 6$ at different values of $(x$, t).

$(\boldsymbol{x},\boldsymbol{t})$	**Absolute error(TPQNPS)**	**Absolute error [12]**	**Absolute error [13]**
(1/2,1/2)	0	1.6191 e−7	0.00006
(1/4,1/4)	2e-17	7.5459 e−8	0.000129
(1/8,1/8)	e-17	1.3338 e−8	0.000070
(1/16,1/16)	e-18	1.7705 e−8	0.0000533
(1/32,1/32)	0	4.6972 e−9	0.00005959
(1/64,1/64)	0	2.9310 e−10	0.00074588



(a) (b)

FIGURE (1)-(a) the exact solution $\eta(x,t)$ and (b) the numerical solution $\eta(x,t)$of example (1).

Example (2) [12, 13]: Consider the following nonlinear two-dimensional Volterra integral equation:

$\eta(x,t) = g(x,t) + \int_0^t \int_0^x \eta^2(y,z)\,dydz, (x,t) \in [0,1) \times [0,1),$

where $g(x,t) = x^2 + t^2 + \frac{1}{45}xt(9x^4 + 10x^2t^2 + 9t^4).$

Exact solution is $\eta(x,t) = x^2 + t^2$.

Table (2) shows that comparison between absolute error of (TPQNPS) and absolute error of [12, 13] for example (2) where$(x,t) = \left(\frac{1}{2^i}, \frac{1}{2^j}\right)$ $and\ i = 0,1,\ldots,5,\ j = 0,1,\ldots,5$.In Figure (2) comparison between exact solution and numerical solution of $\eta(x,t)$for example (2).

TABLE (2): Comparison between the errors of our method with, [12, 13] for example (1) for $n = 5$ at different values of (x, t).

(x, t)	**Absolute error** (TPQNPS)	**Absolute error in Presented method [12]**	**Absolute error [13]**
(1/4,1/4)	0	0	0.00024
(1/8,1/8)	0	2.22045e−16	0.000150
(1/16,1/16)	0	2.22045e−16	0.0000945
(1/32,1/32)	0	0	0.0001369
(1/64,1/64)	0	8.88178e−16	0.0015187

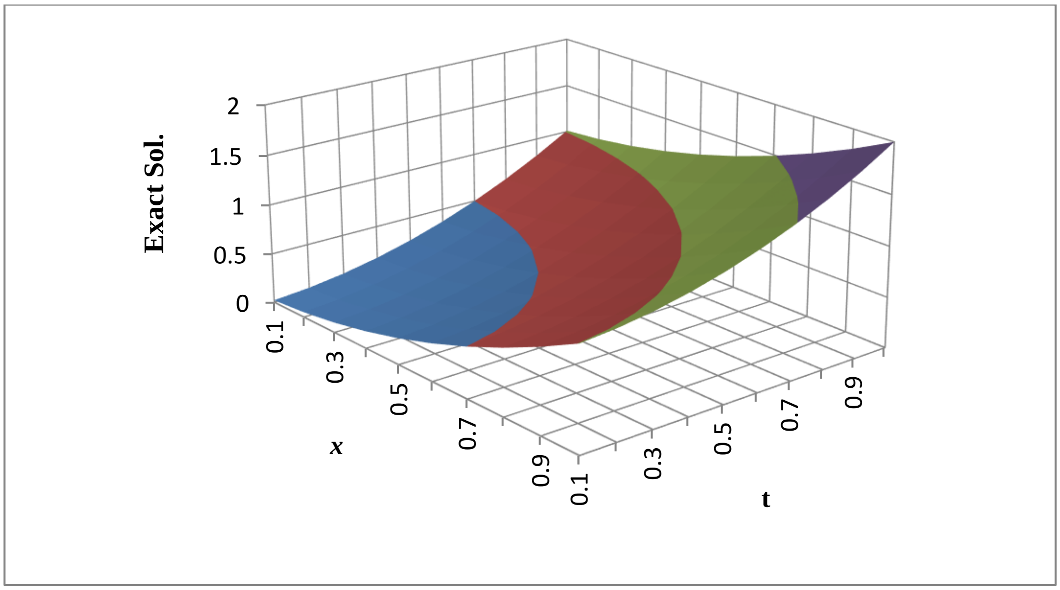


FIGURE (2)-exact solution and numerical solution of $\eta(x,t)$ for example (2).Exact solution is the numerical solution because the numerical solution=0

