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On topological spaces generated by graphs and vice versa

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ABSTRACT

The relationship between Graph Theory and Topological Space has recently developed greatly , as researchers have been able to find solutions to some problems in daily life by transforming the problem into a graph and then generating a topological space and thus facilitating reading the problem and solving it. The researchers also studied the generation of graph from topological spaces.In this article we will present two types of relations on the edges set that generate topological spaces, and we will discuss some properties of this topology, and we will study discuss the method of returning from the topological space to the graphs through using previous relationships.

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1. Introduction

The graph theory is a fundamental mathematical tool for a wide range of applications for example in computers network and Many problems arising in such different fields as industry, chemistry and electrical engineering, marketing, education and management, can be posed as problems from a in graph theory. It can be said that subjects of mathematics in general have arisen it was developed for the need for applications. For example, geometry arose on the banks of the Nile to measure land,regulate agriculture, topology emerged to address engineering problems and Mathematical analysis that does not depend on distances, algebra arose to regulate transactions. Many People thinking topology is far from applications but in the end of the twentieth century and the beginning of the twenty-first century many directions have been added to the previous directions the topology was in such as modern physics, artificial intelligence, and economics are used Topology was used in the fourth decade of the nineteenth century and the biggest proof of the importance of topology. Recently, some researches have created topologies from graphs using various methods. In 2013, SNF Al-khafaji [7] have constructed a topology on graphs and a topology on subgraphs, and in 2018, KA Abdu [1] have constructed applying the topology on digraphs by associate two topologies with the set of edges of any directed graph, called compatible and incompatible edges topologies. Various relationships were used by researchers, as some researchers presented the relationships that connect the topological space with the graph by means of vertices [6,10,11].

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In this paper aims to link the graph theory and topological space, by relations on the graphs that induces new types of topological space to the graph and then apply this space to study some of the properties of this graphs.

2. Preliminaries

In this section, we mentioned some definitions, of graphs and topological space that are needed in this study.

Definition 1. [2] A graph $G = (V(G), E(G), \Psi)$, Where $V(G)$ is a non-empty set called vertex set, (an element in it called vertex) and the set $E(G)$ is a set called edge set, (an element in it called an edge). The end vertices of edge are said to be incident with the edge, and vice versa. Two vertices which are incident with the common edge are adjacent, as are two edges which are incident with the common vertex. If $V \times V$ is considered as the set of unordered pair $\{v , v'\}$'s, then the graph is called an undirected graph. Usually, denote (un), unordered pair of vertices by either vv' or $v'v$ is instead of $\{v , v'\}$. If $V \times V$ is considered as the set of ordered pair $\{v , v'\}$'s, then the graph is called directed- graph, or digraph.

Definition 2. [2] A path P_n is a graph of order n and the size $n-1$ whose all vertices and edges are distanct as a sequence $P_n: x_0 e_1 x_1 e_2 x_2 e_3 \dots e_n x_n$ such that $\Psi_{P_n}(e_i) = x_{i-1}x_i$, $\forall\, i = 1, \dots , n$.. The closed path P_n is called the cycle (denoted by C_n). The simple graph of order n is called complete graph if any two vertices in G are adjacent (denoted by K_n). And it should be known that k_n has exactly $\frac{n(n-1)}{2}$ edges. And the complete graph K_n is $(n-1)$ – regular.If the vertices of a graph $G = (V, E)$ can be split into two disjoint sets X and Y, such that each the edge of a graph G joins the vertex of X and the vertex of Y, then the graph G is the bipartite graph. The complete bipartite graph $K_{n,m}$ is the bipartite graph in which all vertex in X is joined to all vertex in Y by just one edge. It is usual to called $K_{1,n}$ a star, where x is call the center of a star.

Definition 3. [12] Let A be a non-empty set, the set $\tau \subseteq P(A)$ is called a topology on A if the following condition holds:
(a) A and $\emptyset$ belongs to τ.
(b) The intersection of the finite numbers of sets in τ, is belong in τ.
(c) The union of any numbers of the sets in τ belongs to τ.
Then the pair (A, τ) is said to be a topological space.

Definition 4. [12] Let A be a set, a **basis** B for a topology τ on A is a family of subsets of A such that:
(1) For each $x \in A$ there is at least one basis element B containing x.
(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

Main Results

3. *Topological space generated by graphs*

Our main result is that: In these sections we will established new relations to a generated topological space from graph and vice versa.

3.1 The neighborhood Topology of undirected Graph

Definition 5. Let G be a graph . We define the set $N(e) = \{e' \in E(G): e' \textit{ is an adjacent with } e\}$ for each e which will be the open Neighborhood of e . We define a relation on the edge of the graph G (called Edge Neighborhood relation) as: $e \in E(G)$, $N_G(E) = \{N(e)\}_{e \in E(G)}$. Let NB_G be a basis that is generated by the relation $N_G(E)$ above and let $N\tau(G)$ a topology generated by that basis (called Neighborhood Topology) on G.

