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Minimum diameter and $t_F(f, q)$

Suaad Abdulrazzaq Swadi *
Samaher Adnan Abdul-Ghani †
Department of Mathematics
College of Science
University of Basrah
Basra
Iraq

Abstract

For given integer e and q, let $p(e, q)$ (res. $C(e, q)$) indicate to (MDe-PD). The maximum diameter deleting e edges from a connected graph of diameter q is $f(e, q)$, the maximum number $T_F(f, q)$ of edges that have to be delete from a path q. In our study show construction $p(7, 8k-10) = k$ for $k \geq 3$ And $p(7, q') = k$ for $k \geq 3$ and $q' = 8k-6$, where e and q are positive integer also show that $f(4, q) = 5q-4$ where $q \geq 4$ and $f(7, q) = 8q-10$ where $q \geq 5$. Also $1T_F 1(f, q)1 \geq \left\lceil \frac{f-q-3}{q+1} \right\rceil$ Where $1 f(e, q)1 \leq 1q(e+1)1 - e + 3$ and

$$\left\lceil \frac{f-6}{q-2} \right\rceil - 1 \geq T_f(f, q) \geq \begin{cases} \left\lceil \dfrac{f-6}{q+1} \right\rceil & \text{if} \quad f(e,q) \leq (e+1)q+1 \\ \left\lceil \dfrac{f-2}{q-2} \right\rceil - 1 & \text{if} \quad f(e,q) \leq (e+1)q - e + 1 \end{cases}$$

Where $f(e, q) \geq (e+1)q - 2e + 4$

Subject Classification: *05C10, 05C30, 57M15.*

Keywords: *Length of longest path, Addition and deletion problems.*

1. Introduction

Let $V(G)$ is a vertex set and $E(G)$ is the edges set; A graph $G = (V, E)$ be an undirected simple graph [2]. Distance between w and z in G is $d_G(w, z)$

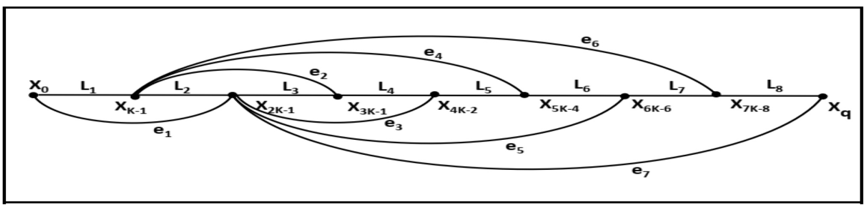
*E-mail: suaad.swadi@uobasrah.edi.iq (Corresponding Author)
†E-mail: samaheradnanmath@gmail.com

which is the length of shortest path joining them. Maximum distance between any two vertices of G is the Diameter which denoted by $d(G)$ [3]. We can see more in [1,4]. In graph theory :given positive integer $e$and q, let $F(e, q)$ indicate to the minimum diameter of an altered graph obtained by adding e- extra edges to a graph with diameter q [3]. Let $p(e, q)$ (res. $C(e, q)$) indicate the minimum diameter of a graph obtained by adding e -edges to a path of length q (MDe-PD). In [5] we can see $F(e,q) = p(e,q)$ for any integer e and q. The convers to edge-addition problem is called edge deletion problem. Let $g(e, q)$ denoted the maximum possible diameter of graph H obtained by deleting $e$edges from $(e + 1)$ - edge-connected graph G with diameter q

Let $f(e, q)$ denoted the maximum diameter of a connected graph obtained after deleting e edges from connected graph with diameter q . exact value of $f(e, q)$ have been obtained for some e or q [6]. Let $T_F(f,q)$ be a maximum number of edges that delete from q (length of path) to changed it to be a graph of diameter at most F [7].

Theorem 2.1 : $p(7, 8k-10) = k \ for \ k \geq 3$

Proof : Let $q = 8k - 10$. We add seven edges :



Construction of theorem 2.1

$e_1 = x_0 x_{2k-1}$, $e_2 = x_{k-1} x_{3k-1}$, $e_3 = x_{2k-1} x_{4k-2}$, $e_4 = x_{k-1} x_{5k-4}$

$e_5 = x_{2k-1} x_{6k-6}$, $e_6 = x_{k-1} x_{7k-8}$, $e_7 = x_{2k-1} x_q$

To path $L = x_0 x_1 \ldots x_q$ which be a simple path. Seven vertices x_{k-1}, x_{2k-1}, x_{3k-1}, x_{4k-2}, x_{5k-4}, x_{6k-6} and x_{7k-8}, partition L into eight segments:

