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Numerical Solutions of Boundary Value Problems by using A new Cubic B-spline Method

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Abstract

In this study, cubic B-spline method is used with a new approximation of the second derivative to find a numerical solution for boundary value problems of the second order. An error analysis was performed for the method and the accuracy of the method was tested via four numerical examples and the results were compared with the exact solution and cubic B-spline method.

Keywords: boundary value problems, error analysis, cubic B-spline, exact solution.

1. Introduction

Splines, especially B-splines, play an important role in the areas of mathematics and engineering today [2],[17]. Splines are popular in computer graphics because of their finesse, flexibility and accuracy. Historically, Isaac Jacob Schoenberg discovered splines in 1946 [6-10], his work motivated other scientists such as Carl de Boor. In the early seventies de Boor [3], [4], [5] discovered a recursive definition for splines. Birkhoff and de Boor (1964) [1] investigated the error bound and convergence of of spline interpolation. Manguia and Bhatta (2015) [18] used cubic B-spline(CBS) functions for solution of second order boundary value problems(BVPs). Reza and Akram [23], applied of cubic B-splines collocation method for solving nonlinear inverse parabolic partial differential equations. Suardi et. al. [26] used the cubic B-spline solution of two-point boundary value problem using HSKSOR iteration and they presented solutions of two-point boundary value problems by using quarter-sweep SOR iteration with cubic B-Spline scheme[27] .

In this study, approximate solutions was found to problems of second order linear arrangement using B-cubes with a new approximation of the second derivative. Lang and X. Xu[16], introduced a new cubic B-spline method for approximating the solution of a class of nonlinear second-order boundary value problem with two dependent variables. His work was a motivation to other mathematicians such Tassaddiq and others [28] to used his method for solve non-linear differential equations arising in visco-elastic flows and hydrodynamic stability problems.

The presented scheme is based on new approximations for the second order derivatives. The approximation for second order derivative is calculated using appropriate linear combinations to approximate the typical B-spline $y''(x)$ at neighbouring values. In the past two decades, several numerical techniques have been used to explore the numerical solution of linear BVP but as far as we know, this new approximation has not been used for this purpose before for solving BVPs. This work is presented as follows. Section 2 is explanation about the cubic B-splines schemes. We presented the new approximation for $y''(x)$ in Section 3.In Section 4, we descripted of the numerical method for new cubic B-spline. The error analysis of the method is described in Section 5. Section 6 tests numerical experiments to demonstrate the feasibility of the proposed method, and this article ends with a conclusion in Section 7.

2. Derivation of the Cubic B-spline Schemes

Let n be a positive integer and $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of $\left[a,b\right]$, $x_i = x_o + ih, i \in \mathbb{Z}$ and $h = \dfrac{b-a}{n}$. The typical third degree B-spline basis functions are defined: [11-14], [24-26]

$$B_i(x) = \frac{1}{6h^3}\begin{cases} (x-x_{i-2})^3 & if \;\; x \in [x_{i-2}, x_{i-1}] \\ -3(x-x_{i-1})^3 + 3h(x-x_{i-1})^2 + 3h^2(x-x_{i-1}) + h^3 & if \;\; x \in [x_{i-1}, x_i] \\ -3(x_{i+1}-x)^3 + 3h(x_{i+1}-x)^2 + 3h^2(x_{i+1}-x) + h^3 & if \;\; x \in [x_i, x_{i+1}] \\ (x_{i+2}-x)^3 & if \;\; x \in [x_{i+1}, x_{i+2}] \\ 0 & if \;\; otherwise \end{cases} \tag{1}$$

Where $i = -1, 2, \cdots, n+1$. For a sufficiently smooth function $y(x)$ there always exists a unique third degree spline $Y(x)$,

$$Y(x) = \sum_{i=-1}^{n+1} c_i B_i(x) \tag{2}$$

which satisfies the prescribed interpolating conditions

$$Y'(a) = y'(a) \; and \; Y'(b) = y'(b) \;\; , i = 0,1,\ldots,n \;\; \text{for all} \;\; Y(x_i) = y(x_i),$$

Where c_i's are finite constants yet to be determined.

