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Finite Element Method for solving Boussinesq model for Describing the Circulation in Khor Al-Zubair Estuarine Lagoon-Northwest of Arabian Gulf

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***Abstract*—** Circulation in a well-mixed semi diurnal tidal inlet, i.e. Khor Al-Zubair Estuarine Lagoon, is studied at the first time based on solving of the Boussinesq equation of dispersion in two dimensions. The finite element method (FEM) ,i.e., Galerkin method, is a most precious method used to solve the oceanographic phenomena like mixing and circulation as used in this study. The observational data used were: Water tidal current and salinity for one tidal cycle (i.e., 13 hours) over a period of 1/5/2020to 30/6/2020 with a freshwater discharge about (84 -94) m3/s. The Galerkin method showed more suitable results for describing the circulation in comparison with actual data. The tidal flow ; the upstream freshwater; boundary friction forcing (from the bottom upward) and wind forcing (from the surface downward) showed the vertical sense to the obtained results of salinity. However, tidal flow; the upstream freshwater; meteorological rainfall flux and meteorological heat flux configured the horizontal sense to the obtained results of salinity. Finally, The Galerkin method presented differences in decimals in comparison between the actual and modeled data.
Keywords- *Finite Element Method, Boussinesq model, Khor Al-Zubair*

I. INTRODUCTION

Khor Al-Zubair, i.e. an extension from Arabian Gulf into the Iraqi land, is influenced at its downstream boundary by the tidal phenomenon from the north-west of Arabian Gulf where the tidal range (the difference between water level at subsequent flood and ebb) during spring tide period about 5 m (Kudair,1999). However, the mean tidal range reaches about 3.2 m (Bakr et al. ,1996). The tides in Khor Al-Zubair are generally mixed type with mainly semidiurnal (i.e., two unequal high and low tides in lunar day. The state of mixing in Khor Al-Zubair is well mixed in the vertical direction (Al-Mahdi, 1990). *An estuarine lagoon :*
It is a coastal transitional environment between the continental land masses and the sea, where the sea water diluted by fresh water coming from continental drainage basin. The circulation is kept going by mixing of seawater with outflowing freshwater. Part of the intruding seawater will mix with freshwater, whereby seawater returned to the sea by estuarine outflow in the upper water layer. This allows 'new' seawater to enter through the lower water layers. In this way, the circulation of seawater is created, which will directed inland along the bottom and seaward along the surface. Mixing is the result of turbulence, which is mainly caused by semidiurnal_tidal flows, spring-neap tidal cycle, boundary friction forcing and wind forcing. Most of the mixing (in the estuary) takes place *vertically* in the water column.Vertical mixing in the estuary occurs at three levels as surface downward by wind forces, the bottom upward by boundary generated turbulence (estuarine and sea boundary mixing), and internally by turbulent mixing caused by the water currents which will drive by tides, wind, and freshwater inflow.
A larger system of equations to model the entire problem, which the FEM then uses variational methods to approximate a solution by minimizing an associated error function. Recently, the finite element method could be considered the best method, in which, it is the first method that modified the stander Galerkin finite

element method. For the marine system characteristics, such as simulation of the circulation phenomenon, numerical modeling most likely is the main technique to be utilized for evaluation. In Iraqi marine waters, very limited works had been done for describing this circulation phenomenon using numerical modeling methods, such as finite differences methods [2]; [3]; [4]; [6] and [7]. However, [1] was one of the very few descriptive studies that have addressed the mixing and circulation phenomena in Khor Al-Zubair channel.

