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Some exact values for $p(t, d)$

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Abstract

Let $G = (V, E)$ be a graph. When a graph is used to model the linkage structure of communication networks, the diameter of a graph gives the length of longest path among all the shortest paths between any two vertices of the graph, $F(t, d)$ denoted the minimum diameter of an altered graph obtained by adding t – extra edges to a graph with diameter d. Let $P(t, d)$ denoted the minimum diameter of a graph obtained by adding t – extra edges to a path with diameter d. Knowing that $F(t, d) = p(t, d)$. In this paper we prove that $p(t, 3t) = 4$, for $t \geq 4$, $p(t, d) = 4$ where ($t \geq 4$ and $3t + 1 \leq d \leq 3t + 3$) and show a construction to $p(t, d) \leq k + 3$ where ($3 \leq t \leq 5$) and $d = k(t + 1) + 8$.

Subject Classification: *05, 94C15*

Keywords: *Diameter, Altered graph, Edge addition.*

1. Introduction

In this paper, $G = (V, E)$ be a simple, finite and undirected graph (without loops and multiple edges) with vertex set V and edge set E [1]. Between any two vertices $u, v \in V(G)$ the distance $d_G(u, v)$ is the length of a shortest path in G joining u and v; the maximum value of $d_G(u, v)$ taken over all pairs of vertices $u, v \in V(G)$ is the diameter $D(G)$, thus a graph is connected if every pair of vertices are joined by path. In the graph model for communication network, the diameter of the graph corresponds to the maximum number of edges over which a message between two

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nods must travel [2] and [5] . The edge – addition and deletion problems are well-known [3] , also in [8] you can known moor.

Let $F(t,d)$ denoted the minimum diameter of an altered graph obtained by adding $t-$extra edges to a graph with diameter d. Determining the exact value of $F(t,d)$ is fairly difficult in general since it has been proved by Schoone, Bodlander and Van Leu Ween [2]. For given integers t ,d and a connected graph G , constructing an altered graph G' of G by adding $t-$extra edges to G such that G' has diameter of at most d in NP-complete. Thus, the problem of determining sharp upper bounds of $F(t,d)$ is of interesting.

Let $p(t,d)$ denoted the minimum diameter of a graph obtained by adding $t-$extra edges to a path with diameter d . In [4] Deng and Xu show that $F(t,d)=p(t,d)$ for any integer t and d .

In theorem 2.1 we prove that the minimum diameter $p(t,d)$ of a graph G formation from a path $P=x_0x_1..x_{3t}$ with diameter d ($d=3t$ and $t\geq 4$ =E by adding $t-$extra edges is= 4 . Clearly that the number of vertices of G is $n(V)=3t+1$, let $H\subseteq V(G)$ is a free vertices (the vertices that is not joined with any extra edge), from making a partition to a set H we can prove that the distance between the vertices of H (for some vertices of H) is greater than or equal 4 which is mean the minimum diameter is greater than or equal 4 .

Theorem 2.1 : $p(t,3t)=4$ *where* $t\geq 4$

Proof : Let $P=x_0x_1..x_{3t}$ be a (x_0,x_{3t}) - path and let G be an altered graph obtain from p by adding $t-$extra edges.

From $\left\lceil\frac{d}{t+1}\right\rceil\leq p(t,d)\leq\left\lceil\frac{d}{t+1}\right\rceil+1$ for $t=4,5$ and $d\geq 4$ by Deng and Xu in[4] and $\left\lceil\frac{d-2}{t+1}\right\rceil\leq p(t,d)\leq\left\lceil\frac{d-2}{t+1}\right\rceil+1$, for $t\geq 6$ and $d\geq 3$ by A.A.Najim and Xu, J. [6] we have==

$$3\leq p(t,3t)\leq 4.......(1)$$

by adding $t-$extra edges to the path P_{3t} we show that $p(t,3t)=d(G)$ is not three because :

Let H be a set of free vertices (the vertices that is not joined with any extra edge) then the number of free vertices is $t+1\leq n(H)\leq 2t$, (if we suppose that we adding $t-$extra edges on the path such that there is no tow edges incident with the same vertex , so we have $2t$ vertices which is incident with $t-$extra edges and since the number of vertices is $3t+1$, so $3t+1-2t=t+1$ is the minimum number of vertices which is free. In another hand if we adding the $t-$extra edges on the path such that the t edges is incident with the same vertex, we have $t+1$ vertices which is incident with $t-$extra edges and since the number of vertices is $3t+1$, so $(3t+1)-(t+1)=2t$ is the maximum number of vertices which is free.