For simplicity, we express the CBS approximations, $Y(x), Y'(x)$ and $Y''(x)$ by Y_j, t_j and T_j, respectively. The cubic B-spline basis function (1) together with (2) and by using Table (1) gives the following relations,

$$Y_j = \sum_{i=j-1}^{i+1} c_i B_i(x) = \frac{1}{6}\left(c_{j-1} + 4c_j + c_{j+1}\right), \tag{3}$$

$$t_j = \sum_{i=j-1}^{j+1} c_i B_i'(x) = \frac{1}{2h}\left(-c_{j-1} + c_{j+1}\right), \tag{4}$$

$$T_j = \sum_{i=j-1}^{j+1} c_i B_i''(x) = \frac{1}{h^2}\left(c_{j-1} - 2c_j + c_{j+1}\right). \tag{5}$$

Moreover ,from (3)-(5) relationships can be created.[7]

$$t_j = y'(x_j) - \frac{1}{180} h^4 y^{(5)}(x_j) + \cdots , \tag{6}$$

$$T_j = y''(x_j) - \frac{1}{12} h^2 y^{(4)}(x_j) + \frac{1}{360} h^4 y^{(6)}(x_j) + \cdots . \tag{7}$$

From (6) and (7) , we have

$\| T_j - y''(x_j) \|_\infty = O(h^2).$ and $\| t_j - y'(x_j) \|_\infty = O(h^4)$

This gives enough motivation to craft a better approximation to, the $y''(x)$.

Table 1: Coefficients of cubic B-spline and its derivative at nodes x_i .

	x_{i-1}	x_i	x_{i+1}	*Else*
$B_i(x)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	*0*
$B_i^{(1)}(x)$	$-\frac{1}{2h}$	*0*	$\frac{1}{2h}$	*0*
$B_i^{(2)}(x)$	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	*0*

3. The New Approximation for $y''(x)$

In order to formulate a new approximation to $y''(x)$, we use (7), to establish the following expression for (T_{j-1}), in knots, $x_j,\ j = 1,2,3,\cdots ,n-1,$ [15-16]

$$T_{j-1} = y''(x_{j-1}) - \frac{1}{12} h^2 y^{(4)}(x_{j-1}) + \frac{1}{360} h^4 y^{(6)}(x_{j-1}) + \cdots ,$$

$$= y''(x_j) - h\, y^{(3)}(x_j) + \frac{5}{12} h^2 y^{(4)}(x_j) - \frac{1}{12} h^3 y^{(5)}(x_j) + \cdots .$$

Similarly,

$$T_{j+1} = y''(x_j) + h\, y^{(3)}(x_j) + \frac{5}{12} h^2 y^{(4)}(x_j) + \frac{1}{12} h^3 y^{(5)}(x_j) + \cdots ,$$

be a new approximation to $y''(x_j)$ such that, T_j let

$$\tilde{T}_j = B_1 T_j + B_2 T_{j-1} + B_3 T_{j+1}. \tag{8}$$

Choosing three parameters B_1, B_2 and B_3 so that the error order of $\tilde{T}_j$ is as high as possible , we obtain

$$B_1 + B_2 + B_3 = 1,$$

$$-B_2 + B_3 = 0,$$

$$-B_1 + 5B_2 + 5B_3 = 0.$$

Hence $B_1 = \dfrac{5}{6}$, and $B_2 = B_3 = \dfrac{1}{12}.$

The expression (8) takes the following form,

$$\tilde{T}_j = B_1 T_j + B_2 T_{j-1} + B_3 T_{j+1} = \frac{1}{12h^2}\left(c_{j-2} + 8c_{j-1} - 18c_j + 8c_{j+1} + c_{j+2}\right). \tag{9}$$

Now we approximate $y''(x)$ at the knot x_0 using four neighboring values, such that.

$$\tilde{T}_0 = B_0 T_0 + B_1 T_1 + B_2 T_2 + B_3 T_3, \tag{10}$$

where.