$L_1 = (x_0, x_{k-1})$, $L_2 = (x_{k-1}, x_{2k-1})$, $L_3 = (x_{2k-1}, x_{3k-1})$, $L_4 = (x_{3k-1}, x_{4k-2})$,

$L_5 = (x_{4k-2}, x_{5k-4})$, $L_6 = (x_{5k-4}, x_{6k-6})$, $L_7 = (x_{6k-6}, x_{7k-8})$, $L_8 = (x_{7k-8}, x_q)$

Now, we will define 28 cycles as follows :

$C^1 = L_1 \bigcup L_2 + e_1$ $C^2 = L_1 \bigcup L_3 + e_1 + e_2$

$C^3 = L_1 \bigcup L_5 + e_3 + e_4 + e_1$ $C^4 = L_1 \bigcup L_8 + e_1 + e_6 + e_7$

$C^5 = L_2 \bigcup L_3 + e_2$ $C^6 = L_2 \bigcup L_5 + e_4 + e_3$

$C^7 = L_2 \bigcup L_7 + e_6 + e_5$ $C^8 = L_3 \bigcup L_4 + e_3$

$C^9 = L_3 \bigcup L_5 + e_2 + e_4 + e_3$ $C^{10} = L_3 \bigcup L_6 + e_2 + e_4 + e_5$

$C^{11} = L_3 \bigcup L_7 + e_2 + e_6 + e_5$ $C^{12} = L_3 \bigcup L_8 + e_2 + e_6 + e_7$

$C^{13} = L_4 \bigcup L_5 + e_2 + e_4$ $C^{14} = L_4 \bigcup L_6 + e_3 + e_5 + e_4 + e_2$

$C^{15} = L_4 \bigcup L_7 + e_3 + e_5 + e_2 + e_6$ $C^{16} = L_4 \bigcup L_8 + e_3 + e_7 + e_2 + e_6$

$C^{17} = L_2 \bigcup L_8 + e_6 + e_7$ $C^{18} = L_5 \bigcup L_6 + e_3 + e_5$

$C^{19} = L_5 \bigcup L_7 + e_3 + e_5 + e_4 + e_6$ $C^{20} = L_5 \bigcup L_8 + e_3 + e_7 + e_4 + e_6$

$C^{21} = L_6 \bigcup L_7 + e_4 + e_6$ $C^{22} = L_6 \bigcup L_8 + e_4 + e_5 + e_6 + e_7$

$C^{23} = L_7 \bigcup L_8 + e_5 + e_7$ $C^{24} = L_1 \bigcup L_4 + e_1 + e_2 + e_3$

$C^{25} = L_1 \bigcup L_6 + e_1 + e_5 + e_4$ $C^{26} = L_1 \bigcup L_7 + e_1 + e_5 + e_6$

$C^{27} = L_2 \bigcup L_4 + e_3 + e_2$ $C^{28} = L_2 \bigcup L_6 + e_5 + e_4$

Their lengths are:

$\varepsilon(L_1) = k-1,\ \varepsilon(L_2) = k,$ $\varepsilon(L_3) = k,\ \varepsilon(L_4) = k-1,$

$\varepsilon(L_5) = k-2,\ \varepsilon(L_6) = k-2,$ $\varepsilon(L_7) = k-2,\ \varepsilon(L_8) = k-2$

Thus, we have:

$\varepsilon(C^1) = 2k$ $\varepsilon(C^{02}) = 2K+1$ $\varepsilon(C^3) = 2k$ $\varepsilon(C^4) = 2k$

$\varepsilon(C^5) = 2k+1$ $\varepsilon(C^6) = 2k$ $\varepsilon(C^{07}) = 2K$ $\varepsilon(C^{08}) = 2K$

$\varepsilon(C^{9}) = 2k+1$	$\varepsilon(C^{010}) = 2K+1$	$\varepsilon(C^{011}) = 2K+1$	$\varepsilon(C^{012}) = 2K+1$
$\varepsilon(C^{13}) = 2k-1$	$\varepsilon(C^{14}) = 2k+1$	$\varepsilon(C^{15}) = 2k+1$	$\varepsilon(C^{16}) = 2k+1$
$\varepsilon(C^{17}) = 2k$	$\varepsilon(C^{018}) = 2K-2$	$\varepsilon(C^{019}) = 2K$	$\varepsilon(C^{20}) = 2k$
$\varepsilon(C^{21}) = 2k-2$	$\varepsilon(C^{22}) = 2k$	$\varepsilon(C^{23}) = 2k-2$	$\varepsilon(C^{024}) = 2K+1$
$\varepsilon(C^{25}) = 2k$	$\varepsilon(C^{26}) = 2k$	$\varepsilon(C^{027}) = 2K+1$	$\varepsilon(C^{028}) = 2K$