Let $K=\{H_i\}_{i\in I}$ is a partition of the set H and let $r=max\,\{n(H_i)\}_{i\in I}$ then $\exists\alpha\in I \ni r=n(H_\alpha)$ so we discuss the cases :

If the maximum degree $t+2, t+1, t, ..., 3$

Let $x_q\in H_\alpha$, $x_{q'}\in N(H_\alpha)$ and $x_j\in H_\beta$ where $N(H_\alpha)$ is a neighbors of H_α And H_β is the set of vertices which is incident with $t-$extra edges

1.1 if $r\in\{6,7,...,3t-4\}$ then there exist $x_\ell\in H_\alpha$ where $d_{H_\alpha}(x_q,x_\ell)\geq 4$, $d_G(x_q,x_\ell)\geq 4$ and hence $d(G)\geq 4$,see figure 1.1.

1.2 **a**. if $r\in\{4,5\}$ then there exist $H_\beta\in k \ni n(H_\beta)\geq 1$ and vertices x_q, x_j where= $d_{H_\alpha}(x_q,x')\leq \dfrac{1}{2}\; d(H_\alpha)$, $\forall x'\in V(H_\alpha)$ and $d_p(x_q,x_j)\geq 4$ such thtat $x_{q'}\in N(H_\alpha)\cap N(H_\beta)$ then $d_G(x_q,x_{q'})\geq 3$ and $d_G(x_{q'},x_j)\geq 1$ thus $d_G(x_q,x_j)\geq 4$ and hence $d(G)\geq 4$, see figure 1.2.a.

1.2. **b.** if $r=t+1=$ and there is not exist $H_\beta\in k$, let $x_q, x_j\in H_\alpha$ and $x_{q'}\in N(H_\beta)=$ where= $d_{H_\alpha}(x_q,x')\leq \dfrac{1}{2}=\; d(H_\alpha), \forall x'\in V(H_\alpha)$, $d_G(x_q,x_{q'})\geq 3$ and $d_G(x_{q'},x_j)\geq 4$ thus $d_G(x_q,x_j)\geq 4$ and hence $d(G)\geq 4$ see, figure 1.2.b.

2. if r = 3, let $d_{H_\alpha}(x_q,x')\leq 1$, $\forall x'\in V(H_\alpha)$:

2.1 if there is $H_\beta\in k \ni n(H_\beta)\geq 2$ and there are two vertices $x_{q'}, x_{j'}$ which are distinct (not free) where $d_p(x_j,x_q)\geq 4$, $d_G(x_q,x_{q'})\geq 2$ and $d_G(x_q,x_{j'})\geq 3$ thus $d_G(x_q,x_j)\geq 4$ and hence $d(G)\geq 4$, see figure 2.1.

2.2 if $n(H_\beta)=1$, $\forall H_B\in k$:

2.2.a if $N(H_\alpha)\cap N(H_\beta)\neq\phi$

That is mean there is $x_{q'}\in N(H_\alpha)\cap N(H_\beta)$ such that $d_G(x_q,x_{q'})\geq 2$, $d_G(x_{q'},x_j)=1$ and $d_G(x_q,x_j)\geq 3$ then there exist $x''\in V(H)$ where $d_p(x_q,x'')\geq 4$ thus $d_G(x_q,x'')\geq 4$ and hence $d(G)\geq 4$,see figure 2.2.a.

2.2.b if $N(H_\alpha)\cap N(H_\beta)=\phi$

That is mean there is $x_{j'}\in N(H_\beta)=$ such that $x_{q'}\neq x_{j'}$ then $d_G(x_q,x_{q'})\geq 2$, $d_G(x_{q'},x_j)\geq 1$ and $d_G(x_j,x_{j'})=1$

So, $d_G(x_q,x_j)\geq 4$ and hence $d(G)\geq 4$,see figure 2.2.b.