$$T_1 = y''(x_0) + h\, y^{(3)}(x_0) + \frac{5}{12} h^2 y^{(4)}(x_0) + \frac{1}{12} h^3 y^{(5)}(x_0) + \cdots ,$$

$$T_2 = y''(x_0) + 2h\, y^{(3)}(x_0) + \frac{23}{12} h^2 y^{(4)} + \frac{7}{6} h^3 y^{(5)}(x_0) + \cdots ,$$

$$T_3 = y''(x_0) + 3h\, y^{(3)}(x_0) + \frac{53}{12} h^2 y^{(4)}(x_0) + \frac{17}{4} h^3 y^{(5)}(x_0) + \cdots .$$

The expression (9) yields the following four equations,

$$B_0 + B_1 + B_2 + B_3 = 1,$$

$$B_1 + 2B_2 + 3B_3 = 0,$$

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$$-B_0+5B_1+23B_2+53B_3=0,$$

$$B_1+14B_2+51B_3=0.$$

Hence $B_0=\frac{7}{6}$, $B_1=-\frac{5}{12}$, $B_2=\frac{1}{3}$ and $B_3=-\frac{1}{12}$.

Using these values in (10), we have

$$\tilde{T}_0=\frac{1}{12h^2}\left(14c_{-1}-33c_0+28c_1-14c_2+6c_3-c_4\right). \tag{11}$$

When working in the same style, rounding is presented at node x_n by

$$\tilde{T}_n=\frac{1}{12h^2}\left(-c_{n-4}+6c_{n-3}-14c_{n-2}+28c_{n-1}-33c_n+14c_{n+1}\right), \tag{12}$$

4. Description of the Numerical Method.

In this section, consider the boundary value problems,

$$p(x)y''(x)+q(x)y'(x)+r(x)y(x)=f(x) \tag{13}$$

with boundary conditions

$$y(a)=\alpha,\ y(b)=\beta.$$

Where $p(x)\neq 0, q(x), r(x)$ and $f(x)$ are continuous real-valued functions on the interval $[a,b]$.

Let $Y(x)$ be the cubic B-spline solution to (14) satisfying the interpolating conditions such that

$$Y(x)=\sum_{i=-1}^{n+1}c_iB_i(x). \tag{15}$$

Discretize Eq.(14) in knots $x_j,\ j=1,2,\ldots,n-1,$ we get,

$$p(x_j)Y''_{k+1}(x_j)+q(x_j)Y'_{k+1}(x_j)+r(x_j)Y_{k+1}(x_j)=f(x_j). \tag{16}$$

Using Eqs.(3)-(4) and (9) in Eq.(16) ,we have

$$p(x_j)\left(\frac{c_{j-2}+8c_{j-1}-18c_j+8c_{j+1}+c_{j+2}}{12h^2}\right)$$

$$+q(x_j)\left(\frac{-c_{j-1}+c_{j+1}}{2h}\right)+r(x_j)\left(\frac{c_{j-1}+4c_j+c_{j+1}}{6}\right)=f(x_j). \tag{17}$$

Similarly, at the knots x_0 and x_n, the following equations are obtained

$$p(x_0)\left(\frac{14c_{-1}-33c_0+28c_1-14c_2+6c_3-c_4}{12h^2}\right)$$

$$+q(x_0)\left(\frac{-c_{-1}+c_1}{2h}\right)+r(x_0)\left(\frac{c_{-1}+4c_0+c_1}{6}\right)=f(x_0), \tag{18}$$

$$p(x_n)\left(\frac{14c_{n-1}-33c_n+28c_{n+1}-14c_{n+2}+6c_{n+3}-c_{n+4}}{12h^2}\right)$$

$$+q(x_n)\left(\frac{-c_{n-1}+c_{n+1}}{2h}\right)+r(x_n)\left(\frac{c_{n-1}+4c_n+c_{n+1}}{6}\right)=f(x_n). \tag{19}$$