Example 1.

Let G be a graph which is shown in figure (1).

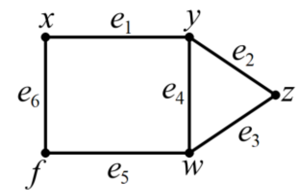


fig.1- A topological graph

We will find the Neighborhood topology $N\tau_G$ of the graph G as follows:

$N(e_1) = \{e_2, e_4, e_6\}$, $N(e_2) = \{e_1, e_3, e_4\}$, $N(e_3) = \{e_2, e_4, e_5\}$, $N(e_4) = \{e_1, e_2, e_3, e_5\}$, $N(e_5) = \{e_3, e_4, e_6\}$,
$N(e_6) = \{e_1, e_5\}$,
$N_G(E) = \{\{e_2, e_4, e_6\}, \{e_1, e_3, e_4\}, \{e_2, e_4, e_5\}, \{e_1, e_2, e_3, e_5\}, \{e_3, e_4, e_6\}, \{e_1, e_5\}\}$

$NB_G(E) = \{\emptyset, \{e_2, e_4, e_6\}, \{e_1, e_3, e_4\}, \{e_2, e_4, e_5\}, \{e_1, e_2, e_3, e_5\}, \{e_3, e_4, e_6\}, \{e_1, e_5\}, \{e_2, e_5\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_4, e_6\}, \{e_5\}, \{e_4\}, \{e_2\}, \{e_1\}, \{e_3\}\}$.

$N\tau_G = \{E, \emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_1, e_5\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_2, e_5\}, \{e_3, e_4\}, \{e_3, e_5\}, \{e_4, e_5\}, \{e_4, e_6\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_5\}, \{e_1, e_3, e_4\}, \{e_1, e_3, e_5\}, \{e_1, e_4, e_5\}, \{e_1, e_4, e_6\}, \{e_2, e_3, e_4\}, \{e_2, e_3, e_5\}, \{e_2, e_4, e_5\}, \{e_2, e_4, e_6\}, \{e_3, e_4, e_5\}, \{e_3, e_4, e_6\}, \{e_4, e_5, e_6\}, \{e_1, e_2, e_3, e_4\}, \{e_1, e_2, e_3, e_5\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_3, e_4, e_5\}, \{e_1, e_3, e_4, e_6\}, \{e_1, e_4, e_5, e_6\}, \{e_2, e_3, e_4, e_5\}, \{e_2, e_3, e_4, e_6\}, \{e_2, e_4, e_5, e_6\}, \{e_3, e_4, e_5, e_6\}, \{e_1, e_2, e_3, e_4, e_5\}, \{e_1, e_2, e_3, e_4, e_6\}, \{e_1, e_2, e_3, e_5, e_6\}, \{e_1, e_2, e_4, e_5, e_6\}, \{e_1, e_3, e_4, e_5, e_6\}, \{e_2, e_3, e_4, e_5, e_6\}\}$.

Theorem 1.

Let K_n be a complete graph, where $E(K_n) = \{e_{ij}\}$, $i = 1,2, \dots, \frac{n(n-1)}{2}$, $j = i+1, i+2, \dots, \frac{n(n-1)}{2}$. Then the topology NT_{K_n} generated by K_n, ($n \neq 4$) is discrete topology on $E(K_n)$.

Proof: The Edge Neighborhoods relation of set $E(K_n) = \{e_{ij}\}$, $i = 1,2, \dots, \frac{n(n-1)}{2}$, $j = i+1, i+2, \dots, \frac{n(n-1)}{2}$:

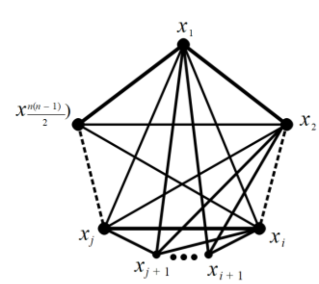
$$N(e) = \{(x, x'), (y, y') \in E(G)\}_{x', y' \in V(G)/\{x,y\}}\} \ \forall e = (x, y) \in E(G).$$

Therefore, $\{e\} \in N_G, \forall\, e \in E(G)$. Then for any $1 \le i \le \frac{n(n-1)}{2}$, $1 \le j \le \frac{n(n-1)}{2}$,

we have $\left(\cap_{k=1, k\neq i, k\neq j}^{\frac{n(n-1)}{n}} N(x_i, x_k) \right) \cap \left(\cap_{s=1, s\neq i, s\neq j}^{\frac{n(n-1)}{n}} N(x_j, x_s) \right) = \{e_{ij}\} = \{(x_i, x_j)\}$

so, $\{e\} \in NB_G$, $\forall\, e \in E(G)$.

