



Stability of nonlinear q -fractional dynamical systems on time scale

Nada K. Mahdi, Ayad R. Khudair *

Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq



ARTICLE INFO

Keywords:

q -fractional calculus
 q -Mittag-Leffler function
Time scale calculus
 q -Mittag-Leffler stability

ABSTRACT

This paper adopts a new terminology, “delta q -Mittag-Leffler stability”, for studying the stability of nonlinear q -fractional dynamical systems on the time scale. In fact, the idea of delta q -Mittag-Leffler stability is inspired by the idea of Mittag-Leffler stability, which is designed to investigate the stability of fractional dynamical systems. The sufficient conditions for delta q -Mittag-Leffler stability of considered dynamical systems with Caputo delta q -derivatives have been introduced.

1. Introduction

Fractional calculus studies fractional-order derivative and integral operators and their applications in real and complex domains.^{1,2} Furthermore, fractional differential equation models have been applied in a variety of fields, including economics,³ automatic control,⁴ epidemiological models,^{5,6} electrical engineering,⁷ and many others. There has been an increase in interest in this topic as a result of its interesting applications in numerous disciplines of science and engineering.⁸⁻¹³

In the early twentieth century, Jackson¹⁴ introduced q -calculus as a study of calculus without limits. Due to the increasing demand for mathematical modeling that incorporates quantum computing, this topic has recently drawn the attention of many researchers. q -calculus establishes a connection between physics and mathematics, where it plays a main role in a variety of physics fields, including high energy physics and nuclear,¹⁵ conformal quantum mechanics,¹⁶ cosmic strings and black holes,¹⁷ and so on. The fundamentals of q -calculus can be found in Kac and Cheung's textbook.¹⁸ The q -differential equations, based on the q -calculus, were established which can describe some special physical processes occurring in quantum dynamics, discrete stochastic processes, discrete dynamical systems, and other areas.^{19,20} The concept of q -fractional calculus was first presented by Al-Salam²¹⁻²³ and then by Agarwal.²⁴ The results of q -fractional integrals, q -fractional derivatives, and the qualitative characteristics of solutions to q -fractional differential equations have all been extensively studied.²⁵⁻²⁹

Recently, with the appearance of time scale calculus,^{30,31} several researchers have begun to pay attention and incorporate time scale approaches into q -fractional calculus.³²⁻³⁴ These findings are mostly related to fractional calculus on the time scale $\mathbb{T}_q := \{q^r : r \in \mathbb{Z}\} \cup \{0\}$, where $0 < q < 1$. Some recent studies^{29,35-38} have focused on q -FDEs with a Caputo nabla q -fractional derivative.

However, the Mittag-Leffler function is crucial in the study of fractional differential equations.³⁹⁻⁴¹ In several papers,⁴²⁻⁴⁵ the stability of linear and nonlinear fractional dynamic systems is investigated. However, to the best of our knowledge, no research has been done on the stability of q -fractional dynamical systems. In Ref. 46, the authors took into consideration two nonlinear dynamical systems in order to illustrate the benefit of utilizing fractional order derivatives rather than integer order derivatives. The integer-order derivative dynamical system turned out to be unstable.

As a result of the aforementioned findings, we present in this paper the delta q -Mittag-Leffler stability theorem for the following nonlinear q -fractional dynamical systems with Caputo delta q -fractional derivatives:

$$\begin{aligned} {}^C D_{A_q, t_0}^\alpha y(t) &= g(t, y(t)) \\ y(t_0) &= y_0, \end{aligned} \tag{1.1}$$

where $\alpha \in (0, 1)$, $t \geq t_0$, $t, t_0 \in \mathbb{T}_q$, $g : \mathbb{T}_q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous in y , and $g(t, 0) = 0$.

2. Preliminaries

The fundamental definitions and results for the q -calculus can be found in Refs. 18, 25, 47-49. Here we shall employ some of those basics on a time scale.

Let \mathbb{T}_q be a time scale for $0 < q < 1$

$$\mathbb{T}_q = \{q^r : r \in \mathbb{Z}\} \cup \{0\},$$

where \mathbb{Z} denotes the integer set. A time scale is, in general, a closed subset of the real numbers.³¹

For the function $g : \mathbb{T}_q \rightarrow \mathbb{R}$, the delta q -derivative of $g(r)$ is defined as

$$\Delta_q g(r) = \frac{g(qr) - g(r)}{(q - 1)r}, \quad r \in \mathbb{T}_q \setminus \{0\}.$$

* Corresponding author.

E-mail addresses: nada20407@yahoo.com (N.K. Mahdi), ayadayad1970@yahoo.com (A.R. Khudair).