The boundary conditions are giving of the following two equations

$$c_{-1}+4c_0+c_1=6\alpha, \tag{20}$$

$$c_{n-1}+4c_n+c_{n+1}=6\beta. \tag{21}$$

In This way they have a system of $(n+3)$ linear equations .Eqs.(17)-(19) which can be written in matrix form as

$$Ac=b. \tag{22}$$

Where A is the coefficients matrix given by

$$A=\begin{pmatrix} 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & & & \\ o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & & & & & \\ a_1 & b_1 & c_1 & d_1 & e_1 & & & & & & \\ 0 & a_2 & b_2 & c_2 & d_2 & e_2 & & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & a_{n+1} & b_{n+1} & c_{n+1} & d_{n+1} & e_{n+1} & & \\ & & & & & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ & & & & & & & & 1 & 4 & 1 \end{pmatrix}$$

where

$o_1 = 14p(x_0) - 6hq(x_0) + 2h^2 r(x_0)$

$o_2 = -33p(x_0) + 8h^2 r(x_0),$

$o_3 = 28p(x_0) + 6hq(x_0) + 2h^2 r(x_0)$

$o_4 = 14p(x_0),$

$o_5 = 6p(x_0),$

$o_6 = -p(x_0),$

$where\ i = 1,2,\dots,n-1,$

$a_i = p(x_i),$

$b_i = 8p(x_i) - 6hq(x_i) + 2h^2 r(x_i),$

$c_i = -18p(x_i) + 8h^2 r(x_i),$

$d_i = 8p(x_i) + 6hq(x_i) + 2h^2 r(x_i),$

$e_i = p(x_i)$

$m_1 = -p(x_n),$

$m_2 = 6p(x_n),$

$m_3 = -14p(x_n),$

$m_4 = 28p(x_n) - 6hq(x_n) + 2h^2 r(x_n),$

$m_5 = -33p(x_n) + 8h^2 r(x_n),$

$m_6 = 14p(x_n) + 6hq(x_n) + 2h^2 r(x_n).$

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and $c=[c_{-1},c_0,c_1,\ldots,c_n,c_{n+1}]^T$, $b=[6\alpha,12h^2f(x_0),12h^2f(x_1),\ldots,12h^2f(x_{n-1}),12h^2f(x_n),6\beta]^T$,

since A is a non-singular matrix, so can solve the system $Ac=b$ for $c_{-1},c_0,c_1,\ldots c_{n-1},c_n,c_{n+1}$ substituting these values in Eq. (15), to get the required approximate solution.

5. Error Analysis

Now, the error analysis is investigated by using the cubic B-spline approximations Eqs.(3)-(5) and Eq.(9) the following relationships can be established

$$h\left[\frac{1}{6}Y'(x_{j-1})+\frac{4}{6}Y'(x_j)+\frac{1}{6}Y'(x_{j+1})\right]=\frac{1}{2}\left[Y(x_{j+1})-Y(x_{j-1})\right], \tag{23}$$

$$h^2Y''(x_j)=\frac{1}{2}\left(7Y(x_{j-1})-8Y(x_j)+Y(x_{j+1})\right)+h\left(Y'(x_{j-1})+2Y'(x_j)\right). \tag{24}$$

Moreover ,we have

$$h^3Y'''(x_j)=12\left[Y(x_j)-Y(x_{j+1})\right]+6h\left[Y'(x_j)+Y'(x_{j+1})\right], \tag{25}$$

$$h^3Y'''(x_j)=12\left[Y(x_{j-1})-Y(x_j)\right]+6h\left[Y'(x_{j-1})+Y'(x_j)\right]. \tag{26}$$

Where $Y'''(x_{j^+})$ *and* $Y'''(x_{j^-})$ indicate approximate values of in $Y'''(x_j)$ in $[x_j,x_{j+1}]$ and $[x_{j-1},x_j]$ respectively.

