Analytical integrations in 3D BEM elastodynamics

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Abstract

Some results on the analytical integration of kernels in hyperbolic problems [1] (acoustics, elastodynamics) for 3D Boundary Element Methods are presented. Adopting polynomial shape functions of arbitrary degree (in space and time) on flat discretizations, integrations are performed both in space and time for Lebesgue integrals working in a local coordinate system. For singular integrals, both a limit to the boundary as well as the finite part of Hadamard [2, 3] approach have been pursued.

Keywords: Boundary elements, dynamics, analytical integrations, finite part of Hadamard.

1 Introduction

Modeling hyperbolic problems by means of boundary integral equations (BIEs) and approximating their solution through boundary element methods (BEM) is firmly established in the academic community as well as in industry. Several well known yet stimulating as well as modern applications and on going research topics can be effectively described via BIEs: to cite but a few, the analysis of ground motion due to moving surface loads induced by high-speed trains [4], the dynamic analysis of the interaction between structures and their surrounding soils, modeled as visco-elastic or porous media [5], the simulation of ultrasonic nondestructive evaluation [6] and of dynamic fracture mechanics in anisotropic media [7].

The present note aims at providing a closed form for analytical integrations involved in 3D BIEs, what seems to be of interest for computational and theoretical purposes. Educational advantages of analytical integrations can also be envisaged, as in [8]. In this note, reference will be made to linear elastodynamics as a prototype of a hyperbolic problem; the boundary integral formulation [9] of Navier's equations of motion stems from Graffi's [10] (see also Wheeler and Sternberg's proof [11]) generalization of Betti's theorem to elastodynamics. Under the hypothesis of vanishing initial conditions and no body forces, the boundary integral representation (BIR) of displacements in the interior of an open domain Ω at time t reads:

$$\mathbf{u}(\mathbf{x},t) = \int_\Gamma \int_0^t \mathbf{G}_{uu}(\mathbf{r}, t-\tau)\mathbf{p}(\mathbf{y},\tau) \mathrm{d}\tau \, \mathrm{d}\Gamma_y - \int_\Gamma \int_0^t \mathbf{G}_{up}(\mathbf{r}, \mathbf{l}(\mathbf{y}), t-\tau)\mathbf{u}(\mathbf{y},\tau) \mathrm{d}\tau \, \mathrm{d}\Gamma_y \, , \qquad \mathbf{x} \in \Omega \quad (1)$$

Here, $\mathbf{r} = \mathbf{x} - \mathbf{y}$ stands for the vector that joins point $\mathbf{y}$ to $\mathbf{x}$. Identity (1) is based on Green's functions (also called kernels) which represent components u_i of the displacement vector $\mathbf{u}$ in a point $\mathbf{x}$ due to: i) a unit force concentrated in space (point $\mathbf{y}$) and time (instant τ) and acting on the unbounded elastic space Ω_∞ in direction j (such functions are gathered in matrix $\mathbf{G}_{uu}$); ii) a unit relative displacement concentrated in time (instant τ) and space (at a point $\mathbf{y}$), crossing a surface with normal $\mathbf{l}(\mathbf{y})$, and acting on the unbounded elastic space Ω_∞ (in direction j) (gathered in matrix $\mathbf{G}_{up}$)).

To obtain an additional integral equation, required for the variational formulation of elastodynamics BIEs [12] as well as by nowadays standard numerical techniques [13], the traction operator can be applied to identity[1] (1), thus obtaining the BIR of tractions on a surface of normal $\mathbf{n}(\mathbf{x})$ in the interior of the domain, i.e. $\mathbf{x} \in \Omega$:

$$\mathbf{p}(\mathbf{x},t) = \int_\Gamma \int_0^t \mathbf{G}_{pu}(\mathbf{r},\mathbf{n}(\mathbf{x}),t-\tau)\mathbf{p}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y - \int_\Gamma \int_0^t \mathbf{G}_{pp}(\mathbf{r},\mathbf{n}(\mathbf{x}),\mathbf{l}(\mathbf{y}),t-\tau)\mathbf{u}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y \quad (2)$$