It is clear that

$$\varepsilon(C^{i}) = 2k,\ i = 1, 3, 4, 6, 7, 8, 17, 19, 20, 22, 25, 26, 28,$$

$$\varepsilon(C^{i}) = 2k+1,\ i = 2, 5, 9, 10, 11, 12, 14, 15, 16, 24, 27.$$

$\varepsilon(C^{i}) = 2k-1$, $i = 13, 18, 21, 23.$ Now, for any pair w and z of $V(G)$, are include at least one cycle of the cycles defined above So, we have

$$Max\,\{q(C^{i}) : 1 \le i \le 28\} \le \left\lfloor \frac{2k+1}{2} \right\rfloor = k$$

We get $p(7,\, 8k-10) \le q(G) \le k$, for any positive integers K. Now, From Deng. [5] we have $p(7,q) \ge \left\lceil \frac{q}{t+1} \right\rceil = k$, for $e = 7$ and $q = 8k-10$.

$$p(7, 8k-10) \ge \left\lceil \frac{8k-10}{8} \right\rceil = k \ \text{ Thus, } p(7, 8k-10) = k.$$

Proposition 2.2 : $p(7, 8k-6) = k$ for $k \ge 3$

Proof : From theorem 2.1 above we have $p(7, 8k-10) = k$ for $k \ge 3$. From Schoon. [7] $p(e, q') \ge p(e, q)$ if $q' \ge q$ we get $p(7, q') \ge p(7, q)$ Where $q' = 8k-6$ and $k \ge 3$, Thus, $p(7, q') \ge k$. From A.A.Najim. [9] $\left\lceil \frac{q-2}{e+1} \right\rceil \le p(e,q) \le \left\lceil \frac{q-2}{e+1} \right\rceil + 1$ for $e = 7$ and $q' = 8k-6$. So, $p(e, q') \le \left\lceil \frac{q'-2}{e+1} \right\rceil + 1 = \left\lceil \frac{8k-6-2}{8} \right\rceil + 1$. We have $p(7, q') \le k$. Thus $p(7, q') = k$

Theorem 2.3 : $f(4, q) = 5q-4$ where $q \ge 4$

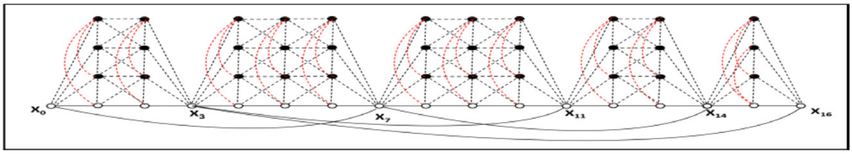


Fig (2.3)

Constriction of $J_{4,q}$ for $q \geq 4$.

Proof : We first prove $f(4,q) \leq 5q-4$, let Gbe any graph with diameter q, and let G' be a connected graph obtained by deleting four edges from G. From AL-Bachary [11] (we have $p(4,5k-4)=k$ for $k \geq 2$). Let q' the diameter of G'; $d(G')=q'$ and let x and y be two vertices of G' such that $d_{G'}(w,z)=q'$ and let be a shortest path $(x=x_0,x_1,\dots,x_q=y)$ of length q' in G'. Partition the vertices in $V(G')$ in tosets V_i, $i=0,1,2,\dots,q'$ $(0 \leq i \leq q')$ by setting $V_i=\{u \mid d(x,u)=i\}$. Notice that all these sets are non empty. Let J and J' be the graphs obtained from G and G' respectively by contracting eachset V_i to a single vertex and removing any loops and duplicate edges. Let $d(J)=j$ and $d(J')=j'$. Then $j \leq q$ and $j' \leq q'$. Since J' simply consist of the path $x_0,x_1\dots,x_q$, we have $J'=q'$. The graph Jcontains the path $x_0,x_1\dots,x_{q'}$, and at most four additional edges(see figure 2.3 for $q=4$). Form Wu [3] we have $j=p(t,f(t,q))$, then $j=p(4,q')=\left\lceil\frac{q'+4}{5}\right\rceil \geq \frac{q'+4}{5}$. So, $\frac{q'+4}{5} \leq j$, then $q' \leq 5j-4$ where $(j \leq q)$. $q' \leq 5j-4$ Thus $f(4,q) \leq 5q-4$.