In addition to a higher order of delta q -derivatives

$$\Delta_q^0 g(r) = g(r), \quad \Delta_q^m g(r) = \Delta_q(\Delta_q^{m-1} g(r)) \quad (m = 1, 2, 3, \dots).$$

The delta q -integral of $g(r)$ is defined as

$$(I_{q,0}g)(r) = \int_0^r g(u) \Delta_q u = (1-q) \sum_{\ell=0}^{\infty} r q^\ell g(r q^\ell), \quad r \in \mathbb{T}_q.$$

and,

$$(I_{q,a}g)(r) = \int_a^r g(u) \Delta_q u = \int_0^r g(u) \Delta_q u - \int_0^a g(u) \Delta_q u, \quad a, r \in \mathbb{T}_q.$$

The basic theorem of calculus specifically applies to the delta q -derivative and delta q -integral as follows:

$$\Delta_q \int_0^r g(u) \Delta_q u = g(r).$$

Additionally, if $g(r)$ is continuous at zero,

$$\int_0^r \Delta_q g(u) \Delta_q u = g(r) - g(0).$$

Also, the following identities will be beneficial

$$\Delta_q \int_a^r g(r, u) \Delta_q u = \int_a^r \Delta_q g(r, u) \Delta_q u + g(qr, r), \quad (2.1)$$

and,

$$\Delta_q \int_r^b g(r, u) \Delta_q u = \int_{qr}^b \Delta_q g(r, u) \Delta_q u - g(r, r).$$

According to the time-scale theorem and the q -calculus theorem in general, the product rule is valid

$$\Delta_q(g(r)v(r)) = v(r)\Delta_q g(r) + g(qr)\Delta_q v(r). \quad (2.2)$$

In time scale integration, the change of variables is valid.³¹ We state an exception as it pertains to the article in the subsequent theorem.

Definition 2.1. The delta q -factorial function is as follows:

If $\ell \in \mathbb{N}$, then

$$(r-u)_q^{(0)} = 1, \quad (r-u)_q^{(\ell)} = \prod_{j=1}^{\ell-1} (r - q^j u).$$

If α is a non positive integer, then

$$(r-u)_q^{(\alpha)} = r^\alpha \prod_{j=0}^{\infty} \frac{1 - \frac{u}{r} q^j}{1 - \frac{u}{r} q^{j+\alpha}}.$$

Several properties are stated for the q -factorial function. By using the definition and a straightforward computation, each property will be verified.

Lemma 2.1. Let $\beta, \gamma \in \mathbb{R}$, then

$$1. \quad (r-u)_q^{(\beta+\gamma)} = (r-u)_q^{(\beta)}(r - q^\beta u)_q^{(\gamma)}.$$

$$2. \quad (\eta r - \eta u)_q^{(\beta)} = \eta^\beta (r-u)_q^{(\beta)}.$$

3. The delta q -derivative of the delta q -factorial function with respect to r is given by the expression

$$\Delta_q(r-u)_q^{(\beta)} = \frac{1 - q^\beta}{1 - q} (r-u)_q^{(\beta-1)}.$$

4. The delta q -derivative of the delta q -factorial function with respect to u is given by the expression

$$\Delta_q(r-u)_q^{(\beta)} = -\frac{1 - q^\beta}{1 - q} (r - qu)_q^{(\beta-1)}.$$

Definition 2.2. For $\beta \in \mathbb{C} \setminus \{-m, m \in \mathbb{N}_0\}$ and $0 < q < 1$, the delta q -Gamma function is

$$\Gamma_q(\beta) = (1-q)_q^{(\beta-1)}(1-q)^{1-\beta},$$

which fulfills

$$\Gamma_q(1+\beta) = \frac{1-q^\beta}{1-q} \Gamma_q(\beta), \quad \Gamma_q(1) = 1. \quad (2.3)$$

For $\beta, \gamma \in \mathbb{C}$, the delta q -Beta function

$$B_q(\beta, \gamma) = \int_0^1 u^{\beta-1} (1-qu)_q^{(\gamma-1)} \Delta_q u.$$

The delta q -Gamma and delta q -Beta defined by

$$B_q(\beta, \gamma) = \frac{\Gamma_q(\beta)\Gamma_q(\gamma)}{\Gamma_q(\beta+\gamma)}.$$

Definition 2.3. Let $C_q^{(m)}[0, b]$ be the space containing all continuous functions with continuous delta q -derivatives up to order $m-1$ on $[a, b]$ is defined as

$$C_q^{(m)}[0, b] = \left\{ g(r) : \Delta_q^\kappa g(r) \in C[0, b], \kappa = 0, 1, \dots, m \right\}.$$

Definition 2.4. The definition of the delta q -exponential function is

$$\begin{aligned} e_{\Delta_q}(r) &= \prod_{\ell=0}^{\infty} (1 - rq^\ell)^{-1} \\ &= \sum_{\ell=0}^{\infty} \frac{r^\ell}{[\ell]!}, \end{aligned}$$

such that $e_{\Delta_q}(0) = 1$.