3. $r = 2$, let x_i be a fixed vertex (the vertex which is not incident with maximum degree)

3.1 if the max degree is $t+1, t, t-1, \ldots, 6$ (interior vertices, $n(H) = 2t$ and $t \le n(H) \le t+2$ or $t+1$ degree (end vertices and $n(k) = t$ or $t+1$)

So, there exist at least 2 - sets of $n(k)$ have 2 - vertices x_q and x_j such that $d_p(x_q, x_i) \ge 2$ and $d_p(x_q, x_j) \ge 4$ then $d_G(x_q, x_j) \ge 4$ and hence $d(G) \ge 4$.

3.2 if the max degree is 5 (interior vertices),= $7 \le n(H) \le 9$ and $4 \le n(k) \le 6$:

3.2. a. if the 2 -sets which is incident with max degree have 2 - vertices and the other edge is:

3.2. a.1. incident by two vertices (or one) with extra edge so, there exist x_q and x_j (which is not close with the vertex joined with other edges) such that $d_p(x_q, x_j) \ge 4$ and $d_G(x_q, x_i) \ge 2$ then $d_G(x_q, x_j) \ge 4$ and hence $= d(G) \ge 4$.

(since $7 \le n(H) \le 9$ and the vertices which is close with other edges is 4 so we have x_j).

3.2. a. 2. is not incident by any vertex with extra edges then there exist x_q and x_j such that $d_p(x_q, x_j) \ge 4$, $d_G(x_j, x_i) \ge 2$ and $d_G(x_q, x_i) \ge 2 =$ then $d_G(x_q, x_j) \ge 4$ and hence $d(G) \ge 4$.

3.2. b. if the if the 2 -sets which is incident with max degree have 2 or 1 vertices and the other edges :

3.2.b.1. is incident by two (or one) vertices with extra edges so, there exist x_q and x_j (which is not clots with the vertices joined with other edges) such that $d_G(x_q, x_j) \ge 4$ and hence $d(G) \ge 4$.

(since $n(H) = 2t - 1$ or $2t$ and the vertices which is close with the other edges is 4 And the vertices which is incident with x_i is 3 so, we have 2 vertices we can choose x_j from it).

3.2. b. 2. Is not incident by any vertex with with extra edges, then there exist x_q and x_j such that $d_p(x_q, x_j) \ge 4$, $d_p(x_q, x_i) \ge 2$ then $d_G(x_q, x_j) \ge 4$ and hence $d(G) \ge 4$.

3.2. c. if the 2-sets which is incident with max degree have 1- vertices and the other edges :

3.2. c. 1. is incident by one vertex with extra edges (incident with 2-vertices with extra edges is not satisfy because H_α is became more than 2- vertices) then we choose 2-sets which is not incident with x_i so, there exist x_q and x_j which is far off the other edges by 2 or more steeps such that $d_p(x_q, x_j) \ge 4$ then $d_G(x_q, x_j) \ge 4$ and hence $d(G) \ge 4$.

(since the number of not free vertices and the free vertices incident with x_i is $2t-2$ then we have $2t-1$ free vertices so, we can choose x_j).

3.2.c.2. is not incident by any vertices with extra edges then there exist x_q and x_j which is incident with x_i and far off the other edges by 2 or more steeps (by two or one side) such that $d_p(x_q, x_j) \geq 4$ then $d_G(x_q, x_j) \geq 4$ and hence $d(G) \geq 4$.