II. MATHEMATICAL DESCRIPTION OF THE MODEL

The adopted model will be Boussinesq equations which used for narrow channel with rectangular cross-section [5]

The hydrodynamic models:

$$\frac{\partial z}{\partial t}+\frac{1}{b}\frac{\partial}{\partial x}\left(b\int_{-z}^{h} U dz\right)=0 \text{ --------------2.1}$$

$$\frac{\partial}{\partial x}(bU)+b\frac{\partial W}{\partial z}=0 \text{ --------------------- 2.2}$$

The dispersion model:

$$\frac{\partial s}{\partial t}+U\frac{\partial s}{\partial x}+W\frac{\partial s}{\partial z}-\frac{\partial}{\partial z}\left(Kz\frac{\partial s}{\partial z}\right)-\frac{1}{b}\frac{\partial}{\partial x}\left(bKx\frac{\partial s}{\partial x}\right)=0 \text{ ------------------ 2.3}$$

Where,

x,z = are the co-ordinate in the plane of undisturbed water surface (+ve) seawards and vertically downwards respectively.

t = is the time co-ordinate

U,W = are the velocity components in the x and z directions respectively

S = is the salinity

h,b = are the depth and width of the channel respectively (function of x Where,

eq. (1) and eq. (2) = are the continuity equations (hydrodynamic model).

eq. (3) = is the salt conservation equation (dispersion model).

III. THE GALERKIN FINITE ELEMENT EQUATION

In this chapter, discussed Galerkin Finite Element Method for solving Boussinesq model ………..

The Boussinesq model:

$$\frac{\partial s}{\partial t}+U\frac{\partial s}{\partial x}+W\frac{\partial s}{\partial z}-\frac{\partial}{\partial z}\left(k_z\frac{\partial s}{\partial z}\right)-\frac{1}{\rho}\frac{\partial}{\partial x}\left(bk_x\frac{\partial s}{\partial x}\right)=0 \text{ -------------(3.1)}$$

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+w\frac{\partial s}{\partial z}-\frac{1}{b}\frac{\partial}{\partial x}\left(bk_x\frac{\partial s}{\partial x}\right)=0$$

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+w\frac{\partial s}{\partial z}-\left(\frac{b}{b}k_x\frac{\partial^2 s}{\partial x^2}t+\frac{k_x}{b}\frac{\partial s}{\partial x}\cdot\frac{\partial b}{\partial x}\right)=0$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+w\frac{\partial S}{\partial z}-k_x\frac{\partial^2 s}{\partial x^2}-\frac{ak_x}{b}\left(\frac{\partial S}{\delta x}\right)=0$$

The weak form is find $s \in V \subseteq H^1$. by the test function v and integer we get

$$\left(\frac{\partial s}{\partial t},v\right)+\left(u\frac{\partial s}{\partial t},v\right)+\left(w\frac{\partial s}{\partial z},v\right)-\left(E\frac{\partial s}{\partial x},v\right)+\left(k_x\frac{\partial s}{\partial x}\cdot\frac{\partial v}{\partial x}\right)=0 \text{ …..(3.2)}$$

Where , $E=\frac{ak_x}{b}$ a=0.018

The fully discrete approximation back ward Euler- Galerkin method :

Find $S_h^n=\left(\frac{S_h^n-S_h^{n-1}}{\Delta t}\right)$

$$\left(\frac{S_h^n-S_h^{n-1}}{\Delta t},v\right)+(u+E)\left(S_{x,h}^n,v\right)+\left(S_{z,h}^n,v\right)-k_x\left(S_{x,h}^n,v_x\right)=0$$

Where, $\Delta t = \tau$

$(S_h^n - S_h^{n-1}, v) + (U + E)\tau(S_{x,h}^n, v) + \tau(S_{z,h}^n, v) - \tau k_x(S_{x,h}^n, v_x) = 0$

$(S_h^n\ , v) - (S_h^{n-1}, v) = \tau k_x(S_{x,h}^n, v_x) - (U + E)\tau(S_{x,h}^n, v) - \tau w(S_{z,h}^n, v)$ ……….(3.3)

The Matrix Form of Boussinesq model:

Let

$S_h^n = \sum_{i=1}^{n} \psi_i \xi_i^n$,

Taking $v = \psi_j$ and substituting in (3.3) we obtain;

$\sum_{i=1}^{n} (\psi_i, \psi_j)\xi_i^n - \sum_{i=1}^{n} (\psi_i, \psi_j)\xi_i^{n-1} = \tau k_x \sum_{i=1}^{n} (\psi_i', \psi_j') - \tau(u + E)(\psi_i', \psi_j)\xi_i^n - \tau w(\psi_i', \psi_j)$ ……………… (3.4)