3. Fractional delta q -integrals and delta q -derivatives

We will use the fractional q -integral and the fractional q -derivative as stated in Refs. 21–24, 26, 27, 33:

In Refs. 22, 24, which defines fractional q -integral of order $\alpha > 0$ with the lower limit of integration being zero, we will use as follows:

$$I_{\Delta_q}^\alpha g(r) = \frac{1}{\Gamma_q(\alpha)} \int_0^r (r - qs)_q^{(\alpha-1)} g(s) \Delta_q s,$$

and for $\alpha_1, \alpha_2 > 0$, then

$$I_{\Delta_q}^{\alpha_2} I_{\Delta_q}^{\alpha_1} g(r) = I_{\Delta_q}^{\alpha_1 + \alpha_2} g(r).$$

Lemma 3.1. If $g(r)$ is defined and finite, then for $0 < \alpha < 1$

$$D_{\Delta_q}^\alpha g(r) = \Delta_q I_{\Delta_q}^{\alpha-1} g(r),$$

where $r \in [q^a, q^b] \subset \mathbb{T}_q$ with $a, b \in \mathbb{N} \cup \{0\}$, $b < a$.

Proof. Begin with the left-hand side of the equality

$$\begin{aligned} \Delta_q I_{\Delta_q}^{\alpha-1} g(r) &= \Delta_q \left[\frac{1}{\Gamma_q(1-\alpha)} \int_0^r (r - qu)_q^{(-\alpha)} f(u) \Delta_q u \right] \\ &= \frac{1}{\Gamma_q(1-\alpha)} \int_0^r \frac{1 - q^{-\alpha}}{1 - q} (r - qu)_q^{(-\alpha-1)} g(u) \Delta_q u \\ &\quad + (qr - qr)_q^{(-\alpha)} g(r) \\ &= \frac{1}{\Gamma_q(-\alpha)} \int_0^r (r - qu)_q^{(-\alpha-1)} g(u) \Delta_q u \\ &= D_{\Delta_q}^\alpha g(r). \end{aligned}$$

Remark 3.1. If $\alpha > 0$ and $n-1 < \alpha < n$ for a positive integer n , we extend the idea of the proof of Lemma 3.1 and write

$$\begin{aligned} D_{\Delta_q}^\alpha g(r) &= D_{\Delta_q}^{n-(n-\alpha)} g(r) \\ &= \Delta_q^n (I_{\Delta_q}^{-(n-\alpha)} g(r)), \end{aligned}$$

on $r \in \mathbb{T}_q$.

In Refs. 21, 26, 27, 33, defines the fractional q -integral with the lower limit of integration being nonzero. We will use it as follows:

$$\begin{aligned} I_{\Delta_q, r_0}^0 g(r) &= g(r), \\ I_{\Delta_q, r_0}^\alpha g(r) &= \frac{1}{\Gamma_q(\alpha)} \int_{r_0}^r (r - qs)_q^{(\alpha-1)} g(s) \Delta_q s. \end{aligned}$$

Some of the properties of the previously defined the delta q -integral

$$(I_{\Delta_q, r_0}^{\alpha_2} I_{\Delta_q, r_0}^{\alpha_1} g)(r) = (I_{\Delta_q, r_0}^{\alpha_1 + \alpha_2} g)(r), \quad r_0 < r,$$

where $\alpha_1, \alpha_2 > 0$.

Definition 3.1. For $g : \mathbb{T}_q \rightarrow \mathbb{R}$ at $r_0 \in \mathbb{T}_q$, the fractional delta q -derivative of the Riemann–Liouville type of order $\alpha \geq 0$ is given by:

$$D_{\Delta_q, r_0}^\alpha g(r) = \Delta_q^m I_{\Delta_q, r_0}^{m-\alpha} g(r),$$

where $m = [\alpha] + 1$.

For $0 < r_0 < r$, $\alpha \in \mathbb{R}^+$. Then

$$D_{\Delta_q, r_0}^\alpha I_{\Delta_q, r_0}^\alpha g(r) = g(r). \quad (3.1)$$

We start by stating and proving some important preliminary lemmas. Then we define Caputo delta q -fractional derivatives and show how they are related to fractional derivatives.

Lemma 3.2. The following equality holds for any value of $\alpha > 0$:

$$I_{\Delta_q, r_0}^\alpha \Delta_q g(r) = \Delta_q I_{\Delta_q, r_0}^\alpha - \frac{(r - r_0)_q^{(\alpha-1)}}{\Gamma_q(\alpha)} g(r_0).$$

Proof. Using (2.2) and Lemma 2.1, one can have the following result:

$$\Delta_q \left((r - u)_q^{(\alpha-1)} g(u) \right) = (r - qu)_q^{(\alpha-1)} \Delta_q g(u) - \frac{1 - q^{\alpha-1}}{1 - q} (r - qu)_q^{(\alpha-2)} g(u). \quad (3.2)$$

Applying Eq. (3.2) leads to

$$I_{\Delta_q, r_0}^\alpha \Delta_q g(r) = \frac{(r - u)_q^{(\alpha-1)}}{\Gamma_q(\alpha)} g(u) \Big|_{r_0}^r + \frac{1 - q^{\alpha-1}}{1 - q} \int_{r_0}^r (r - qu)_q^{(\alpha-2)} g(u) \Delta_q u,$$

or,

$$I_{\Delta_q, r_0}^\alpha \Delta_q g(r) = -\frac{(r - r_0)_q^{(\alpha-1)}}{\Gamma_q(\alpha)} g(r_0) + \frac{1 - q^{\alpha-1}}{1 - q} \int_{r_0}^r (r - qu)_q^{(\alpha-2)} g(u) \Delta_q u.$$

Now, use Lemma 2.1, Eq. (2.1) and the identity (2.3), we find that

$$\Delta_q I_{\Delta_q, r_0}^\alpha g(r) = \frac{1 - q^{\alpha-1}}{1 - q} \int_{r_0}^r (r - qu)_q^{(\alpha-2)} g(u) \Delta_q u,$$

which means the proof is now complete.