Such a BIR involves Green's functions (collected in matrices $\mathbf{G}_{pu}$ and $\mathbf{G}_{pp}$) which describe components (p_i) of the traction vector $\mathbf{p}$ on a surface of normal $\mathbf{n}(\mathbf{x})$ due to: i) a unit force concentrated in space (point $\mathbf{y}$) and time (instant τ) and acting on the unbounded elastic space Ω_∞ in direction j; ii) a unit relative displacement concentrated in space (at a point $\mathbf{y}$), crossing a surface with normal $\mathbf{l}(\mathbf{y})$ and acting at instant τ on the unbounded elastic space Ω_∞ (in direction j).

BIEs for the linear elastic problem can be derived from BIRs (1) (thus obtaining the so-called "displacement equation") and (2) (so that the "traction equation" comes out) by performing the space boundary limit[2] $\Omega \ni \mathbf{x} \rightarrow \mathbf{x} \in \Gamma$. In the limit process, after integration in time, singularities of Green's functions are triggered off: their singularity-orders show to be equivalent to the elastostatic case[3]. Assuming smooth boundaries, after imposing the fulfillment of the displacement equation on Dirichlet boundary Γ_u and of the traction equation on Neumann boundary Γ_p, the following linear boundary integral problem (omitting the arguments of Green's functions for paucity of space) comes out:

$$\int_{\Gamma_u} \int_0^t \mathbf{G}_{uu}\,\mathbf{p}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y - ⨍_{\Gamma_p} \int_0^t \mathbf{G}_{up}\,\mathbf{u}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y = \mathbf{f}^u(\mathbf{x},t)\ , \quad \mathbf{x} \in \Gamma_u \quad (3)$$

$$- ⨍_{\Gamma_u} \int_0^t \mathbf{G}_{pu}\,\mathbf{p}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y + ⨎_{\Gamma_p} \int_0^t \mathbf{G}_{pp}\,\mathbf{u}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y = \mathbf{f}^p(\mathbf{x},t)\ , \quad \mathbf{x} \in \Gamma_p \quad (4)$$

Vectors $\mathbf{f}^i$, $i = u, p$, that gather all data are the following:

$$\begin{aligned}
\mathbf{f}^u(\mathbf{x},t) &= \frac{1}{2}\bar{\mathbf{u}}(\mathbf{x},t) - \int_{\Gamma_p}\int_0^t \mathbf{G}_{uu}\bar{\mathbf{p}}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y + ⨍_{\Gamma_u}\int_0^t \mathbf{G}_{up}\bar{\mathbf{u}}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y\ , \quad \mathbf{x} \in \Gamma_u \\
\mathbf{f}^p(\mathbf{x},t) &= -\frac{1}{2}\bar{\mathbf{p}}(\mathbf{x},t) + ⨍_{\Gamma_p}\int_0^t \mathbf{G}_{pu}\bar{\mathbf{p}}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y - \int_{\Gamma_u}\int_0^t \mathbf{G}_{pp}\bar{\mathbf{u}}(\mathbf{y},\tau)\mathrm{d}\tau\,\mathrm{d}\Gamma_y\ , \quad \mathbf{x} \in \Gamma_p
\end{aligned}$$

Integral problem (3-4) can be written in the compact form:

$$\mathcal{L}\,[\,y\,] = f \quad (5)$$

with all terms defined by comparison. Unknown vector y is made of tractions (Neumann data) $\mathbf{p}$ on the Dirichlet boundary Γ_u and displacements (Dirichlet data) $\mathbf{u}$ on the Neumann boundary Γ_p. Let $h > 0$ be a parameter and let $[\mathbf{p}_h(\mathbf{y},\tau), \mathbf{u}_h(\mathbf{y},\tau)]^\top \overset{def}{=} y_h \in Y_{\mathcal{L}h}$ be an approximation of the unknown vector field y, denoting with $Y_{\mathcal{L}h}$ a family of finite dimensional subspaces of $Y_\mathcal{L}$ such that

$$\forall y \in Y_\mathcal{L}\ , \inf_{y_h \in Y_{\mathcal{L}h}} ||y - y_h|| \rightarrow 0 \quad as \quad h \rightarrow 0 \quad (6)$$