$B\xi_i^n + \tau(k_x A - LC)\xi_i^n = B\xi_i^{n-1}$ ……………………………….(3.5)

Where,

$B = \int_0^h \int_0^h (\psi_i, \psi_j) dxdz$

$A = \int_0^h \int_0^h \psi_i' \psi_j' dxdz$

$C = \int_0^h \int_0^h \psi_i' \psi_j dxdz$;

And $L = (u + E)$

IV. Material and Methods

Water current datasets:

By using an Acoustic Doppler Current Profiler (ADCP), the measurements of water currents velocities have been done over 13 hour time period of tidal currents. Five datasets of tidal currents velocities have measured at the selected cross-sections along the Khor Al-Zubair channel (Figure 1). The surface and bottom values of the currents at the middle of each cross-section have been taken and drawn as shown in the Figures (2.2-2.6). Current velocity data see Tables (2.1-2.5) have been drawn and simulated in the numerical model that depends partly on the boundary condition of the model.

Table Current velocity measurements for surface and bottom at cross-section (1-5)

Time	S1	S2	S3	S4	S5
1	1.48	0.81	0.47	0.59	1.5
2	1.14	1.1	0.7	0.84	1.29
3	0.98	1.1	0.72	0.4	0.9
4	0.75	0.8	0.81	0.54	0.4
5	0.48	0.33	0.55	0.57	0.3
6	0.38	0.31	0.36	0.49	0.1
7	0.13	0.29	0.11	0.13	0.31
8	0.03	0.43	0.2	0.25	0.57
9	0.32	0.64	0.35	0.46	0.88
10	0.7	0.73	0.63	0.89	0.77
11	0.91	0.77	0.6	1.12	0.61
12	0.75	0.61	0.67	1.1	0.2
13	0.63	0.63	0.68	0.8	0.17

Table Current velocity measurements for bottom at cross-section (1-5)

Time	S1	S2	S3	S4	S5
1	1.33	0.73	0.4	0.59	1.01
2	1.08	0.85	0.56	0.6	0.98
3	0.77	0.99	0.73	0.62	0.74
4	0.34	0.66	0.66	0.35	0.22
5	0.21	0.24	0.46	0.35	0.23
6	0.13	0.06	0.34	0.48	0.3
7	0.22	0.33	0.22	0.41	0.64
8	0.52	0.61	0.3	0.7	0.79
9	0.77	0.66	0.35	0.11	0.68
10	0.82	0.62	0.55	0.44	0.41
11	0.92	0.53	0.64	0.82	0.4
12	0.77	0.54	0.55	0.66	0.39
13	0.75	0.39	0.52	0.58	0.27

Table Longitude and Latitude coordinates of the locations of the studies cross- sections.

Section Nos.	**X**	**Y**	**Longitude**	**Latitude**
1	778822.8582	3340172.802	47° 53' 41.9851'' E	30° 9' 39.7968'' N
2	777896.3101	3343627.752	47° 53' 10.6585'' E	30° 11' 32.660'' N
3	776608.9621	3345463.818	47° 52' 24.3063'' E	30° 12' 33.2907'' N
4	767079.0595	3366390.665	47° 46' 47.4313'' E	30° 24' 0.0167'' N
5	770271.4366	3359101.096	47° 48' 40.1557'' E	30° 20' 0.9279'' N