Theorem 3.1. The following equality holds for any real $\alpha > 0$ and any positive integer v such that $\alpha - v + 1$ is not negative integer or 0. More specifically, $\alpha > v - 1$

$$I_{\Delta_q, r_0}^\alpha \Delta_q^v g(r) = \Delta_q^v I_{\Delta_q, r_0}^\alpha g(r) - \sum_{\ell=0}^{v-1} \frac{(r - r_0)_q^{(\alpha-v+\ell)}}{\Gamma_q(\alpha + \ell - v + 1)} \Delta_q^\ell g(r_0).$$

Proof. Using Lemma 3.2, Lemma (2.2) and (2.3), the proof can be achieved by following inductively on v .

If we substitute α for $1 - \alpha$ in Lemma 3.2, the q -delta fractional derivatives of Riemann and Caputo can thus be related.

Definition 3.2. For $g : \mathbb{T}_q \rightarrow \mathbb{R}$ at $r_0 \in \mathbb{T}_q$, the Caputo delta q -fractional derivatives of order $\alpha \geq 0$ is

$$\begin{aligned} C D_{\Delta_q, r_0}^\alpha g(r) &= I_{\Delta_q, r_0}^{m-\alpha} \Delta_q^m g(r) \\ &= \frac{1}{\Gamma_q(m-\alpha)} \int_{r_0}^r (r - qu)_q^{(m-\alpha-1)} \Delta_q^m g(u) \Delta_q u, \end{aligned}$$

where $m = [\alpha] + 1$.

Such that the sequential Caputo delta q -fractional derivatives $C D_{\Delta_q, r_0}^{m\alpha}$, $m \in \mathbb{N}$ is

$$C D_{\Delta_q, r_0}^{m\alpha} = C D_{\Delta_q, r_0}^\alpha \dots C D_{\Delta_q, r_0}^\alpha (m - \text{times}).$$

For $\alpha \in \mathbb{R}^+$. Then

$$C D_{\Delta_q, r_0}^\alpha I_{\Delta_q, r_0}^\alpha g(r) = g(r), \quad r \in \mathbb{T}_q, \quad r > r_0.$$

Theorem 3.2. For any $0 < \alpha < 1$, we have

$$C D_{\Delta_q, r_0}^\alpha g(r) = D_{\Delta_q, r_0}^\alpha g(r) - \frac{(r - r_0)_q^{-\alpha}}{\Gamma_q(1 - \alpha)} g(r_0)$$

Lemma 3.3. Let $\alpha > 0$ and g is defined in suitable domains. Then

$$I_{\Delta_q, r_0}^\alpha C D_{\Delta_q, r_0}^\alpha g(r) = g(r) - \sum_{\ell=0}^{\eta-1} \frac{(r - r_0)_q^{(\ell)}}{\Gamma_q(\ell + 1)} \Delta_q^\ell g(r_0),$$

and if $0 < \alpha \leq 1$, then

$$I_{\Delta_q, r_0}^\alpha C D_{\Delta_q, r_0}^\alpha g(r) = g(r) - g(r_0). \quad (3.3)$$

The proof is given by the definition of Caputo delta q -fractional derivatives, Eq. (3.1), Lemma 3.2 and Theorem 3.1.

In order to solve the linear q -fractional equations, it is essential to know the identity:

$$I_{\Delta_q, r_0}^\alpha (x - r_0)_q^\mu = \frac{\Gamma_q(\mu + 1)}{\Gamma_q(\alpha + \mu + 1)} (x - r_0)_q^{\mu+\alpha}, \quad 0 < r_0 < x < r, \quad (3.4)$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.

Definition 3.3. For $\zeta, \zeta_0 \in \mathbb{C}$, the delta q -Mittag-Leffler function is

$${}_4 E_{\alpha, \beta}(\lambda, \zeta - \zeta_0) = \sum_{\eta=0}^{\infty} \lambda^\eta \frac{(\zeta - \zeta_0)^{\alpha\eta}}{\Gamma_q(\alpha\eta + \beta)}, \quad (3.5)$$

when $\beta = 1$, we use ${}_4 E_\alpha(\lambda, \zeta - \zeta_0) = {}_{4,1} E_\alpha(\lambda, \zeta - \zeta_0)$.

Example 3.1. Consider the Caputo delta q -fractional differential equation

$$\begin{aligned} C D_{\Delta_q, r_0}^\alpha y(r) &= \lambda y(r) + g(r), \\ y(r_0) &= c_0, \end{aligned} \quad (3.6)$$

where $0 < \alpha \leq 1$. If we apply I_{Δ_q, r_0}^α on Eq. (3.6) then by the help of Eq. (3.3) we see that

$$y(r) = c_0 + \lambda I_{\Delta_q, r_0}^\alpha y(r) + I_{\Delta_q, r_0}^\alpha g(r).$$

Utilizing the successive approximation method to obtain a clear and explicit solution.