Now, wanted to prove $f(4,q) \geq 5q-4$, from schoone [7] we have $f(e,q) \geq (e+1)q-e=5q-4$, if q is even and From Wu and Xu [10] $f(e,q)=(e+1)q-2e+4=5q-4$, if q is odd .

Thus $f(4,q) \geq 5q-4$. $f(4,q)=5q-4$

Theorem 2.4 : $f(7,q)=8q-10$ *where* $q \geq 5$

Proof : We want to prove that $f(7,q) \geq 8q-10$, we only need to construct a $8-edge$ -connected graph with diameter q such that its diameter increases to at least $8q-10$ when its seven edge are deleted (see[12]) . Let $R_{7,q}$ be a graph obtaining from a path of length $(8q-10)$, $p=x_0,x_1,\dots,x_{8q-10}$, plus seven extra edges: $e_1=x_0x_{2q-1}$, $e_2=x_{q-1}x_{3q-1}$, $e_3=x_{2q-1}x_{4q-2}$, $e_4=x_{q-1}x_{5q-4}$

$e_5 = x_{2q-1}x_{6q-6}$, $e_6 = x_{q-1}x_{7q-8}$, $e_7 = x_{2q-1}x_{8k-10}$. The prove is similarity to the prove of theorem 2.3

Theorem 2.5 : $T_F(f,q) = \left\lfloor \frac{f-q-3}{q+1} \right\rfloor$

Proof : From Chung and Garey in 1984 [4] we have $F(e,q) \le q(e+1) - e + 3$

Put $f = F(e,q)$, So, $f \le q(e+1) - e + 3$. $f - q - 3 \le (q+1)e$, then $t \ge \frac{f-q-3}{q-1}$, Thus $T_F \ge \left\lfloor \frac{f-q-3}{q-1} \right\rfloor$

Theorem 2.6 :

$$\left\lceil \frac{f-6}{q-2} \right\rceil - 1 \ge T_F(f,q) \ge \begin{cases} \left\lceil \frac{f-q}{q+1} \right\rceil \text{ iff } (e,q) \le (e+1)q + e \\ \left\lceil \frac{f-2}{q+2} \right\rceil - 1 \text{ iff } (e,q) \le (e+1)q - e + 1 \end{cases}$$

Proof : from Wu and Xu in 2006 [10] we have $f(e,q) \ge (e+1)q - 2e + 4$ for $e \ge 4$ and $q \ge 3$ is odd

Put $f = f(e,q)$, So $f \ge (e+1)q - 2e + 4$

$$f - q - 4 \ge e(q-2)$$

$$e \le \frac{f-q-4}{q-2} \text{ then } e \le \frac{f-6}{q-2} - 1 \quad e \le \frac{f-6}{q-2} - 1$$

Thus $T_F \le \left\lceil \frac{f-6}{q-2} \right\rceil - 1$ From A.A. Najim and J.M. Xu 2006 [8] we have

$$f(e,q) \le \begin{cases} (e+1)q & \text{if } p(e,q) = \dfrac{q-2}{e+1} \\ (e+1)q - e + 1 & \text{if } p(e,q) = \dfrac{q-2}{e+1} + 1 \end{cases}$$

for $e \ge 4$ and $q \ge 3$.

Now, if $f(e,q) \le (e+1)q + e$

Put $f = f(e,q)$ we get $f \le (e+1)\,q + e$, then $e \ge \frac{f-q}{q+1}$, Thus $T_F \ge \left\lfloor \frac{f-q}{q+1} \right\rfloor$

If $f(e,q) \le (e+1)\,q - e + 1$

Put $f = f(e,q)$ we get $f \leq (e+1)\,q - e + 1$

$f - q - 1 \leq e(q-1)$, So, $e \geq \frac{f-q-1}{q-1}$ Thus $T_F \geq \left\lfloor \frac{f-2}{q-1} \right\rfloor - 1$

So

$$\left\lceil \frac{f-6}{q-2} \right\rceil - 1 \geq T_F(f,q) \geq \begin{cases} \left\lceil \frac{f-q}{q+1} \right\rceil \text{ iff}(e,q) \leq (e+1)q + e \\ \left\lceil \frac{f-2}{q+2} \right\rceil - 1 \text{ iff}(e,q) \leq (e+1)q - e + 1 \end{cases}$$

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