Discretization (6) allows to transform integral problem (5) into a set of algebraic equations. Several techniques have been developed to this aim: the collocation boundary element method (CBEM) [15],

[1] The above introduced kernels are infinitely smooth in their domain, which is the whole space $\mathbb{R}^3$ with exception of the origin (that is $\mathbf{x} \neq \mathbf{y}$)

[2] In the traction equation (4) the boundary limit must be taken at a smooth point $\mathbf{x}$ with a well defined normal vector $\mathbf{n}(\mathbf{x})$. Strong and hypersingular kernels generate free terms in the limit process such that $\boldsymbol{\chi}_\Gamma^u(\mathbf{x}) = \boldsymbol{\chi}_\Gamma^p(\mathbf{x}) = \frac{1}{2}\mathbb{1}$ for smooth boundaries, whereas special cares are required for the discrete problem [14].

[3] Kernel $\mathbf{G}_{uu}$ shows an integrable singularity (named "weak"); kernels $\mathbf{G}_{up}$ and $\mathbf{G}_{pu}$ present a strong singularity $O(r^{-2})$; kernel $\mathbf{G}_{pp}$ is usually named hypersingular, because it shows a singularity (of $O(r^{-3})$) greater than the dimension of the integral.

the hypersingular collocation approach (HBEM) [16], the Galerkin [17] method (GBEM), the dual BEM [13], the convolutive variational method [12], the variational formulation in extended sense [18]. All aforementioned method require the evaluation of "integrals" of the form:

$$\int_{\Gamma_s}\int_0^t \mathbf{G}_{rs}(\mathbf{x},\mathbf{y},t-\tau)\,\chi_h(\mathbf{y},\tau)\,\mathrm{d}\Gamma_y \mathrm{d}\tau \qquad \mathrm{r=u,\ s=u,p} \tag{7}$$

denoting with $\chi_h(\mathbf{y},\tau)$ *scalar* shape functions for modeling the components of approximation y_h of the unknown vector fields along $\partial\Omega\times[0,T]$. For CBEM and HBEM, point $\{\mathbf{x},t\}$ in integral (7) belongs to a selected set of collocation points $\mathbf{x}_i^*\in\partial\Omega$, $t_j^*\in[0,T]$; for other techniques, point $\{\mathbf{x},t\}$ takes different meanings.

2 Shape functions

The assumption of time-space variable separation is taken, namely:

$$\chi_h(\mathbf{y},\tau)=\phi_n(\mathbf{y})\omega_k(\tau) \tag{8}$$

where $\omega_k(\tau)$ is assumed to be a polynomial in τ.

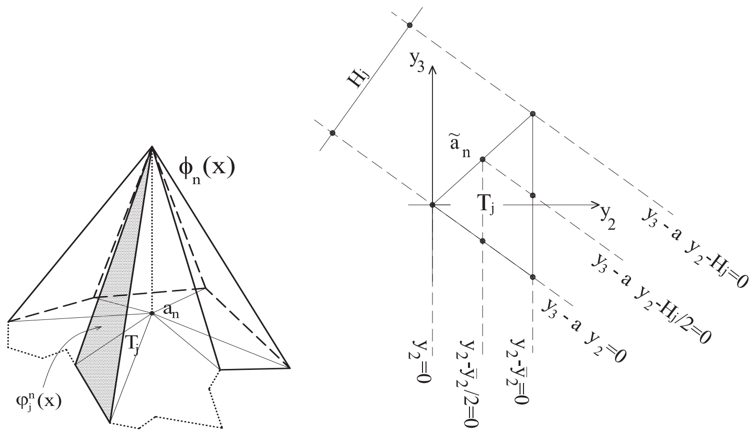

Figure 1: a) Local $\varphi_j^n(\mathbf{x})$ and global $\phi_n(\mathbf{x})$ shape functions. b) Local coordinate system $\mathcal{L}_\triangle$. The extension to quadrilateral or mixed triangular-quadrilateral tassellation is straightforward.