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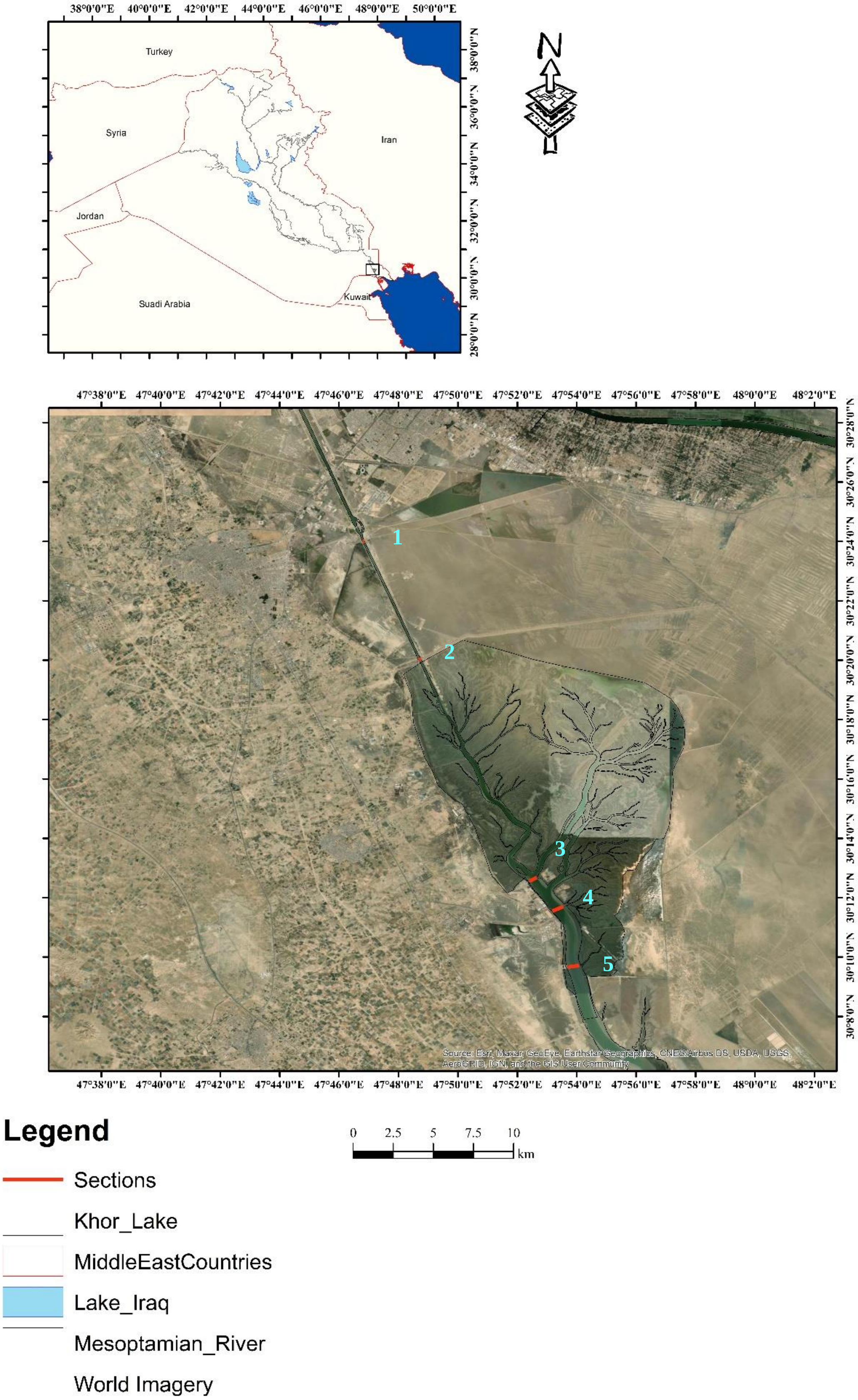

V. Results and Discussions

The presented models (i.e. equation no. 2.3) and the methodology steps displayed in methodology part allow the investigation of the circulation in Khor Al-Zubair. Matching between the results of the measured and observed spatial patterns of salinity at each cross-section are expected. Therefore, it is essential to examine the dispersion equation of the Boussinesq model for describing the circulation over the period (01/05/2021 to 30/06/2021) by using the Galerkin finite element method.

The results may be reasonable for most cross-sections in the study area. Because the Khor Al-Zubair is an area of permanent exchange of air and water masses of different physical features, which results in significant variability from day to day and from year to year. All of the obtained results have to be discussed in terms of the oceanographic physics and the precision of the method used (i.e., at the first time to solve the dispersion equation of Boussinesq model in Khor Al-Zubair Lagoon in the cntext of climate change).