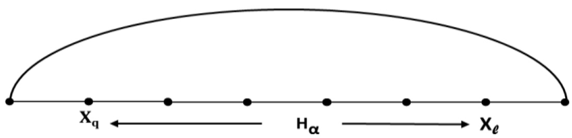


Figure 1.1

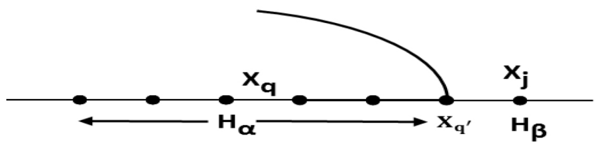


Figure 1.2.a

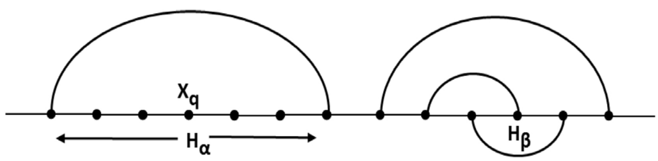


Figure 1.2.b

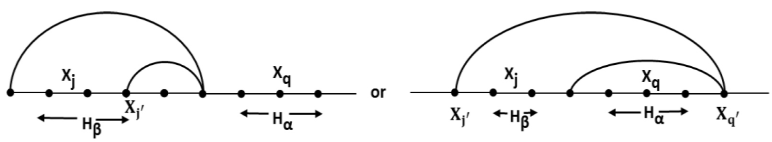


Figure 2.1

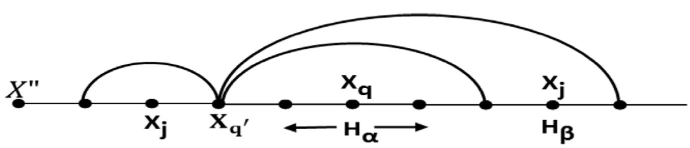


Figure 2.2.a

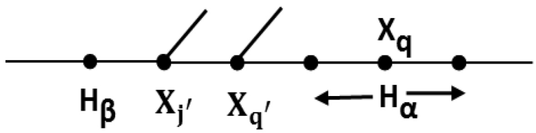


Figure 2.2. b

4. $r = 2$

4.1. If the maximum degree $t+2, t+1, \ldots, 6$ (end or interior vertex) and $4,5$ (end vertex and maximum degree 4 is not done at $d = 12$) . Is no satisfying since H_α contain vertices greater than or equal 2 .

Hint: the vertices which is incident with $t-edge$ is $t+1$ and the free vertices which is close with the incident to $t-edge$ is $t+1$ then we have $2t+2$ and hence $3t+1-(2t+2) = t-1$ free vertices.

4.2. If the maximum degree 5 (interior vertex), $n(H) = t+3$ and $n(k) = t+3$

Then there exist x_q and x_j is not a neighborhood to the vertex with maximum degree, and there is no common edge between neighborhood x_q and x_j , also $d_p(x_q, x_j) \geq 4$, so x_q it need at least $4 =$ steeps in the graph to reach x_j (x_q need one steep to reach vertex join with extra edge then to vertex with max degree after that reach to vertex join with another extra edge and finally reach to x_j so $d_G(x_q, x_j) \geq 4$ and hence $d(G) \geq 4$.

4.3. Maximum degree is 4 (interior vertex,= $t+2 \leq n(H) \leq t+3$, $t+2 \leq n(k) \leq t+3 = = =$ and maximum degree is (interior or end vertex,= $t+1 \leq n(H) \leq t+2$, $t+1 \leq n(k) \leq t+2 = = =$ then there exist x_q and x_j such that $d_p(x_q, x_j) \geq 4$, is not a neighborhood to the vertex incident with maximum degree and there is not common edge between neighborhood of x_q and x_j then $d_G(x_q, x_j) \geq 4$ and hence $d(G) \geq 4$.

From above results $p(t, 3t) = d(G) \geq 4$.........(2) and from equation (1)

We have $p(t, 3t) = 4$ where $t \geq 4$. ■

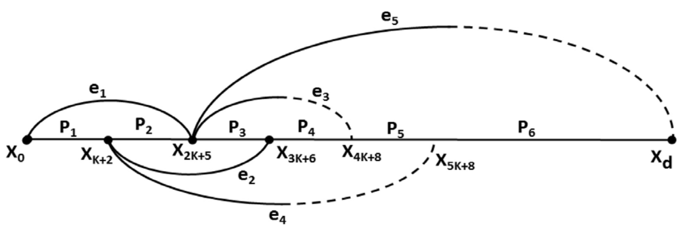
Proposition 2.2 : $p(t,d) = 4$ where $t \geq 4$ and $3t+1 \leq d \leq 3t+3$

Proof : From theorem 2.1 above we have $p(t, 3t) = 4$ where $t \geq 4$

From Schoon. [6] $p(t,d) \geq p(t,d')$ if= $d \geq d'$

We get $p(t,d) \geq 4$ for $t \geq 4$ and $3t+1 \leq d \leq 3t+3$

On other hand, from Deng and Xu [4] $p(t,d) \leq \left\lceil \frac{d}{t+1} \right\rceil + 1$ where $t = 4,5$ and $d \geq 4$, we get $p(t,d) \leq 4$