Then $N\tau_{K_n} = P(E)$. When $P(E)$ is the power set of edge set E:



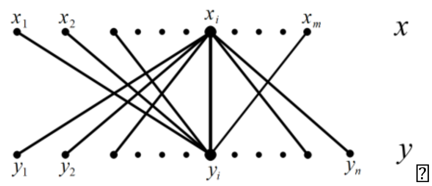
It is seen has been proved that $N\tau_{K_n}$ is discrete topology on E.

Theorem 2.

Let $K_{n,m} = (V, E)$ be a complete bipartite graph. Then the topological space generated by $K_{n,m}$ is a discrete.

Proof:

Let $K_{n,m}$ be a complete bipartite graph with bipartite $\{X, Y\}$ such that $|X| \geq 3$, $|Y| \geq 3$,



Let $x, x' \in X, y, y' \in Y$

Then $N(x, y') \cap N(x', y) = \{(x, y)\}$, for all $x, x', y, y' \in N(G)$.

$N(e) = \{(x, y_j), (x_i, y) \in E(G): for\ any\ 1 \leq i \leq m , 1 \leq j \leq n , x_i \in X/\{x\},\ y_j \in Y/\{y\}\}$,$\forall\ e = (x, y) \in E(G)$

We have $\left(\cap_{k=1,K\neq i}^{n} N(x_i, y_k)\right) \cap \left(\cap_{s=1,S\neq j}^{m} N(x_s, y_j)\right) = \{e_{ij}\} = \{(x_i, y_j)\}$

Then $\{e\} \in NB_G$,$\forall\ e \in E(G)$. Then $N\tau_{K_{n,m}}$ is a discrete topology on E.

Theorem 3.

Let C_n be a cycle graph whose edges set is $E = \{e_1, e_2, \dots, e_n\}$ where $n \geq 3$ $(n \neq 4)$ Then the topological space generated by the cycle C_n is a discrete topological space.

Proof:

The neighborhoods relation of edges set $E = \{e_1, e_2, \dots, e_n\}$.is

$$N(e_i) = \begin{cases} \{e_n, e_2\} & i = 1 \\ \{e_{i-1}, e_{i+1}\} & 2 \leq i \leq n-1 \\ \{e_{i-1}, e_1\} & i = n \end{cases}$$

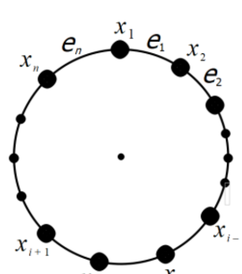
So, the basis is,

$NB_G = \left\{\{e_{n-1}\}, \{e_n\}, \{e_{n-1}, e_1\}, \{e_n, e_2\}\right\} \cup \left\{\{e_i\}, \{e_i, e_{i+2}\}\right\}_{i=1}^{n-2}$

$N(e_i) \cap N(e_{i+2}) = \{e_{i+1}\}$,$\forall\ i = 1,2, \dots, n-2$.

$N(e_{n-1}) \cap N(e_1) = \{e_n\}, N(e_n) \cap N(e_2) = \{e_1\}$

$\{e\} \in NB_G$,$\forall\ e \in E(G)$.



Then the topology generated by C_n is

$N\tau_{C_n} = P(E) = \{\varphi, E\} \cup \{\{a_1, a_2, \dots, a_k\}: 1 \le k \le n-1\}$.

It is seen has been proved that $N\tau_{C_n}$ is discrete topology on E.

Remark 1.

Let P_n be a path of length n. The relation of **Neighborhood Edges** on $E(P_n) = \{1, 2, 3, \dots, n\}$ is $N(E) = \{\{2\}, \{n-1\}, \{1,3\}, \{2,4\}, \{3,5\}, \dots, \{n-2, n\}\}$

The basis is $NB_G = \{\varphi\} \cup \{\{i+1\}, \{i, i+2\}\}_{i=1}^{n-2}$

The Topology of that relation on E is define as:

$N\tau_G = \{\varphi\} \cup \{\{a_1, a_2, \dots, a_k\} : 1 \le k \le n-1\} \cup \{\{n-(k-1), n-(k-2), \dots, n-1, n\}: 2 \le k \le n\}$

where, for $1 \le r \le k$,

$$a_r = \begin{cases} 2, 3, \dots, n-1, & r = 1, k = 1 \\ 1, 2, \dots, n-k, & r = 1, k > 1 \\ 2, & r = 2, a_1 = 1, k > 2 \\ 3, & (r = 2, a_1 = 1, k = 2) \text{ or } \\ & (r = 3, a_1 = 1, a_2 = 2, k > 2) \\ a_{(r-1)} + 1, a_{(r-1)} + 2, \dots, n - (k - (r-1)), & 2 \le r \le k, \begin{pmatrix} (r = 2, k = 2, a_1 \ne 1) & or \\ (r = 3, k = 3, a_1 \ne 1, a_2 \ne 2) \end{pmatrix} \\ n, & r = k > 2, (a_{r-2} = n-2 \text{ or } a_{r-1} = n-2) \end{cases}$$