Set $y_0(r) = c_0$, and

$$y_m(r) = c_0 + \lambda I_{\Delta_q, r_0}^\alpha y_{m-1}(r) + I_{\Delta_q, r_0}^\alpha g(r), \quad m = 1, 2, 3, \dots$$

For $m = 1$, we have the power formula Eq. (3.4)

$$y_1(r) = c_0 \left[1 + \frac{\lambda(r - r_0)_q^{(\alpha)}}{\Gamma_q(\alpha + 1)} \right] + I_{\Delta_q, r_0}^\alpha g(r).$$

For $m = 2$, we also see that

$$\begin{aligned} y_2(r) &= c_0 + \lambda c_0 I_{\Delta_q, r_0}^\alpha \left[1 + \frac{(r - r_0)_q^\alpha}{\Gamma_q(\alpha + 1)} \right] + I_{\Delta_q, r_0}^\alpha g(r) + \lambda I_{\Delta_q, r_0}^{2\alpha} g(r) \\ &= c_0 \left[1 + \frac{\lambda(r - r_0)_q^\alpha}{\Gamma_q(\alpha + 1)} + \frac{\lambda^2 (r - r_0)_q^{2\alpha}}{\Gamma_q(2\alpha + 1)} \right] + I_{\Delta_q, r_0}^\alpha g(r) + \lambda I_{\Delta_q, r_0}^{2\alpha} g(r). \end{aligned}$$

If we follow an inductive process and let $m \rightarrow \infty$, we arrive at the solution

$$\begin{aligned} y(r) &= c_0 \left[1 + \sum_{\eta=1}^{\infty} \frac{\lambda^\eta (r - r_0)_q^{\eta\alpha}}{\Gamma_q(\eta\alpha + 1)} \right] + \int_{r_0}^r \left[\sum_{\eta=1}^{\infty} \frac{\lambda^{\eta-1}}{\Gamma_q(\eta\alpha)} (r - q\vartheta)_q^{\eta\alpha-1} \right] g(\vartheta) \Delta_q \vartheta \\ &= c_0 \left[1 + \sum_{\eta=1}^{\infty} \frac{\lambda^\eta (r - r_0)_q^{\eta\alpha}}{\Gamma_q(\eta\alpha + 1)} \right] \\ &\quad + \int_{r_0}^r \left[\sum_{\eta=1}^{\infty} \frac{\lambda^{\eta-1}}{\Gamma_q(\eta\alpha + \alpha)} (r - q\vartheta)_q^{\eta\alpha + (\alpha-1)} \right] g(\vartheta) \Delta_q \vartheta \\ &= c_0 \left[1 + \sum_{\eta=1}^{\infty} \frac{\lambda^\eta (r - r_0)_q^{\eta\alpha}}{\Gamma_q(\eta\alpha + 1)} \right] \\ &\quad + \int_{r_0}^r (r - q\vartheta)_q^{(\alpha-1)} \left[\sum_{\eta=1}^{\infty} \frac{\lambda^\eta}{\Gamma_q(\eta\alpha + \alpha)} (r - q^\alpha \vartheta)_q^{(\alpha\eta)} \right] g(\vartheta) \Delta_q \vartheta. \end{aligned}$$

Using Eq. (3.5), we have

$$y(r) = c_0 \Delta_q E_\alpha(\lambda, r - r_0) + \int_{r_0}^r (r - q\vartheta)_q^{\alpha-1} \Delta_q E_{\alpha,\alpha}(\lambda, r - q^\alpha \vartheta) g(\vartheta) \Delta_q \vartheta.$$

A Caputo delta q -fractional initial value problem (3.6) has been solved, and its solution is expressed by means of a newly introduced delta q -Mittag-Leffler function (3.5).

4. The delta q -Mittag-Leffler stability

For nonlinear q -fractional dynamical systems on the time scale (1.1), we define the delta q -Mittag-Leffler stability, which is equivalent to the definition of the q -fractional dynamical systems' Mittag-Leffler stability in Ref. 46.

Definition 4.1. The system's trivial solution (1.1), $y(t)$, is said to be delta q -Mittag-Leffler stable if

$$\|y(t)\| \leq \left[\mathcal{M}(y(t_0)) \Delta_q E_\alpha(-\lambda, t - t_0) \right]^c, \quad (4.1)$$

where $\lambda \geq 0$, $c > 0$, $\mathcal{M}(0) = 0$, $\mathcal{M}(y) > 0$ and $\mathcal{M}(y)$ is locally Lipschitz for $y \in S_\eta = \{y \in \mathbb{R}^n : \|y\| < \eta\} \subset \mathbb{R}^n$.

We observe that condition (4.1) expands the definition of "classical exponential stability".