Let Γ_h be a triangulation of boundary Γ, $T_j\subset\Gamma_h$ its generic triangle and $\mathbf{a}_n$ a generic node of Γ_h. Collect in set $\mathcal{T}_n := \{T_j\ s.t.\ \mathbf{a}_n\in T_j\}$ all triangles of Γ_h sharing node $\mathbf{a}_n$ (see figure 1-a). Choose over T_j a local (lagrangian) basis $\boldsymbol{\varphi} := \{\varphi_j^1,\varphi_j^2,...,\varphi_j^{M(j)}\}$ and denote with $\varphi_j^{n(j)}$ the unique element of $\boldsymbol{\varphi}$ such that $\varphi_j^{n(j)}(\mathbf{a}_n)=1$. Define shape function $\phi_n(\mathbf{x})$ (see figure 1-a) as a piecewise continuous function over Γ_h whose value is zero outside $\mathcal{T}_n$, as follows:

$$\phi_n\in C^0(\Gamma_h) \qquad supp(\phi_n)=\mathcal{T}_n \qquad \phi_n|_{T_j}=\varphi_j^{n(j)} \tag{9}$$

A suitable choice of an orthogonal cartesian coordinate system allows an effective representation for $\varphi_j^{n(j)}(\mathbf{y})$. Let $\mathcal{L}_\triangle\equiv\{y_1,y_2,y_3\}$ define a local coordinate system such that: i) a vertex of T_j is the origin; ii) the plane $y_1 = 0$ contains T_j; iii) the plane $y_3 = 0$ is orthogonal to the side of T_j opposite to the origin. In $\mathcal{L}_\triangle$, T_j is defined by:

$$T_j := \{\mathbf{y} \in \mathbb{R}^3 \text{ s.t. } y_1 = 0\,;\, 0 \le y_2 \le \bar{y}_2\,;\, a y_2 - y_3 \le 0\,;\, b y_2 - y_3 \ge 0\}$$

where a and b denote the slopes of the two sides of T_j that cross the origin (see figure 1-b). Selecting arbitrarily one of these two sides, say $y_3 - a\,y_2 = 0$, denote with H_j the height of T_j, namely the segment orthogonal to a side emanating from the vertex opposite to it - see figure 1-b. Denoting with $\mathbf{d} = \mathbf{y} - \mathbf{x}$, $r = ||\mathbf{d}||$, shape functions can be readily expressed in terms of H_j in the form:

$$\varphi_j^n(\mathbf{y}) = \mathbf{d}_3^\top\, \mathbf{T}_3^\top\, \Lambda_j^n\, \mathbf{T}_2\, \mathbf{d}_2 \tag{10}$$

where $\mathbf{d}_2 = \left\{d_2^i\right\}_{i=0,1,\ldots,m}$, $\mathbf{d}_3 = \left\{d_3^i\right\}_{i=0,1,\ldots,m}$,

$$\{\mathbf{T}_k\}_{i,j} = \left(\begin{array}{c} i-1 \\ j-1 \end{array}\right) x_k^{(i-j)} \qquad k = 2,3 \quad i,j = 1,2,....$$

and matrix of constants Λ_j^n depends on node $\mathbf{a}_n$. For linear shape functions and with reference to the node at the origin,

$$\varphi_j^n(\mathbf{y}) = \begin{bmatrix} 1 & -\frac{1}{\bar{y}_2} \end{bmatrix} \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ d_2 \end{bmatrix} \tag{11}$$