Example 2. Let G be a graph

1- Let $G = P_4$ (as it is shown in figure 2). Then

Fig.2- P_4

$N(e_1) = \{e_2\}, N(e_2) = \{e_1, e_3\}, N(e_3) = \{e_2, e_4\}, N(e_4) = \{e_3\}$
$N_G(E) = \{\{e_2\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_3\}\}$
$NB_G(E) = \{\emptyset, \{e_2\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_3\}\}$.

$N\tau_G = \{E, \emptyset, \{e_2\}, \{e_3\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}\}$.

2- Let $G = P_9$ (as it is shown in figure 3). Then

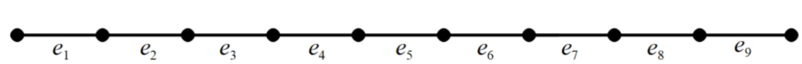


Fig.3- P_9

$N(e_1) = \{e_2\}, N(e_2) = \{e_1, e_3\}, N(e_3) = \{e_2, e_4\}, N(e_4) = \{e_3, e_5\}, N(e_5) = \{e_4, e_6\}$,
$N(e_6) = \{e_5, e_7\}, N(e_7) = \{e_6, e_8\}, N(e_8) = \{e_7, e_9\}, N(e_9) = \{e_8\}$
$N_G(E) = \{\{e_2\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_3, e_5\}, \{e_4, e_6\}, \{e_5, e_7\}, \{e_6, e_8\}, \{e_7, e_9\}, \{e_8\}\}$

$NB_G(E) = \{\emptyset, \{e_2\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_3, e_5\}, \{e_4, e_6\}, \{e_5, e_7\}, \{e_6, e_8\}, \{e_8\}, \{e_7\}, \{e_3\}, \{e_4\}, \{e_5\}, \{e_6\}\}.$

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$, \{e_2, e_3, e_4, e_5, e_6, e_8\}, \{e_2, e_3, e_4, e_5, e_7, e_9\}, \{e_2, e_3, e_4, e_6, e_7, e_8\}, \{e_2, e_3, e_4, e_6, e_7, e_9\}, \{e_2, e_3, e_5, e_6, e_7, e_8\}, \{e_2, e_3, e_5, e_6, e_7, e_9\},$
$\{e_2, e_4, e_5, e_6, e_7, e_8\}, \{e_2, e_4, e_5, e_6, e_7, e_9\}, \{e_3, e_4, e_5, e_6, e_7, e_8\}, \{e_3, e_4, e_5, e_6, e_7, e_9\}, \{e_4, e_5, e_6, e_7, e_8, e_9\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\},$
$\{e_1, e_2, e_3, e_4, e_5, e_6, e_8\}, \{e_1, e_2, e_3, e_4, e_5, e_7, e_9\}, \{e_1, e_2, e_3, e_4, e_6, e_7, e_8\}, \{e_1, e_2, e_3, e_4, e_6, e_7, e_9\},$
$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}, \{e_1, e_2, e_3, e_4, e_7, e_8, e_9\}, \{e_1, e_3, e_4, e_5, e_6, e_7, e_8\}, \{e_2, e_3, e_4, e_5, e_6, e_7, e_8\},$
$\{e_2, e_3, e_4, e_5, e_6, e_7, e_9\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}, \{e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}\}.$

3.2 The Neighborhood Topology of directed Graph

Definition 6. Let $G = (V(G), E(G))$ be directed graph. Then for each $e \in E(G)$, we define the out and in neighborhood of e by

$$N^+(e) = \left\{ \begin{matrix} e' \in E(G): There\ is\ a\ vertix\ \ x\ suchthat\ e\ is\ out - \\ going\ of\ x\ and\ e' is\ in - comming\ of\ x \end{matrix} \right\}\ and\ N'(e) = N^+(e) \cup \{e\}.$$

$$N^-(e) = \left\{ \begin{matrix} e' \in E(G): There\ is\ a\ vertix\ \ x\ suchthat\ e'\ is\ out - \\ going\ of\ x\ and\ e\ is\ in - comming\ of\ x \end{matrix} \right\}\ and\ N''(e) = N^-(e) \cup \{e\}$$

We define a base NB_G using the finite intersection of members of B. From the arbitrary union of members of NB_G,

we have the topological structure $N\tau_G$ on G.