Theorem 4.1. If a scalar function $V(t, y) \in C[\mathbb{T}_q \times S_\eta, \mathbb{R}_+]$ and positive constants ξ_1 , ξ_2 and ξ_3 exist such that

$$\xi_1 \|y\|^2 \leq V(t, y) \leq \xi_2 \|y\|^2, \quad (4.2)$$

and

$${}^C D_{\Delta_q,t_0}^\alpha V(t, y) \leq -\xi_3 \|y\|^2, \quad (4.3)$$

at all $(t, y) \in \mathbb{T}_q \times S_\eta$, $t \geq t_0$, then the system's trivial solution (1.1) is delta q -Mittag-Leffler stable.

Proof. Let $y(t) = y(t, t_0, y_0)$ be any solution of the system (1.1). In accordance with conditions (4.2) and (4.3), we have

$${}^C D_{\Delta_q,t_0}^\alpha V(t, y) \leq -\frac{\xi_3}{\xi_2} V(t, y).$$

As a result, at all $t \in \mathbb{T}_q$, $t \geq t_0$, there is a function $\Xi(t) \geq 0$ such that

$${}^C D_{\Delta_q,t_0}^\alpha V(t, y) = -\Xi(t) - \frac{\xi_3}{\xi_2} V(t, y). \quad (4.4)$$

Therefore, Eq. (4.4) is in the form of Eq. (1.1) and has the solution

$$\begin{aligned} V(t, y(t)) &= V(t_0, y_0) \Delta_q E_\alpha\left(-\frac{\xi_3}{\xi_2}, t - t_0\right) \\ &\quad - \int_{t_0}^t (t - q\vartheta)_q^{\alpha-1} \Delta_q E_{\alpha,\alpha}(\lambda, t - q^\alpha \vartheta) g(\vartheta) \Delta_q \vartheta. \end{aligned}$$

Consequently,

$$V(t, y(t)) \leq V(t_0, y_0) \Delta_q E_\alpha\left(-\frac{\xi_3}{\xi_2}, t - t_0\right).$$

Once more from condition (4.2), we have

$$\|y(t)\|^2 \leq \frac{1}{\xi_1} V(t, y),$$

and

$$V(t_0, y_0) \leq \xi_2 \|y_0\|^2.$$

Therefore, we have

$$\begin{aligned} \|x(t)\|^2 &\leq \frac{1}{\xi_1} V(t, y) \\ &\leq \frac{1}{\xi_1} V(t_0, y_0) \Delta_q E_\alpha\left(-\frac{\xi_3}{\xi_2}, t - t_0\right) \leq \frac{\xi_2}{\xi_1} \|y_0\|^2 \Delta_q E_\alpha\left(-\frac{\xi_3}{\xi_2}, t - t_0\right). \end{aligned}$$

which is

$$\|y(t)\|^2 \leq \sqrt{\frac{\xi_2}{\xi_1}} \|y_0\| \left[\Delta_q E_\alpha\left(-\frac{\xi_3}{\xi_2}, t - t_0\right) \right]^{\frac{1}{2}}.$$

As a result, $y(t)$ is delta q -Mittag-Leffler stable.

Theorem 4.2. If a scalar function $V(t, y) \in C[\mathbb{T}_q \times \mathbb{R}^n, \mathbb{R}_+]$ and positive constants ξ_1 , ξ_2 and ξ_3 exists such that

$$\xi_1 \|y\|^2 \leq V(t, y) \leq \xi_2 \|y\|^2,$$

and

$${}^C D_{\Delta_q,t_0}^\alpha V(t, y) \leq -\xi_3 \|y\|^2,$$

at all $t \geq t_0$ and $(t, y) \in \mathbb{T}_q \times \mathbb{R}^n$, then the system's trivial solution (1.1) is globally delta q -Mittag-Leffler stable.

Proof. Similar to the proof of Theorem 4.1, we still have the estimate

$$\|y(t)\|^2 \leq \sqrt{\frac{\xi_2}{\xi_1}} \|y_0\| \left[\Delta_q E_\alpha\left(-\frac{\xi_3}{\xi_2}, t - t_0\right) \right]^{\frac{1}{2}},$$

for all $t \geq t_0$ and $(t, y) \in \mathbb{T}_q \times \mathbb{R}^n$. As a result, $y(t)$ is globally delta q -Mittag-Leffler stable.

Lemma 4.1. If $V(t_0, y(t_0)) \geq 0$, then for $0 < \alpha \leq 1$ we have ${}^C D_{\Delta_q,t_0}^\alpha V(t, y) \leq D_{\Delta_q,t_0}^\alpha V(t, y)$ for $t \geq t_0$.

Proof. From Eq. 3.2, the result can be clearly observed. It resembles the one from the classical case.

The related theorems for the Riemann delta q -fractional derivative are shown below, taking into account Eq. 3.2.

Theorem 4.3. If a scalar function $V(t, y) \in C[\mathbb{T}_q \times S_\eta, \mathbb{R}_+]$ and positive constants ξ_1 , ξ_2 and ξ_3 exists such that

$$\xi_1 \|y\|^2 \leq V(t, y) \leq \xi_2 \|y\|^2, \quad (4.5)$$

and

$$D_{\Delta_q,t_0}^\alpha V(t, y) \leq -\xi_3 \|y\|^2, \quad (4.6)$$

at all $t \geq t_0$ and $(t, y) \in \mathbb{T}_q \times S_\eta$, then the system's trivial solution (1.1) is delta q -Mittag-Leffler stable.