Description (9) of shape functions extends straightforwardly to quadrilateral panels Q_j, and representation can be made effectively in a local coordinate system $\mathcal{L}_\square \equiv \{y_1, y_2, y_3\}$ such that: i) barycenter of Q_j is the origin; ii) the plane $y_1 = 0$ contains Q_j; iii) planes $y_2 = 0$ and $y_3 = 0$ are orthogonal to the sides of Q_j. In $\mathcal{L}_\square$, Q_j is defined by:

$$Q_j := \left\{\mathbf{y} \in \mathbb{R}^3 \text{ t.c. } y_1 = 0;\, -a \le y_2 \le a;\, -b \le y_3 \le b\right\}$$

where a and b denote half the length of the two sides of Q_j. By the binomial expansion rule, it is straightforward to get:

$$\varphi_j^n(\mathbf{y}) = \mathbf{a}_h^\top \mathbf{Q}_2^\top \mathbf{d}_2\, \mathbf{d}_3^\top \mathbf{Q}_3 \mathbf{a}_k \tag{12}$$

where $\mathbf{a}_h$, $\mathbf{a}_k$ are vector of constants and

$$\{\mathbf{Q}_k\}_{i,j} := \binom{j-1}{i-1} x_k^{j-i} \qquad k = 2,3 \quad i,j = 1,2,...$$

3 Main result

3.1 Analytical integration in time

Focusing on discretization (12) and exploiting a well known convolution property, integral (7) can be recast in the form:

$$\sum_\alpha \omega_\alpha\, \mathbf{a}_h^\top \mathbf{Q}_2^\top\, \mathbb{K}_{rs}(\mathbf{x}, t)\, \mathbf{Q}_3 \mathbf{a}_k \qquad \mathrm{r = u,\ s = u, p}$$

where:

$$\mathbb{K}_{rs}(\mathbf{x}, t) = \int_{Q_j} \mathbf{d}_2 \int_0^t \mathbf{G}_{rs}(\mathbf{x}, \mathbf{y}, \tau)\, (t-\tau)^\alpha\, \mathrm{d}\tau\, \mathbf{d}_3^\top\, \mathrm{d}\Gamma_y = \widehat{\mathbb{K}}^{rs}(\mathbf{x}, d_2, d_3; t)\, \Big|_{d_3=-b-x_3}^{d_3=b-x_3} \Big|_{d_2=-a-x_2}^{d_2=a-x_2} \tag{13}$$

Integration in time, because of the nature of Green's functions, is given in terms of the following outcome of distributions theory:

$$\int_0^t (t-\tau)^\alpha \frac{\partial^k}{\partial \tau^k} \mathrm{H}\,(\tau - \beta)\, \mathrm{d}\tau = \prod_{j=\inf \Xi}^{sup \Xi} (\alpha - j)^{\mathrm{sgn}(j)}\ (\tau - \beta)^{\alpha + 1 - k}\, \mathrm{H}\,(t - \beta) \tag{14}$$

where: i) $\mathrm{H}(\,)$ is the Heaviside distribution, $\delta(\,) = \dot{\mathrm{H}}(\,)$ is the Dirac distribution; ii) $\beta > 0$ holds $\frac{r}{c_T}$ or $\frac{r}{c_L}$, with c_L (c_T) the dilatational (shear) wave speed; iii) $k \in \mathbb{Z}$ can be negative, in this case indicating a primitive of order $|k|$ of $\mathrm{H}(\,)$; iv) $\Xi \subset \mathbb{Z}$ is the set of numbers between 0 and $k-1$ including both of them: for instance, $k = 2 \Rightarrow \Xi = \{0,1\}$, $k = 1 \Rightarrow \Xi = \{0\}$, $k = 0 \Rightarrow \Xi = \{-1,0\}$, $k = -1 \Rightarrow \Xi = \{-2,-1,0\}$; v) function $\mathrm{sgn} : \mathbb{Z} \to -1, 0, 1$ is defined as:

$$\mathrm{sgn}(j) = \begin{cases} -1 & \text{if } j < 0 \\ 0 & \text{if } j = 0 \\ 1 & \text{if } j > 0 \end{cases}$$