Example 3. Let G be directed graph as it is shown in figure (5).

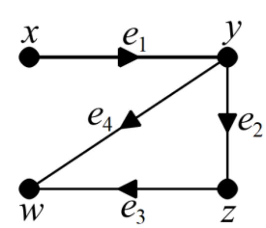


Fig.5- The directed Graph G.

$N^+(e_1) = \{e_2, e_4\}, N^+(e_2) = \{e_3\}, N^+(e_3) = \emptyset, N^+(e_4) = \emptyset$
$N'(e_1) = \{e_1\} \cup \{e_2, e_4\} = \{e_1, e_2, e_4\}, N'(e_2) = \{e_3\} \cup \{e_2\} = \{e_2, e_3\}$
$N'(e_3) = \emptyset \cup \{e_3\} = \{e_3\}, N'(e_4) = \emptyset \cup \{e_4\} = \{e_4\}$
$N'(e) = \{\{e_3\}, \{e_4\}, \{e_2, e_3\}, \{e_1, e_2, e_4\}\}$
$NB_G = \{\emptyset, \{e_2\}, \{e_3\}, \{e_4\}, \{e_2, e_3\}, \{e_1, e_2, e_4\}\}$
$N\tau_G = \{E, \emptyset, \{e_2\}, \{e_3\}, \{e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_2, e_3, e_4\}\}$
$N^-(e_1) = \emptyset, N^-(e_2) = \{e_1\}, N^-(e_3) = \{e_2\}, N^-(e_4) = \{e_1\}$

$N''(e_1) = \emptyset \cup \{e_1\} = \{ e_1\}$, $N''(e_2) = \{ e_1\} \cup \{e_2\} = \{ e_1, e_2\}$
$N''(e_3) = \{e_2\} \cup \{e_3\} = \{ e_2, e_3\}$, $N''(e_4) = \{ e_1\} \cup \{e_4\} = \{ e_1, e_4\}$

$N''(e) = \{\{ e_1\} , \{ e_1, e_2\} , \{ e_2, e_3\} , \{ e_1, e_4\}\}$
$NB_G = \{\emptyset, \{ e_1\} ,\{ e_4\}, \{ e_1, e_2\} , \{ e_1, e_4\}, \{ e_2, e_3\}\}$
$N\tau_G = \{E, \emptyset, \{ e_1\} , \{ e_4\}, \{ e_1, e_2\} , \{ e_1, e_4\}, \{ e_2, e_3\}, \{ e_1, e_2, e_3\}, \{ e_1, e_2, e_4\}, \{e_2, e_3, e_4\}\}$.
$N' \cup N'' = \{\{ e_3\}, \{ e_4\}, \{ e_2, e_3\},\{ e_1, e_2, e_4\}, \{ e_1\} , \{ e_1, e_2\} , \{ e_1, e_4\}\}$
$NB_G = \{\emptyset, \{ e_1\} , \{ e_2\}, \{ e_3\},\{ e_4\}, \{ e_1, e_2\}, \{ e_1, e_4\}, \{ e_2, e_3\},\{ e_1, e_2, e_4\}\}$
$N^{\pm}\tau(G) = \{E, \emptyset\} \cup \{\{ e_1\} , \{ e_2\}, \{ e_3\},\{e_4\}, \{ e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}\}$.

4. Induced Graphs by Topological Space

In this section, we will study new forms of graphs that we generate from topology by using the methods that we mentioned previously.

4.1 Using the Neighborhood Topology of undirect and direct Graph

In this method, we choose the largest proper open subset, then we form all the possible induced vertex-subgraphs using the inverse of the neighborhoods method shown in section 3.1 then we choose another subset with the same specifications and we repeat what we did above based on the induced vertex- subgraphs, where and we delete from it what those which are does not satisfying the neighborhood method and then we take another subsets and so on until all the proper open subset are finished. The result is a set of data, one of which is the original statement from which the topology was generated.

Example 4. Let $G = (V, E)$ be an undirected graph as it is shown in Figure (6).

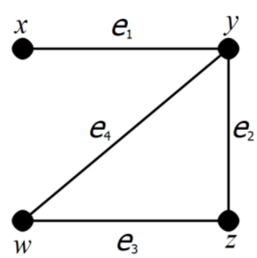


Fig.6- an undirected Graph G

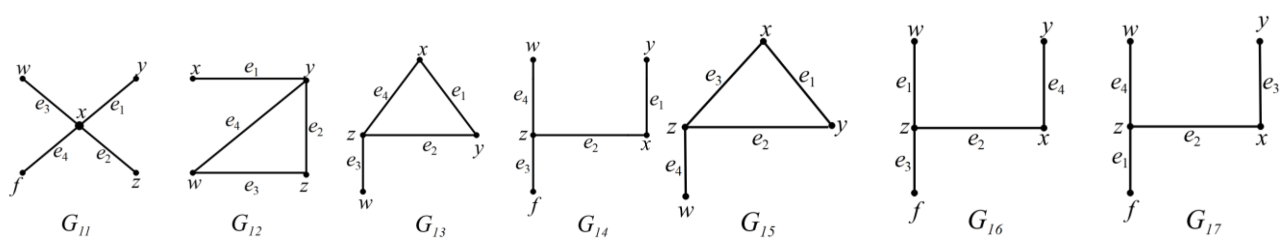
$N\tau_G = \{E, \emptyset\} \cup \{\{e_2\}, \{e_4\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_3, e_4\}\}$.