Example (3) [2, 13]: Consider the following nonlinear two-dimensional Volterra integral equation:
$\eta(x,t) = g(x,t) + \int_0^t \int_0^x (x + t - z - y)\eta^2(y,z)\, dydz, 0 \leq x, t \leq 1,$
where, $g(x,t) = x + t - \frac{1}{12}xt(x^3 + 4x^2t + 4xt^2 + t^3)$.
Exact solution is $\eta(x,t) = x + t$.
Table (3) shows that comparison between absolute error of (TPQNPS) and absolute error of [2, 13] for example (3) where$(x,t) = (\frac{1}{2^i}, \frac{1}{2^j})$ $and\ i = 0,1, \dots, 6, j = 0,1, \dots, 6$.In Figure (3) comparison between exact solution and numerical solution of $\eta(x,t)$ for example (3).

Table (3): Comparison between the errors of our method with [2, 13] for example (3) for $n = 6$ at different values of (x, t).

(x, t)	**Absolute error(TPQNPS)**	**Absolute error [2]**	**Absolute error in Presented method [13]**
(1/2,1/2)	0	3.1 e-2	0.00266
(1/4,1/4)	0	3.1 e-2	0.00011
(1/8,1/8)	0	3.1 e-2	0.00005
(1/16,1/16)	0	3.1 e-2	0.000101
(1/32,1/32)	0	3.1 e-2	0.000125
(1/64,1/64)	0	2.2 e-9	0.016704

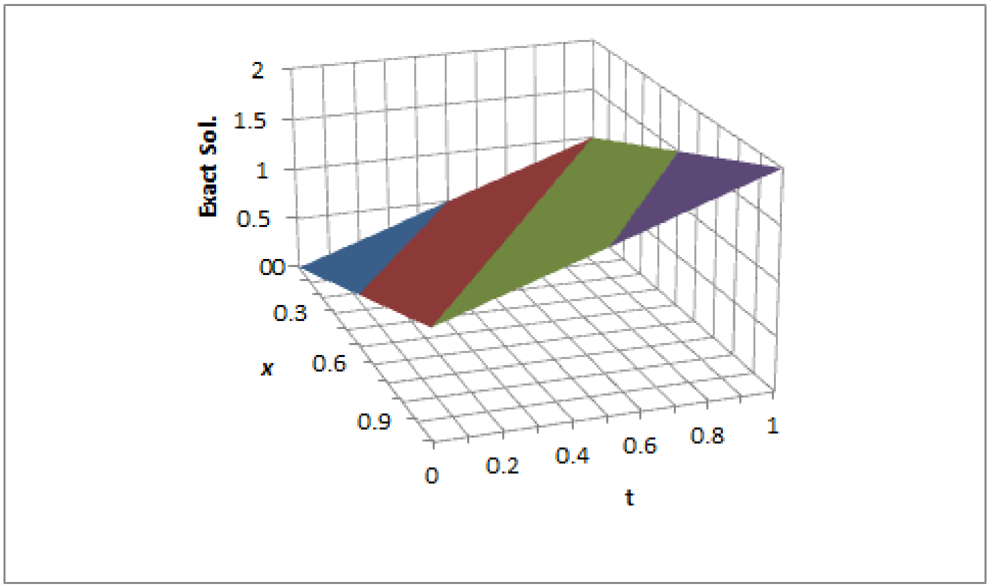


FIGURE (3)-exact solution and numerical solution of $\eta(x,t)$ for example (3). Exact solution is the numerical solution because the numerical solution=0

When you add a fourth example, the paper pages increase.

CONCLUSIONS

In this paper, the quadratic non polynomial method is used to find the approximation solution of nonlinear 2D Volterra integral equations, the results clarified a very high agreement with the exact solution for the examples. It is a new idea which based adopted on the using of the 2D Volterra integral equations and it derivatives by used tensor product. The toughness of the proposed method obtains the simple applicability, accurate and efficient to solve 2D Volterra integral equations by making a compare with other methods, this is introduced in the tables and figures**.**

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