$E^{\lambda}\left(Y'(x_j)\right)=Y'(x_{j+\lambda}),\lambda\in Z,$ Using the operator notation

Equation (19) can also be written as

$h\left[\frac{1}{6}E^{-1}+\frac{4}{6}+\frac{1}{6}E\right]Y'(x_j)=\frac{1}{2}\left[E-E^{-1}\right]y(x_j),$ Hence

$$hS'(x_j)=3\left(E-E^{-1}\right)\left[E^{-1}+4+E\right]^{-1}s(x_j), \tag{27}$$

Using $E=e^{hD}$, $D=\frac{d}{dx}$, we can get it

$$E+E^{-1}=e^{hD}+e^{-hD}=2\left[1+\frac{h^2D^2}{2!}+\frac{h^4D^4}{4!}+\frac{h^6D^6}{6!}+\cdots\right],$$

$$E-E^{-1}=e^{hD}-e^{-hD}=2\left[hD+\frac{h^3D^3}{3!}+\frac{h^5D^5}{5!}+\frac{h^7D^7}{7!}+\cdots\right].$$

Therefore, Eq. (27) can be expressed as.

$$Y'\left(x_j\right)=\left(D+\frac{h^2D^3}{3!}+\frac{h^4D^5}{5!}+\cdots\right)\left[1+\left(\frac{h^2D^2}{6}+\frac{h^4D^4}{72}+\frac{h^6D^6}{2160}+\cdots\right)\right]^{-1}y\left(x_j\right),$$

Simplify, we get.

$$Y'\left(x_j\right)=\left(D-\frac{h^4D^5}{180}+\frac{h^6D^7}{1512}-\cdots\right)y\left(x_j\right),$$

Hence

$$Y'\left(x_j\right)=y'\left(x_j\right)-\frac{1}{180}h^4y^{(5)}\left(x_j\right)+\cdots, \tag{28}$$

Similarly, writing Eq. (20) in operator notation we have

$$h^2Y''\left(x_j\right)=\frac{1}{2}\left[7E^{-1}-8+E\right]y\left(x_j\right)+h\left[E^{-1}+2\right]y'\left(x_j\right),$$

$$=\left(-3hD+2h^2D^2-\frac{h^3D^3}{2}+\frac{h^4D^4}{6}-\frac{h^5D^5}{40}+\frac{h^6D^6}{180}-\cdots\right)y\left(x_j\right)$$

$$+\left(3h-h^2D+\frac{h^3D^2}{2}-\frac{h^4D^3}{6}+\frac{h^5D^4}{24}-\frac{h^6D^5}{120}+\cdots\right)y'\left(x_j\right).$$

Simplify the relationship above, we have.

$$Y''\left(x_j\right)=y''\left(x_j\right)+\frac{1}{60}h^3y^{(5)}\left(x_j\right)-\frac{1}{360}h^4y^{(6)}\left(x_j\right)+\cdots . \tag{29}$$

Using the same method in Eq.(21) it can also be written,

$$Y'''\left(x_j\right)\cong\frac{1}{2}\left[y'''\left(x_{j^+}\right)+y'''\left(x_{j^-}\right)\right]=y'''\left(x_j\right)+\frac{1}{12}h^2y^{(5)}\left(x_j\right)+\cdots . \tag{30}$$

Let us define the term error $e(x)=Y(x)-y(x)$, using relations (24) and (26) in the Taylor series expand $e(x)$ we get

$$e\left(x_j+\theta h\right)=\frac{\theta(5\theta-2)(\theta+1)}{360}h^5y^{(5)}\left(x_j\right)-\frac{\theta^2}{720}h^6y^{(6)}\left(x_j\right)+\ldots \quad . \tag{31}$$

Where $\theta\in\left[0,1\right]$, from Eq. (31) The new B-spline approximation is $O\left(h^5\right)$ accurate.

6. Numerical Examples

In this section we illustrate the numerical techniques discussed in the previous sections by the following two boundary value problems of Eqs.(1-2) , in order to illustrate the comparative performance of our method over other existing methods. We now consider four numerical examples to illustrate the comparative performance of our method. All calculations are implemented by Maple.