To compare the measured and observed surface spatial patterns of salinity at each cross-section within one tidal period (i.e., 13 hours) over the studied period, the results are presented in the following figures and tables to show the precision of the Galerkin finite element method.

Regarding cross-sections (1;2;3;4 and 5), the surface salinity changes can be noted in the flowing figures (2.1:2.5) for one tidal period respectively.

Hence, during the first half tidal period: It have significant ascending tendency at sections (1 and 2), see Figures (2.1:2.2). However, a significant descending tendency were more promounced at sections (3;4 and 5),see Figures (2.3:2.5). Next, during the second half tidal period: It have significant ascending tendency at all sections.

These results indicate to the location of Khor Al-Zubair Lagoon under the influence of the water masses of each of the Arabian Gulf and the upstream fershining water.

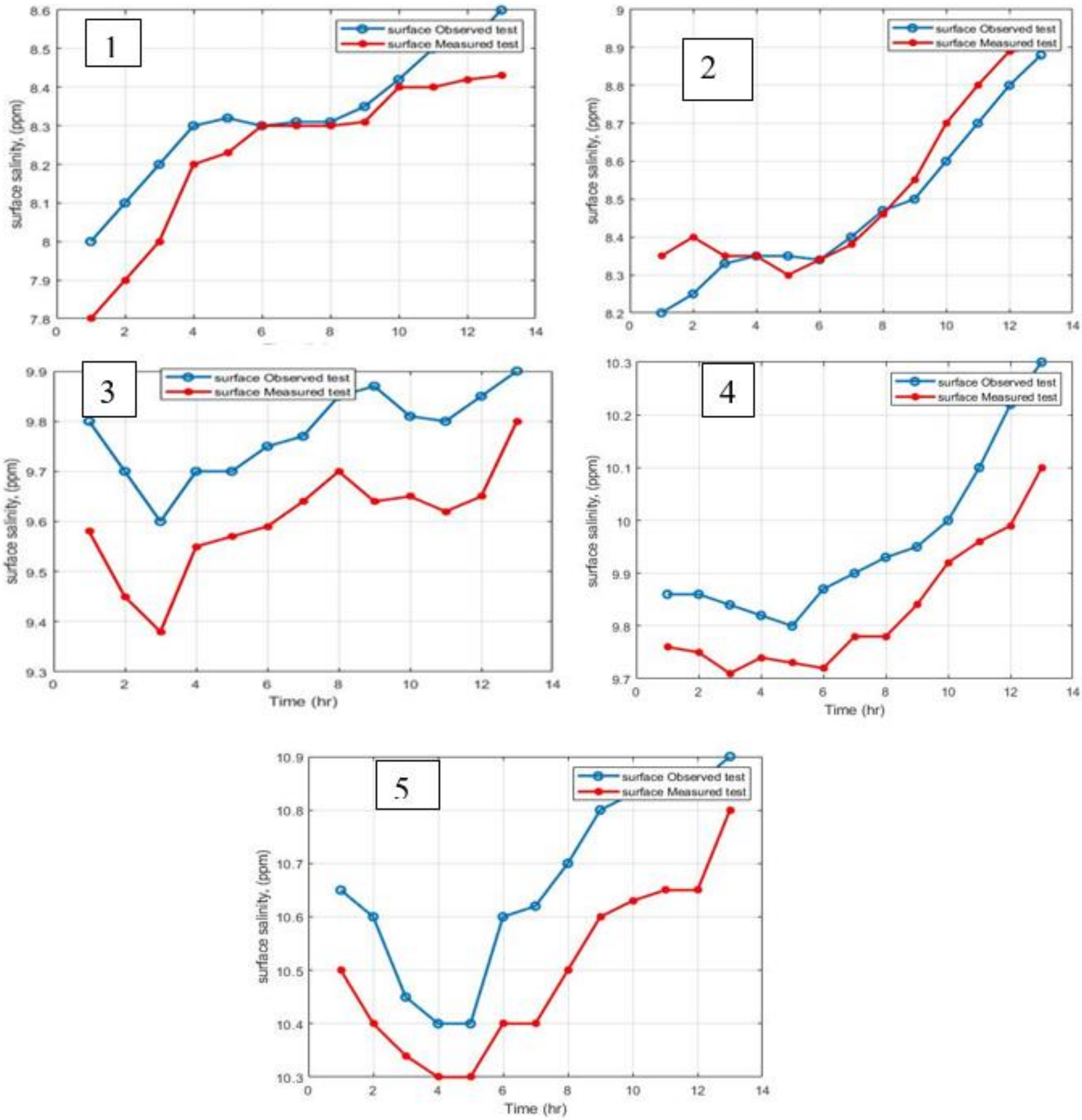

Figure 2: Salinity of measured and observed surface data at cross-section 1-5

Also to compare the measured and observed bottom spatial patterns of salinity at each cross-section, the results are presented in the following figures and tables to show the precision of the Galerkin finite element method. Regarding cross-sections (1;2;3;4 and 5), the bottom salinity changes can be noted in the flowing figures (3.1:3.5) for one tidal period respectively. Hence, during the first and the second half tidal period It have significant ascending tendency and descending tendency at all sections.
These results indicate to the location of Khor Al-Zubair Lagoon under the influence of the water masses of each of the Arabian Gulf and the upstream freshening water.