Proof. From Eq. (4.5), we have $V(t_0, y_0) \geq 0$. Consequently, from Lemma 4.1, we have

$${}^C D_{\Delta_q,t_0}^\alpha V(t, y) \leq D_{\Delta_q,t_0}^\alpha V(t, y), \quad \forall t \geq t_0.$$

Thus, we have from Eq. (4.6)

$${}^C D_{\Delta_q,t_0}^\alpha V(t, y) \leq -\xi_3 \|y\|^2.$$

The system's trivial solution (1.1) is, therefore, delta q -Mittag-Leffler stable because Theorem 4.1 hypotheses are satisfied.

Theorem 4.4. If a scalar function $V(t, y) \in C[\mathbb{T}_q \times \mathbb{R}^n, \mathbb{R}_+]$ and positive constants ξ_1, ξ_2 and ξ_3 exists such that

$$\xi_1 \|y\|^2 \leq V(t, y) \leq \xi_2 \|y\|^2,$$

and

$$D_{q,t_0}^\alpha V(t, y) \leq -\xi_3 \|y\|^2,$$

at all $t \geq t_0$ and $(t, y) \in \mathbb{T}_q \times \mathbb{R}^n$, then the system's trivial solution (1.1) is globally delta q -Mittag-Leffler stable.

Proof. The proof is similar to the theorem's proof 4.4.

5. Conclusions

Recently, the concept of Mittag-Leffler stability has been introduced to study the stability of fractional dynamical systems. Inspired by this, we introduce the concept of delta q -Mittag-Leffler stability for q -fractional dynamical systems. Moreover, sufficient conditions for delta q -Mittag-Leffler stability of nonlinear q -fractional dynamical systems with Caputo delta q -fractional derivatives have been introduced. In addition, to the best of our knowledge, there are no studies covering this kind of stability. Therefore, we expect that this type of stability will be very useful in control theory and other areas of mathematics, physics, and engineering when q -fractional nonlinear dynamical systems are taken into account.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