Example 1: We consider a linear boundary value problem with constant coefficients :[18]

$$y''(x)+y'(x)-6y(x)=x,$$

with boundary conditions $y(0)=0, y(1)=1,$

The exact solution to boundary value problem is

$$y(x)=\frac{(43-e^2)e^{-3x}-(43-e^{-3})e^{2x}}{36(e^{-3}-e^2)}-\frac{1}{6}x-\frac{1}{36}.$$

The numerical result of the example (1) are presented in the Table (2) for with $n=20$ .In Table 3 the observed maximum absolute errors and compared our result with the results given in cubic b-spline method [18]. Figure 1 shows the comparison of the exact and numerical solutions for $n=20$.

Table 2: The numerical solutions and exact solution of example (1).

x	New Cubic B-Spline	Cubic B-Spline[18]
0	0	0
0.2	5.59E-8	2.3534E-5
0.3	6.23 E-8	4.41179E-5
0.4	6.06 E-8	6.46773E-5
0.5	5.44 E-8	8.19815E-5
0.6	4.57 E-8	9.30536E-5
0.7	3.59 E-8	9.47169E-5
0.8	2.54 E-8	8.31905E-5
0.9	1.52 E-8	5.36906E-5
1	0	0

Table 3: Comparison of the error proposed method with CBS[18] for example(1) .

x	New Cubic B-Spline	Exact Solution
0	0	0
0.2	0.1074285058	0.1074285617
0.3	0.1636254812	0.1636255435
0.4	0.2267411540	0.2267412146
0.5	0.3006953149	0.3006953693
0.6	0.3896566891	0.3896567348
0.7	0.4982584629	0.4982584988
0.8	0.6318199536	0.6318199790
0.9	0.796586555	0.7965865702
1	1	1

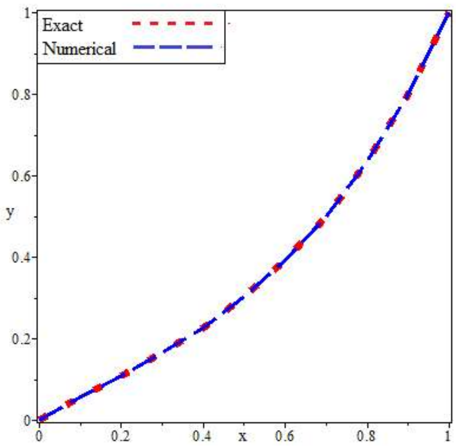

Figure 1 : Comparison of the exact and the proposed method of example(1) for n=20

Example 2: We consider a linear boundary value problem with constant coefficients[18],

$$y''(x)+2y'(x)+5y(x)=6\cos(2x)-7\sin(2x), \quad \text{for } 0<x<\frac{\pi}{4},$$

with boundary conditions

$$y(0)=4,\ y(\frac{\pi}{4})=1.$$

The exact solution to boundary value problem is

$$y(x)=2(1+e^{-x})\cos(2x)+\sin(2x).$$

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The numerical result of the example (2) are presented in the Table 4 compared our result with the exact solution. In Table 5 the observed maximum absolute errors and compared our result with the results given in cubic B-spline method [18]. Figure 2 shows the comparison of the exact and numerical solutions for $n = 20$.

Table 4: The numerical solutions and exact solution of example (2) .

x	New Cubic B-Spline	Exact Solution
$\frac{\pi}{80}$	3.989348208	3.9893481701
$\frac{3\pi}{80}$	3.906796607	3.9067967056
$\frac{5\pi}{80}$	3.748792026	3.7487922376
$\frac{7\pi}{80}$	3.523205708	3.5232056151
$\frac{9\pi}{80}$	3.238294433	3.2382892895
$\frac{11\pi}{80}$	2.902583355	2.9025837374
$\frac{13\pi}{80}$	2.524830455	2.5248342470
$\frac{15\pi}{80}$	2.113912251	2.1139139602
$\frac{17\pi}{80}$	1.678750121	1.6787494845
$\frac{19\pi}{80}$	1.228243494	1.2282459716

Table 5: Comparison of the error proposed method with CBS[18] for example(2) .

