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# Generalized $q$-difference equation of the generalized $q$-operator ${}_r\Phi_s(\theta)$ and its application in $q$-integrals

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ABSTRACT

In this paper, we employ the $q$-difference equation technique to generalize some well-known $q$-integrals such as the extension of Askey–Roy $q$-integral, Andrews–Askey $q$-integral, and Askey–Wilson $q$-integral.

## 1. Introduction

The notations used in Ref. 1 are employed in this paper, and we suppose that $|q| < 1$.

The $q$-shifted factorial is defined by Ref. 1:

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \qquad and \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The basic hypergeometric series ${}_r\phi_s$ is given by Ref. 1:

$${}_r\phi_s\left(\begin{matrix} a_0, a_1, \dots, a_{r-1} \\ b_1, b_2, \dots, b_s \end{matrix} ;q,x\right) = \sum_{n=0}^{\infty} \frac{(a_0,\dots,a_{r-1};q)_n}{(q,b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

where $q \neq 0$ when $r > s+1$. Note that

$${}_{s+1}\phi_s\left(\begin{matrix} a_1, a_2, \dots, a_{s+1} \\ b_1, b_2, \dots, b_s \end{matrix} ;q,x\right) = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\dots,a_{s+1};q)_n}{(q,b_1,b_2,\dots,b_s;q)_n} x^n, \qquad |x|<1.$$

The multiple $q$-shifted factorials is:

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n \dots (a_m,q)_n.$$

The $q$-binomial coefficients is given by Ref. 1:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \qquad 0 \leqslant k \leqslant n.$$

The following identity, will be used in this paper[1]:

$$(q/a;q)_n = (aq^{-n};q)_n\left(-\frac{q}{a}\right)^n q^{\binom{n}{2}}. \tag{1.1}$$

The $q$-Chu–Vandermonde sum is

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix} ;q,cq^n/a\right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{1.2}$$

The $q$-differential operator is defined by Refs. 2–5

$$D_q\{f(x)\} = \frac{f(x)-f(xq)}{x}. \tag{1.3}$$

The $q$-operator $\theta$ is defined by Refs. 6, 7:

$$\theta\{f(a)\} = \frac{f(aq^{-1})-f(a)}{aq^{-1}}.$$

The $\theta$ operator working with a variable $a$ will signified by $\theta_a$.

The homogeneous $q$-difference operator $D_{xy}$ is described as follows[8,9]:

$$D_{xy}\{f(x,y)\} = \frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}.$$

The Thomae–Jackson $q$-integral[1,10,11] is

$$\int_c^d f(x)d_qx = (1-q)\sum_{n=0}^{\infty}[df(q^nd)-cf(q^nc)]q^n.$$

In 1981, Andrews and Askey[12] presented the following $q$-integral:

**Theorem 1.1** ([12]). *For* $max\{|ac|,|ad|,|bc|,|bd|\} < 1$*, we have*

$$\int_c^d \frac{(qx/c,qx/d;q)_\infty}{(ax,bx;q)_\infty}d_qx = \frac{d(1-q)(q,qd/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}. \tag{1.4}$$

In 1985, Askey and Wilson[13] obtained the next $q$-integral:

$$\frac{1}{2\pi i}\int_C \frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty}\frac{dz}{z} = \frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{1.5}$$

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