- Kilbas AA. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006.
- Podlubny I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*. San Diego, CA: Academic Press; 1999.
- Traore A, Sene N. Model of economic growth in the context of fractional derivative. *Alex Eng J*. 2020;59(6):4843–4850.
- Wei J. The controllability of fractional control systems with control delay. *J Comput Appl Math*. 2012;64(10):3153–3159.
- Lazima ZA, Khalaf SL. Optimal control design of the in-vivo HIV fractional model. *Iraqi J Sci*. 2022;63(9):3877–3888.
- Khalaf SL, Kadhim MS, Khudair AR. Studying of COVID-19 fractional model: Stability analysis. *Partial Differ Equ Appl Math*. 2023;7:100470.
- Moaddy K, Radwan A, Salama K, Momani S, Hashim I. The fractional-order modeling and synchronization of electrically coupled neuron systems. *Comput Appl Math*. 2012;64(10):3329–3339.
- Khudair AR. On solving non-homogeneous fractional differential equations of Euler type. *J Comput Appl Math*. 2013;32(3):577–584.
- Khalaf SL, Khudair AR. Particular solution of linear sequential fractional differential equation with constant coefficients by inverse fractional differential operators. *Differ Equ Dyn Syst*. 2017;25(3):373–383.
- Arshad S, Baleanu D, Tang Y. Fractional differential equations with bio-medical applications. In: Baleanu D, Lopes AM, eds. *Applications in Engineering, Life and Social Sciences, Part A*. Berlin, Boston: De Gruyter; 2019:1–20.
- Alchikh R, Khuri S. Numerical solution of a fractional differential equation arising in optics. *Optik*. 2020;208:163911.
- Abdeljawad T, Banerjee S, Wu G. Discrete tempered fractional calculus for new chaotic systems with short memory and image encryption. *Optik*. 2020;218:163698.
- Jalil AFA, Khudair AR. Toward solving fractional differential equations via solving ordinary differential equations. *Comput Appl Math*. 2022;41(1):37. 12 pp.
- F.H. Jackson. XI.—On q -functions and a certain difference operator. *Earth Environ Sci Trans R Soc Edinb*. 1909;46(2):253–281.
- Lavagno A, Swamy P. q -deformed structures and nonextensive statistics: a comparative study. *Physica A*. 2002;305(1–2):310–315.
- Youm D. q -Deformed conformal quantum mechanics. *Phys Rev D*. 2000;62(9):095009.
- Page DN. Information in black hole radiation. *Phys Rev Lett*. 1993;71(23):3743–3746.
- Kac V, Cheung P. *Quantum Calculus*. New York, NY: Springer; 2002.
- Abdeljawad T, Alzabut J, Baleanu D. A generalized q -fractional gronwall inequality and its applications to nonlinear delay q -fractional difference systems. *J Inequal Appl*. 2016;2016:240. 13 pp.
- Hajiseyedazizi SN, Samei ME, Alzabut J, Yu-ming C. On multi-step methods for singular fractional q -integro-differential equations. *Open Math*. 2021;19(1):1378–1405.
- Al-Salam WA. q -analogues of Cauchy's formulas. *Proc Amer Math Soc*. 1966;17(3):616–621.
- Al-Salam WA. Some fractional q -integrals and q -derivatives. *Proc Edinb Math Soc*. 1966;15(2):135–140.
- Al-Salam WA, Verma A. A fractional Leibniz q -formula. *Pacific J Math*. 1975;60(2):1–9.
- Agarwal RP. Certain fractional q -integrals and q -derivatives. *Math Proc Cambridge Philos Soc*. 1969;66(2):365–370.
- Ernst T. *The History of q -Calculus and a New Method*. Uppsala, Sweden: Uppsala University; 2000.
- Rajkovic P, Marinkovic S, Stankovic M. On q -analogues of Caputo derivative and Mittag-Leffler function. *Fract Calc Appl Anal*. 2007;10(4):359–373.
- Annaby MH, Mansour ZS. *q -Fractional Calculus and Equations*. Berlin, Heidelberg: Springer; 2012.
- Almeida R, Martins N. Existence results for fractional q -difference equations of order with three-point boundary conditions. *Commun Nonlinear Sci Numer Simul*. 2014;19(6):1675–1685.
- Rangaig NA, Cairolena T, Pada VCC. On the existence of the solution for q -Caputo fractional boundary value problem. *Appl Math Phys*. 2017;5(3):99–102.
- Hilger S. Analysis on measure chains — A unified approach to continuous and discrete calculus. *Results Math*. 1990;18(1–2):18–56.
- Bohner M, Peterson A. *Dynamic Equations on Time Scales: An Introduction with Applications*. New York, NY: Springer; 2001;
- Martin B, Gusein G, Allan P. Introduction to the time scales calculus. In: Bohner M, Peterson A, eds. *Advances in Dynamic Equations on Time Scales*. Birkhäuser Boston; 2003:1–15.
- Atici FM, Eloe PW. Fractional q -calculus on a time scale. *J Nonlinear Math Phys*. 2007;14(3):341–352.
- Predrag R, Sladjana M, Miomir S. Fractional integrals and derivatives in q -calculus. *Appl Anal Discrete Math*. 2007;1(1):311–323.
- Abdeljawad T, Baleanu D. Caputo q -fractional initial value problems and a q -analogue Mittag-Leffler function. *Commun Nonlinear Sci Numer Simul*. 2011;16(12):4682–4688.
- Abdeljawad T, Benli B, Baleanu D. A generalized q -Mittag-Leffler function by q -Caputo fractional linear equations. *Abstr Appl Anal*. 2012;2012:546062.
- Wu GC, Baleanu D. New applications of the variational iteration method - from differential equations to q -fractional difference equations. *Adv Differential Equations*. 2013;2013:21. 16 pp.
- Salahshour S, Ahmadian A, Chan CS. Successive approximation method for Caputo q -fractional IVPs. *Commun Nonlinear Sci Numer Simul*. 2015;24(1–3):153–158.
- Wang G, Sudsutad W, Zhang L, Tariboon J. Monotone iterative technique for a nonlinear fractional q -difference equation of Caputo type. *Adv Difference Equ*. 2016;2016:211. 11 pp.
- Fernandez A, Mohammed P. Hermite-hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. *Math Methods Appl Sci*. 2020;44(10):8414–8431.
- Mohammed PO, Goodrich CS, Hamasalh FK, Kashuri A, Hamed YS. On positivity and monotonicity analysis for discrete fractional operators with discrete Mittag-Leffler kernel. *Math Methods Appl Sci*. 2022;45(10):6391–6410.
- Mohammed PO, Hamasalh FK, Abdeljawad T. Difference monotonicity analysis on discrete fractional operators with discrete generalized Mittag-Leffler kernels. *Adv Differ Equ*. 2021;2021:213. 16 pp.
- Chen J, Xu D, Shafai B. On sufficient conditions for stability independent of delay. *IEEE Trans Automat Control*. 1995;40(9):1675–1680.
- Gu K, Kharitonov VL, Chen J. *Stability of Time-Delay Systems*. Birkhäuser Boston; 2003.
- Momani S, Hadid S. Lyapunov stability solutions of fractional integrodifferential equations. *Int J Math Math Sci*. 2004;2004(47):2503–2507.
- Zhang X. Some results of linear fractional order time-delay system. *Appl Math Comput*. 2008;197(1):407–411.
- Li Y, Chen Y, Podlubny I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability. *J Comput Appl Math*. 2010;35(5):1810–1821.
- Jackson FH. q -difference equations. *Amer J Math*. 1910;32(4):305–314.
- Jackson DO, Fukuda T, Dunn O, Majors E, Skill-based C. On q -definite integrals. *Quart J Pure Appl Math*. 1910;41:193–203.
- Askey R. The q -Gamma and q -Beta functions. *Appl Anal*. 1978;8(2):125–141.