Table 1: Salinity of measured and observed surface data at cross-section 1-5.

Time(hr)	S1		S2		S3		S4		S5	
	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)
1	7.8	8	8.35	8.2	9.58	9.8	9.76	9.86	10.5	10.65
2	7.9	8.1	8.4	8.25	9.45	9.7	9.75	9.86	10.4	10.6
3	8	8.2	8.35	8.33	9.38	9.6	9.71	9.84	10.34	10.45
4	8.2	8.3	8.35	8.35	9.55	9.7	9.74	9.82	10.3	10.4
5	8.23	8.32	8.3	8.35	9.57	9.7	9.73	9.8	10.3	10.4
6	8.3	8.3	8.34	8.34	9.59	9.75	9.72	9.87	10.4	10.6
7	8.3	8.31	8.38	8.4	9.64	9.77	9.78	9.9	10.4	10.62
8	8.3	8.31	8.46	8.47	9.7	9.85	9.78	9.93	10.5	10.7
9	8.31	8.35	8.55	8.5	9.64	9.87	9.84	9.95	10.6	10.8
10	8.4	8.42	8.7	8.6	9.65	9.81	9.92	10	10.63	10.83
11	8.4	8.5	8.8	8.7	9.62	9.8	9.96	10.1	10.65	10.84
12	8.42	8.51	8.89	8.8	9.65	9.85	9.99	10.22	10.65	10.85
13	8.43	8.6	8.9	8.88	9.8	9.9	10.1	10.3	10.8	10.9

Table 1: Salinity of measured and observed bottom data at cross-section 1-5.

Time(hr)	S1		S2		S3		S4		S5	
	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)	Measured salinity (ppm)	Observed salinity (ppm)
1	8.3	8.2	8.5	8.8	9.7	9.75	9.7	9.95	10.68	10.8
2	8.34	8.26	8.4	8.7	9.68	9.64	9.6	9.9	10.65	10.75
3	8.4	8.3	8.3	8.65	9.5	9.5	9.5	9.86	10.55	10.67
4	8.4	8.35	8.35	8.64	9.6	9.6	9.45	9.8	10.55	10.65
5	8.5	8.4	8.24	8.6	9.75	9.64	9.5	9.85	10.6	10.8
6	8.65	8.55	8.18	8.65	9.81	9.79	9.6	9.88	10.65	10.82
7	8.6	8.6	8.35	8.41	9.74	9.64	9.65	9.91	10.7	10.9
8	8.58	8.55	8.59	8.75	9.7	9.7	9.7	9.96	10.72	10.92
9	8.8	8.61	8.85	8.9	9.8	9.8	9.75	10.1	10.8	10.95
10	8.9	8.7	8.95	9.2	9.9	9.9	9.8	10.2	10.95	11
11	8.95	8.81	9	9.3	9.94	10	9.75	10	11.1	11.2
12	9	8.9	9.34	9.45	9.98	10.1	9.8	10.2	11.25	11.35
13	9.1	9	9.4	9.55	9.86	10	9.9	10.3	11.35	11.5