Is an induced topology from graph with size 4. To find that graph:

1- we choose the largest proper open subset

$H_1 = \{e_1, e_3, e_4\}$

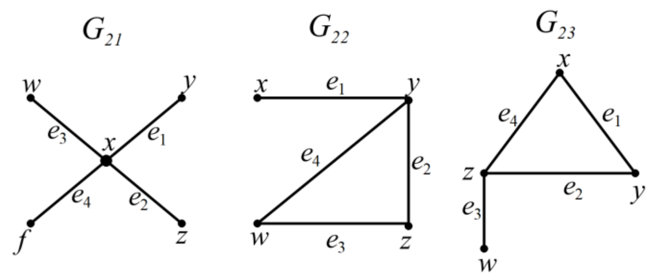
2- we find all graph satisfying H_1 is neighborhood set of the edge e_2.



3- We choose another open proper subset

$H_2 = \{e_1, e_2, e_3\}$

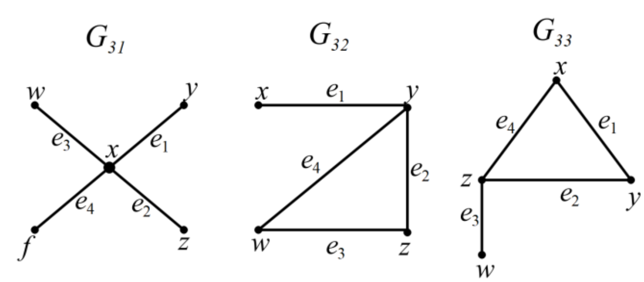
4- We choose from graphs $G_{11}, \dots, G_{17}$ only that graphs satisfy H_2 as neighborhood set of e_4:



5- We choose another open proper subset

$H_3 = \{e_2, e_4\}$

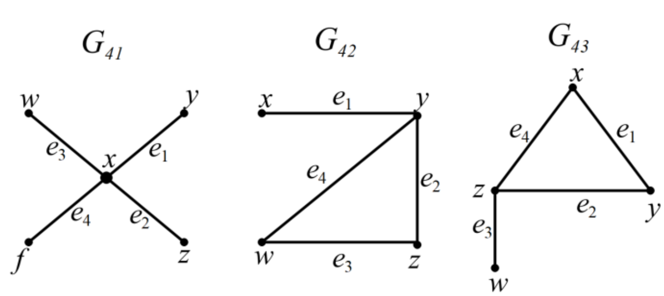
6- We choose from graphs G_{21}, G_{22}, and G_{23} only that graphs satisfy H_3 as neighborhood set of e_1 and e_3:



7- We choose another open proper subset

$H_4 = \{e_1, e_3\}$

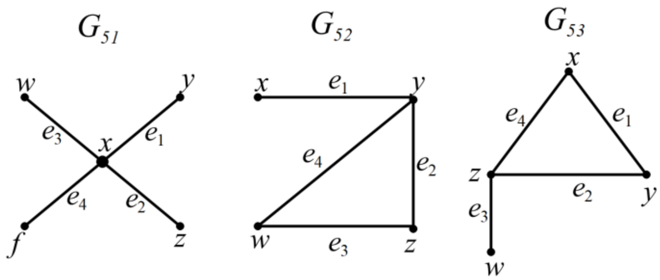
8- We choose from graphs G_{31}, G_{32}, and G_{33} only that graphs satisfy H_4 as neighborhood set of $e_2$and e_4:



9- We choose another open proper subset

$H_5 = \{e_4\}$

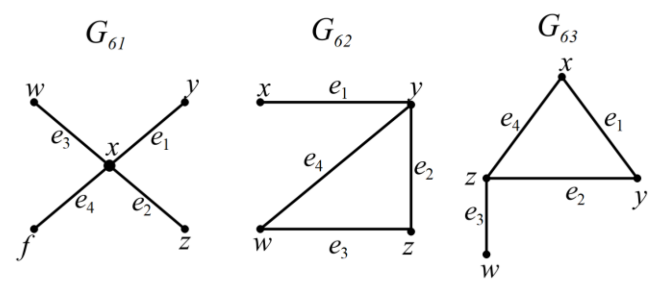
10- We choose from graphs $G_{41}, G_{42},$ and G_{43} only that graphs satisfy H_5 as neighborhood set of $e_1, e_2$and e_3:



11- We choose another open proper subset

$H_6 = \{e_2\}$

12- We choose from graphs G_{51}, G_{52}, and G_{53} only that graphs satisfy H_6 as neighborhood set of $e_1, e_3$and e_4:



Then, this is the graph generated by topological spaces.