3.2 Analytical integration in space

In view of Green's functions contributions, terms of the following kind must be dealt with in order to evaluate $\mathbb{K}_{rs}(\mathbf{x}, t)$:

$$\int_{-a-x_2}^{a-x_2} \int_{-b-x_3}^{b-x_3} \frac{d_3^k}{r^{2m+1}}\, \mathrm{d}d_3 \qquad k, m \in \mathbb{N}_0$$

The identity:

$$\frac{x^{2k}}{\alpha^2 + x^2} = (-1)^k \frac{\alpha^{2k}}{\alpha^2 + x^2} + \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} (\alpha^2 + x^2)^{j-1} (\alpha^2)^{k-j} \tag{15}$$

which comes out from the binomial expansion rule, permits to obtain the following recursive relationship, that seems to be useful for analytical integrations:

$$\frac{d_3^k}{r^{2m+1}} = (-\alpha^2)^{\widehat{k}} \frac{d_3^{k_{[2]}}}{r^{2m+1}} + \sum_{j=1}^{\widehat{k}} \binom{\widehat{k}}{j} (-1)^{\widehat{k}-j} \sum_{h=0}^{j-1} \binom{j-1}{h} \alpha^{2(\widehat{k}-1-h)} \frac{d_3^{2h+k_{[2]}}}{r^{2m-1}} \qquad k, m \in \mathbb{N}_0 \tag{16}$$

where $\alpha^2 = d_1^2 + d_2^2$ is the squared projection of the distance on the plane $d_3 = 0$. Here and in the rest of the paper $\widehat{k} = k \div 2$ stands for the integer division, whereas $k_{[2]} = k - 2\widehat{k}$ is its remainder. Making use of formula (16) and reference to [19] for details, $\widehat{\mathbb{K}}^{rs}(\mathbf{x}, d_2, d_3; t)$ can be reduced to the sum of a set of basic integrals and its final expression, reads:

$$\widehat{\mathbb{K}}^{rs} = \mathbb{L}_2^{rs} \log(d_2 + r) + \mathbb{L}_3^{rs} \log(d_3 + r) + \mathbb{A}^{rs} \operatorname{arctanh} \frac{d_3}{r} + \mathbb{I}^{rs} I_{\square}^{r^{-3}}() + \mathbb{R}^{rs} r + \mathbb{P}^{rs} + \mathbb{S}^{rs} \frac{1}{r} + \mathbb{H}^{rs} \frac{1}{r^3} \tag{17}$$

where: i) $I_{\square}^{r^{-3}}(\mathbf{x}, d_2, d_3)$ is the Lebesgue integral of function $\frac{1}{r^3}$ over Q_j, discussed in details in [20]; ii) $\mathbb{L}_2^{rs}, \mathbb{L}_3^{rs}, \mathbb{A}^{rs}, \mathbb{I}^{rs}, \mathbb{R}^{rs}, \mathbb{P}^{rs}, \mathbb{S}^{rs}, \mathbb{H}^{rs}$ are polynomial matrices of the same order of $\mathbb{K}^{rs}$ whose expression can be found in [19] for constant and linear shape functions in time and space.

4 Concluding Remarks

Analytical integrations have been performed for both the singular and the regular part of $\widehat{\mathbb{K}}^{rs}(\mathbf{x}, d_2, d_3; t)$: for paucity of space, discussion on singularity issues has been here omitted and the reader is referred to [19] for details. The proposed outcomes are exhaustive for the collocation approach as well as for the post-process reconstruction of primal and dual fields (temperature and flux, displacement and stress). It seems to be of interest for the dual, the Galerkin, and the variational technique as well, because it firmly distinguishes the weakly singular terms relevant to the outer integral and the singular terms in the outer integration process. Besides accuracy and computational efficiency, the availability of closed form (17) entails the possibility of analytical manipulations - see e.g. [21] - which are hardly possible with alternative approaches. Obtained results may have influence on extremely modern and stimulating applications, e.g. [22] but need to be extended in order to comply with very promising techniques for time marching schemes [23] to which hypothesis (8) does not apply.

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