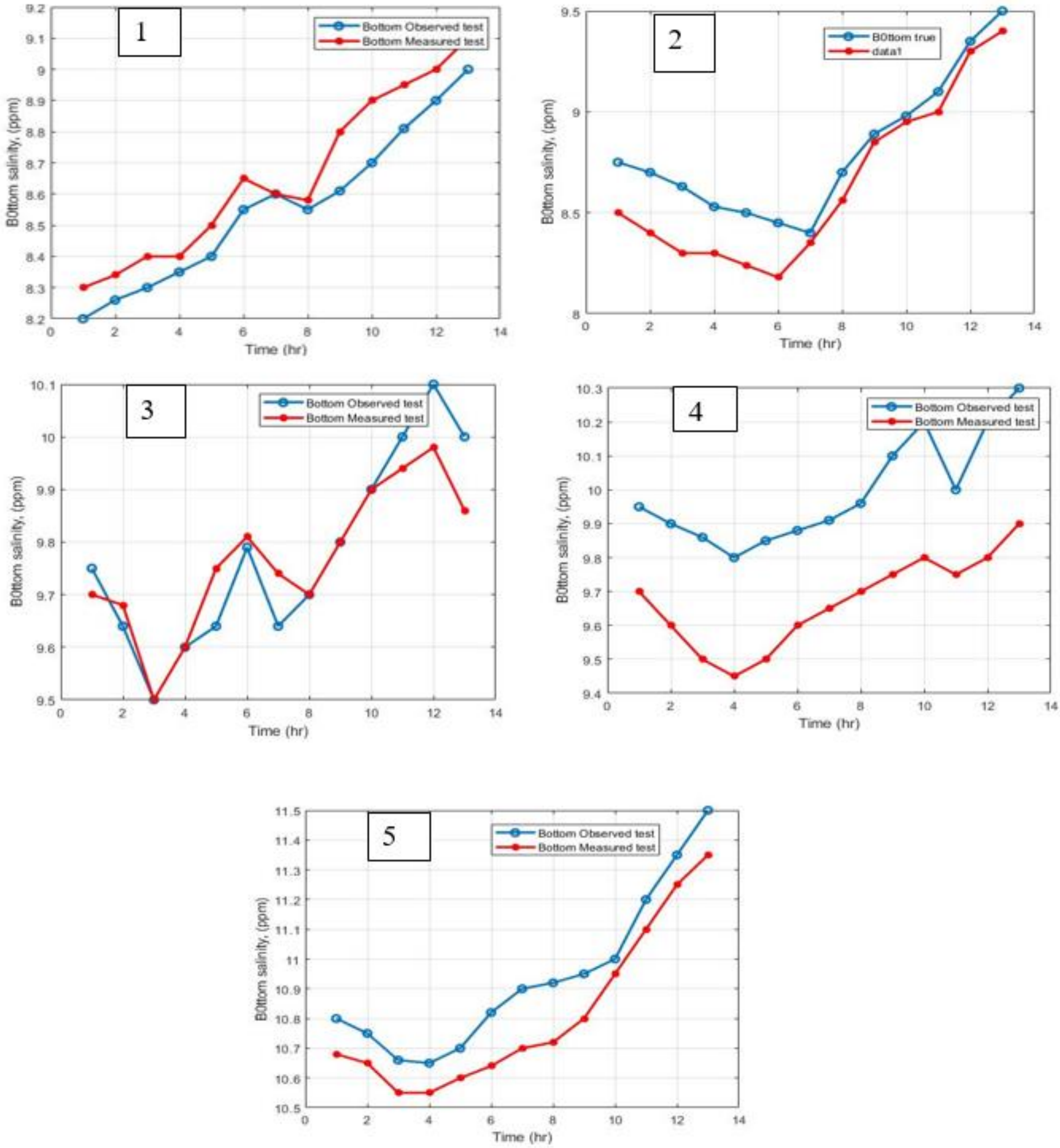

Figure 3: Salinity of measured and observed bottom data at cross-section 15

VI. Discussion

The Galerkin finite element method showed more suitable results in comparison with actual data. In addition,the Boussinesq equation (i.e. dispersion equation) showed more suitable results for describing the circulation in Khor Al-Zubair estuarine lagoon-Northwest Arabian Gulf in comparison with actual data. In which, the estuary could be defined as a transition region between the upstream fresh water and the downstream salty water of the Arabian Gulf.

Since the Khor Al-Zubair water body respond to different oceanographic; meterologic; hydrographical and chemical events, the salinity changes becomes a proxy for such events.

The salinity change in a water column at each cross-section could be attributed to the influence of mixing, in which the circulation is kept going by mixing of salty water with the upstream fresh water vertically in the water column. As a location of the Khor Al-Zubair drainage basin between the maritime and continental climatic conditions, it is strongly influenced by the Northwest winds. That leads to seasonal and interannual variations in terms of meterology. On the other hand, the climatic and physiographic conditions could be considerd the main factors controlling upstream fresh water change in the drainage basin.

In a vertical sense to the obtained results of salinity**,** it could be think that the Khor Al-Zubair water body with a vertical gradient of salinity (at each cross-section) as estuary.

The ascending and the descending tendencies to the surface and bottom results showed spatially non-uniform patterns in terms of magnitude of salinity in the different sections. These patterns along the Khor Al-Zubair channel could be related to the turbulent mixing which is caused by the water currents what are driven by: tidal flow (mainly); the upstream freshwater; boundary friction forcing (from the bottom upward) and wind forcing (from the surface downward), which yields a stronger mixing at the water coulumn.

In a horizontal sense to the obtained results of salinity**,** it could be think that the Khor Al-Zubair water body with a longiudinal gradient of salinity (along channel) as estuary.