Construction of theorem 2.3

And from Nijim and Xu [7] $p(t,d) \le \left\lceil \frac{d-2}{t+1} \right\rceil + 1$ where $t \ge 6$ and $d \ge 3$

We get $p(t,d) \le 4$

So, $p(t,d) = 4$ ■

Theorem 2.3 : $p(t,d) \le k+3$ *for* $k \ge 2$, $3 \le t \le 5$ *and* $d = k(t+1)+8$.

Proof: Let G be an altered graph construct from a single path $P = x_0 x_1 ... x_d$ plus t – extra edgessuch that the diameter of G is $p(t,d)$

Let $d = k(t+1)+8$ we add= t – extra edges $(3 \le t \le 5)$:

$e_1 = x_0 x_{2k+5}$ $\quad e_4 = x_{k+2} x_{5k+8}$

$e_2 = x_{k+2} x_{3k+6}$ $\quad e_5 = x_{2k+5} x_d$

$e_3 = x_{k+5} x_{4k+8}$

Now , the end – vertices of these edges divide P into $t+1$ segments :

$P_1 = (x_0, x_{k+2})$ $\quad P_3 = (x_{2k+5}, x_{3k+6})$

$P_2 = (x_{k+2}, x_{2k+5})$ $\quad P_4 = (x_{3k+6}, x_{4k+8})$

$P_5 = (x_{4k+8}, x_{5k+8})$ $\quad P_6 = (x_{5k+8}, x_{6k+8})$

It is easy to see that :

$\varepsilon(p_2) = k+3$ $\quad \varepsilon(p_i) = k+2$ where $i = 1,4$.

$\varepsilon(p_3) = k+1$ $\quad \varepsilon(p_i) = k$ where $i = 5,6$

We will prove the distance between any two vertices x and y of $V(G)$, is less than or equal $k+3$.

Now we define $\frac{t(t+1)}{2}$ cycles , $c^1, c^2, ..., c^{\frac{t(t+1)}{2}}$ *as* :

$C_t^1 = p_1 \cup p_2 + e_1$ $\quad C_t^9 = p_3 \cup p_5 + e_2 + e_3 + e_4$

$C_t^2 = p_1 \cup p_3 + e_1 + e_2$ $\quad C_t^{10} = p_4 \cup p_5 + e_2 + e_4$

$C_t^3 = p_2 \cup p_3 + e_2$ $\quad C_t^{11} = p_1 \cup p_6 + e_1 + e_4 + e_5$

$C_t^4 = p_1 \cup p_4 + e_1 + e_2 + e_3$ $\quad C_t^{12} = p_2 \cup p_6 + e_4 + e_5$

$C_t^5 = p_2 \cup p_4 + e_2 + e_3$ $\quad C_t^{13} = p_3 \cup p_6 + e_2 + e_4 + e_5$

$C_t^6 = p_3 \cup p_4 + e_3$
$C_t^7 = p_1 \cup p_5 + e_1 + e_3 + e_4$
$C_t^8 = p_2 \cup p_5 + e_3 + e_4$
$C_t^{14} = p_4 \cup p_6 + e_2 + e_3 + e_4 + e_5$
$C_t^{15} = p_5 \cup p_6 + e_3 + e_5$

Their lengths are :

$\varepsilon(C_t^i) \le 2k + 6$ where $i = 1,4,14$.
$\varepsilon(C_t^i) \le 2k + 5$ where $i = 2,3,7,8,11,12$.
$\varepsilon(C_t^5) = 2k + 7$
$\varepsilon(C_t^i) \le 2k + 4$ where $i = 6,9,10,13$
$\varepsilon(C_t^{15}) = 2k + 2$

Now, clearly that any two vertices x and y of $V(G)$, are contained in cycle

$c^{\frac{t(t+1)}{2}}$ define above the fact : $\max\{d(C_t^i) : 1 \le i \le 15\} \le \left\lfloor \frac{2k+5}{2} \right\rfloor$
$= k + 3$

So, $p(t, k(t+1)+8) \le k + 3$ ■

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