x	New Cubic B-Spline	Cubic B-Spline[18]
$\frac{\pi}{80}$	3.8E-8	2.0634E-5
$\frac{3\pi}{80}$	9.9 E-8	4.8130E-5
$\frac{5\pi}{80}$	2.12 E-7	6.0894E-5
$\frac{7\pi}{80}$	9.3 E-8	6.2779E-5
$\frac{9\pi}{80}$	5.143E-6	5.70988E-5
$\frac{11\pi}{80}$	3.82 E-7	4.67074E-5
$\frac{13\pi}{80}$	3.792E-6	3.40587E-5
$\frac{15\pi}{80}$	1.709E-6	2.12666E-5
$\frac{17\pi}{80}$	6.37 E-7	1.01538E-5
$\frac{19\pi}{80}$	2.478E-6	2.2885E-5

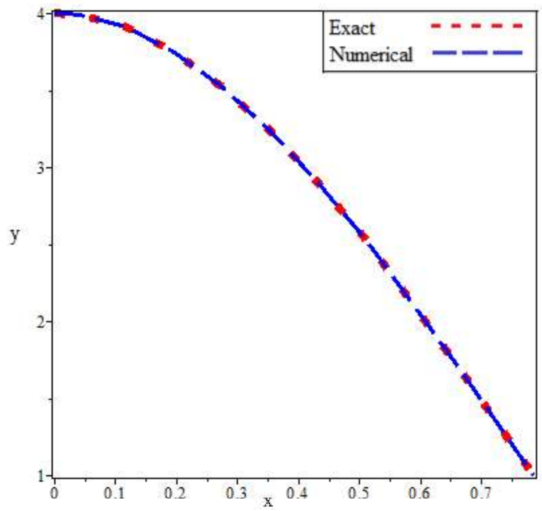

Figure 2 : Comparison of the exact and the proposed method of example(2) for n=20.

Example 3: We consider a linear boundary value problem with constant coefficients[18]

$x^2 y''(x) + 3xy'(x) + 3y = 0$ for $1 < x < 2$,

with boundary conditions

$y(1) = 5,\ y(2) = 0.$

The exact solution to boundary value problem is

$$y(x) = \frac{5}{x}[\cos(\sqrt{2}\ln x) - \cot(\sqrt{2}\ln 2)\sin(\sqrt{2}\ln x)].$$

The numerical result of the example (3) are presented in the Table (6) for with. In Table 7 the observed maximum absolute errors and compared our result with the results given in cubic B-spline method [18]. Figure 3 shows the comparison of the exact and numerical solutions for $n = 20.$

Table 6: The numerical solutions and exact solution of example (3).

x	New Cubic B-Spline	Exact Solution
1.1	4.094768326	4.0947693502
1.2	3.316711309	3.3167126115
1.3	2.649607254	2.6496084276
1.4	2.077976455	2.0779773959
1.5	1.587980746	1.5879814418
1.6	1.167624994	1.1676254805
1.7	0.806670353	0.8066706529
1.8	0.496442085	0.4964422651
1.9	0.229613526	0.2296136048

Table 7: Comparison of the error proposed method with CBS [18] for example(3) .

x	New Cubic B-Spline	Cubic B-Spline[18]
1.1	1.024E-6	1.609202E-4
1.2	1.303E-6	3.065565E-4
1.3	1.174E-6	3.980724E-4
1.4	9.41E-7	4.327606E-4
1.5	6.96E-7	4.197742E-4
1.6	4.86E-7	3.707492E-4
1.7	2.999E-7	2.964084E-4
1.8	1.801E-7	2.055654E-4
1.9	7.88E-8	1.050575E-4

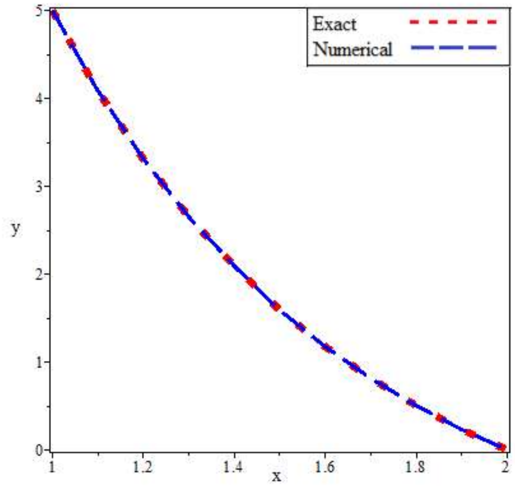

Figure 3 : Comparison of the exact and the proposed method of example(3) for n=20.