Example 5.

From example 3, we have the induced topology

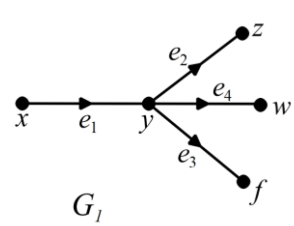
$$N^{\pm}\tau_G = \{E, \emptyset\} \cup \{\{ e_1\}, \{ e_2\}, \{ e_3\}, \{e_4\}, \{ e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}\}.$$

Is an induced topology from graph with size 4. To find that graph:

1- we choose the largest proper open subset

$H_1 = \{e_2, e_3, e_4\}$

2- we find all graph satisfying H_1 is neighborhood set of the edge e_1.



3- we choose the largest proper open subset

$H_2 = \{ e_1, e_2, e_4\}$

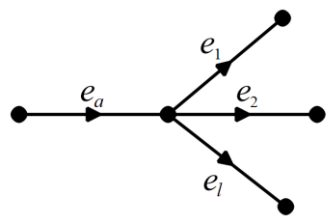
All the graphs are not satisfy that H_2 . So, the inverse neighborhood method is failed.

Theorem 4.

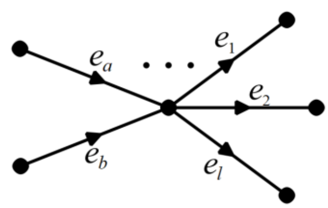
From the induced topological space $N^{\pm}\tau_G$ of a digraph G cannot find the digraph G.

Proof:

Let $E(G) = \{e_1, \dots, e_n\}$ and $H = \{e_1, \dots, e_k\}$,$1 \leq k \leq n$ be a largest proper open subset of the topology $N^{+}\tau_G$.
If we applied the inverse neighborhood $N^{\pm}\tau_G$ method to find the graph G, then
them is at least one edge $e \in H^C$ satisfy $N^{+}(e) = H$.
Then we have the graphs



$a = l + 1, l + 2, \dots , n.$



$a, b = l + 1, l + 2, \dots , n.\ \ a \neq b$

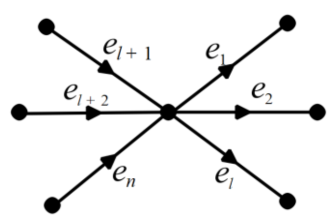


Fig (a)

Since H is proper open subset of $N^{+}\tau_G$ then H^C is edge proper open subset of $N^{+}\tau_G$.
Now, we apply the invers out-neighborhood method for H^C then there at least out edge $e' \in H$ satisfy $N^{+}(e') = H^C$.
i.e.

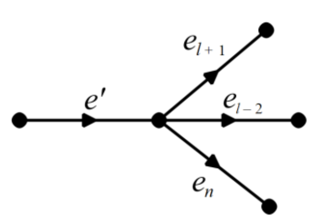


Fig (b)

Then from Figure (a) and Figure (b) then apply edges $e \in H^C$ and $e' \in H$ sets

and

But this is contradiction.
So, cannot find the digraph G.

Remark 2.

If the induced topology is the discrete then we can applied the inverse neighborhood method because there is no graph satisfying all the open proper subset of the topology. For example

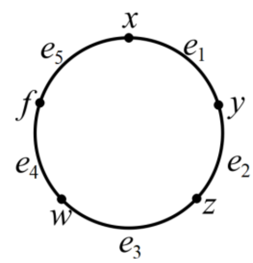


Fig7. C_5

$N(e_1) = \{e_2, e_5\}$, $N(e_2) = \{e_1, e_3\}$, $N(e_3) = \{e_2, e_4\}$, $N(e_4) = \{e_3, e_5\}$, $N(e_5) = \{e_1, e_4\}$,

$N_G(E) = \{ \{e_2, e_5\} , \{e_1, e_3\} , \{e_2, e_4\} , \{e_3, e_5\} , \{e_1, e_4\}\}$

$NB_G = \{\emptyset, \{e_2, e_5\} , \{e_1, e_3\} , \{e_2, e_4\} , \{e_3, e_5\} , \{e_1, e_4\}, \{e_5\}, \{e_4\}, \{e_2\}, \{e_1\}, \{e_3\}\}$