Where, the generation of the horizontal gradients of salinity along the Khor Al-Zubair channel (i.e., shallow waters) coud be kept by turbulent mixing which is caused by the water currents what are driven by: tidal flow; the upstream freshwater; meterological rainfall flux and meterological heat flux, which yields a stronger freshwater and high temperature than over the Arabian Gulf (i.e., deeper waters).

In particular, all the previous results of [2]; [3]; [6];[7] ;[15] and[16], confirmed the influence of the tidal flow is a non linear during the flood and ebb tides in Khor Al-Zubair Lagoon.

Therefore, the main reason behind the spatial diversification among the salinity distribution at all sections could be related to the differences in the strength of the tidal flow and the upstream freshwater.

The discrepancies relative to other studies: The obtained results may not fully be consistent with the studies of [2]; [3] and [6]who found a precious matching between the measured and observed values of salinity in Khor Al-Zubair , i.e., due to the increased values of the upstream freshwater nowadays.

Where, a significant total linear freshwater trend could beattributed to the change in natural and anthropogenic climate changes as well as the change in physiographic conditions of the catchment area.

VII. Conclusions

In the present study, we solved at the first time the Boussinesq equation of dispersion by using the Galerkin finite element method for describing the circulation in Khor Al-Zubair Estuarine Lagoon. That is, to simulate the estuarine tidal circulation in terms of the horizontal and vertical sense to the salinity distribution. Despite the simplicity of the used model , it was possible to reproduce data could be simulated a real data of salinity.

With the help of the model, the Galerkin method presented differences in decimals in comparison between the actual and modeled data as well as in comparison with the previous studies. Based on the model, the obtained results of the salinity could be considered as a proxy distribution to the influence of the tides and upstream fresh water (i.e., the main drivers to the current velocity)

The model used showed further and very interesting results of circulation in a well-mixed estuary in the context of climate change.

Finally we can say, the model as non-linear partial differential equation sets to describe a non-linear circulation in two dimensions by using five cross-sections in the study area.

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