Example 4: We consider a linear boundary value problem with constant coefficients,[18]

$$xy''(x)+y'(x)=x \quad \text{for} \quad 1<x<2,$$

with boundary conditions

$$y(1)=1,\ y(2)=1.$$

The exact solution to boundary value problem is

$$y(x)=\frac{x^2}{4}-\frac{3\ln x}{4\ln 2}+\frac{3}{4}.$$

The numerical result of the example (4) are presented in the Table 8 for with .In Table 9 the observed maximum absolute errors and compared our result with the results given in cubic B-spline method [18]. Figure 4 shows the comparison of the exact and numerical solutions for $n=20.$

Table 8: The numerical solutions and exact solution of example (4).

x	New Cubic B-Spline	Exact Solution
1.1	0.9493723880	0.9493723572
1.2	0.9127242439	0.9127241956
1.3	0.8886163346	0.8886162826
1.4	0.8759299325	0.8759298796
1.5	0.8737781718	0.8737781245
1.6	0.8814461109	0.8814460712
1.7	0.8983489704	0.8983489402
1.8	0.9240023401	0.9240023201
1.9	0.9580004464	0.9580004361

Table 9: Comparison of the error proposed method with CBS [18] for example(4) .

x	New Cubic B-Spline	Cubic B-Spline[18]
1.1	3.08E-8	2.38675E-5
1.2	4.83 E-8	3.66902E-5
1.3	5.20 E-8	4.21471E-5
1.4	5.29 E-8	4.25917E-5
1.5	4.73 E-8	3.95759E-5
1.6	3.97 E-8	3.41494E-5
1.7	3.02 E-8	270371E-5
1.8	2.00 E-8	1.87491E-5
1.9	1.03 E-8	9.6491E-5

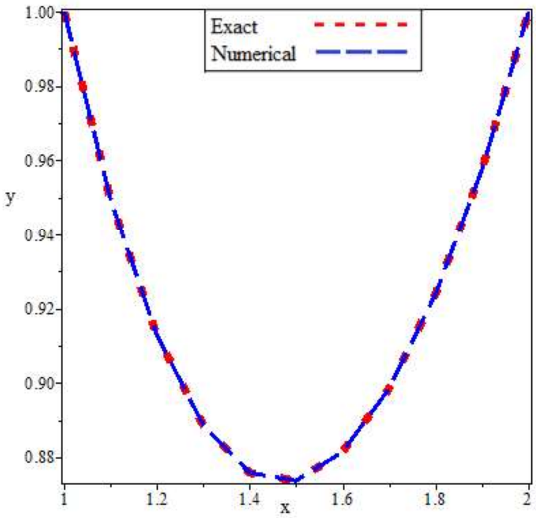

Figure 4 : Comparison of the exact and the proposed method of example(4) for n=20.

7. Conclusion

The cubic B-spline method with a new approximation of the second derivative is developed for the approximate solution of second order two –point BVPs in this paper. Four examples are considered for numerical illustration of the method. Numerical result are presented in Tables (2), (4), (6), and (8) and compared with the exact solutions. We also compared the results with the (CBS) method [18] in Tables (3), (5), (7), and (9) and It can be concluded that this method is quite suitable, accurate.

The obtained numerical results show that the proposed methods maintain a high accuracy which make them are very encouraging for dealing with the solution of this type of two point boundary value problems.

المستخلص

في هذه الدراسة ، تم استخدام طريقة B-spline المكعبة مع تقريب جديد للمشتقة الثانية لإيجاد حل عددي لمسائل القيم المحددة من الدرجة الثانية. تم إجراء تحليل الخطأ للطريقة وتم اختبار دقة الطريقة باربعة أمثلة عددية وقورنت النتائج مع الحل الدقيق وطريقة B-spline المكعبة

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