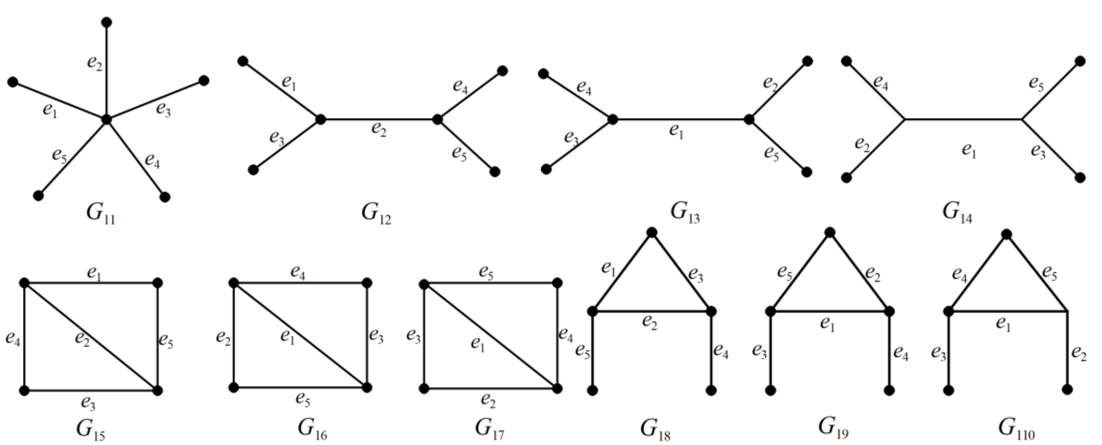
$N\tau_G = \{E, \emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\},\{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_1, e_5\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_2, e_5\}, \{e_3, e_4\}, \{e_3, e_5\},$

$\{e_4, e_5\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_5\},\{e_1, e_3, e_4\}, \{e_1, e_3, e_5\},\{e_1, e_4, e_5\}, \{e_2, e_3, e_4\}, \{e_2, e_3, e_5\}, \{e_2, e_4, e_5\}, \{e_3, e_4, e_5\}$

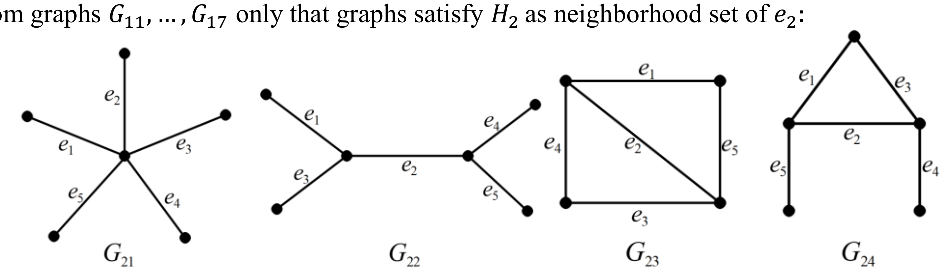
$\{e_1, e_2, e_3, e_4\}, \{e_1, e_2, e_3, e_5\}, \{e_1, e_3, e_4, e_5\}, \{e_2, e_3, e_4, e_5\}\}$

Is an induced topology from graph with size 5. To find that graph:

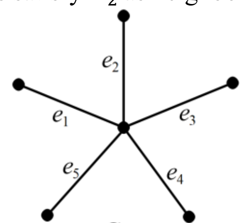
1- we choose the largest proper open subset
$H_1 = \{e_2, e_3, e_4, e_5\}$
2- we find all graph satisfying H_1 is neighborhood set of the edge e_1.



3- we choose the largest proper open subset
$H_2 = \{e_1, e_3, e_4, e_5\}$
4- We choose from graphs $G_{11}, \ldots, G_{17}$ only that graphs satisfy H_2 as neighborhood set of e_2:

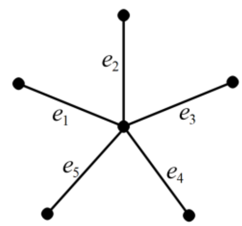


5- we choose the largest proper open subset
$H_3 = \{e_1, e_2, e_3, e_5\}$,
6- We choose from graphs $G_{21}, \dots, G_{24}$ only that graphs satisfy H_2 as neighborhood set of e_4:



G_3

7- we choose the largest proper open subset
$H_3 = \{e_1, e_2, e_3, e_4\}, \dots, H_{29} = \{e_1, e_2, e_3\}$
8- All the generated graph by topological spaces is star.



5-Counclusion

In this paper it is shown that topologies can be generated by simple undirected and directed graphs. It is studied topologies generated by certain graphs. Therefore, it is seen shown that there is a topology generated by every simple undirected and directed graph. And it is shown that graphs can be generated by topological space by the neighborhood relations on edges.

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