



A novel numerical method for solving optimal control problems using fourth-degree hat functions



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ABSTRACT

This paper focuses on solving a class of nonlinear optimal control problems by constructing novel hat functions based on fourth-order polynomials, namely, fourth-degree hat functions (FDHF). The FDHF are used in this method to approximate the state equations and cost function. In fact, the FDHF enable us to turn the optimal control problem under consideration into a nonlinear optimal control problem with unknown coefficients that is easy to solve by any numerical method. The main advantages of the proposed method are its simplicity, ease of application, low computational expense, and avoidance of numerical integration. Several examples have been discussed to demonstrate the efficacy and applicability of the suggested method.

1. Introduction

We live in an epoch when science and technology are rapidly evolving, and new technology has increased the demand for automation, which can be seen in spacecraft, artificial intelligence machinery, automobile manufacturing, and other areas. As a result, the theory of systems and control faces greater challenges in such circumstances.^{1–9} The Optimal Control theory first became manifest in the 1950s, although its utility was associated with problems with a long history and was motivated by space problems, particularly the Space Race between the Soviet Union at the time (USSR) and the United States of America (USA) to excel in space exploration. Control theory began as a means of obtaining tools for the analysis and synthesis of control systems, so early research focused on the centrifugal system, simple regulating devices for industrial processes, electronic amplifiers, and manageable firefighting systems. With the development of the theory of optimal control, it became clear that this theory could be applied to a large variety of different technical and non-technical systems in a large number of nonlinear problems.^{10–15}

Recently, optimal control theory (also known as the optimization problem) has become a well-developed mathematical field that has numerous applications in science, engineering, medicine, economics, finance, and a variety of other disciplines.^{16–21} Control theory's main goal is to influence a system's behavior in order to achieve certain objectives, such as cost minimization or maximization, achieving a target, stabilizing the system, and so on. For modeling purposes, it is natural to restrict the state space of a given controlled system governed by ordinary or stochastic differential equations, taking into account the

presence of state constraints. In this case, the controller is only allowed to act using control inputs that comply with the constraints. Controls like these are referred to as admissible controls. The so-called controller determines which action to take based on the actual observations. The controller can be, for instance, a person, a mechanical device, or a computer. Take, for example, one of the most well-known controllers, the thermostat. A thermostat can be used to control the temperature in a room. To accomplish this, the thermostat measures the temperature in the room and, if it is too high, reduces the amount of hot water passing through the radiator, while if it is too low, the amount is increased instead. The temperature in the room is maintained at the desired level in this manner.

Optimal control, which has been extensively researched over the past few decades, is a fundamental part of control theory. Optimal control is a branch of control theory strictly related to the optimization of a system. The goal of these problems is to find a control strategy that meets a certain optimality criterion. This criterion is usually expressed as a cost, which is a function that depends on the control input selection. An ordinary differential equation's solution that allows for some parameter selection is referred to as a controlled process. Consequently, the solution's trajectory is determined. Every trajectory has a cost, and the objective of the optimal control problem is to reduce this cost over all possible control parameter choices.

There are three types of well-known optimal control problems: the Bolza, Lagrange, and Mayer problems.²² They appear to differ in the formulation of the function to be optimized. However, they are equivalent, and every problem can be transformed into either of the other two forms. These optimal control problems can be solved using a variety of

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numerical methods. These numerical methods are divided into direct and indirect methods.²³ With regard to indirect methods, there are two common indirect methods: the Pontryagin Maximum (Minimum) Principle method and the Dynamic Programming method created by Bellman.^{24–26} Indirect methods look for solutions to the necessary conditions. All problems that arise in the calculus of variations are subject to the Maximum (Minimum) Principle of Pontryagin. On the other hand, Dynamic Programming Problems (DPP) are popular problems in the management of natural resources, finance, and economics.^{27–29} It is important to note that the optimal strategies (actions or controls) and value functions are solutions to a specific optimal control problem that is solved by the DPP approach. Furthermore, solutions are usually calculated by numerical approximation methods, which vary in complexity and computational requirements.^{30–33} In direct methods, state and control variables are determined first, and then the optimal control problem is treated as a nonlinear optimization problem. The direct methods are searching for the minimum of the objective function or the Lagrange function. Numerous researchers have employed direct numerical techniques to solve optimal control problems, including the finite difference method,³⁴ the finite element method ?, the Monte-Carlo method,³⁵ the hybridizable discontinuous Galerkin method,^{36,37} the adaptive Radau collocation method,³⁸ the Legendre wavelet collocation method,³⁹ the B-spline collocation method,⁴⁰ the Chebyshev wavelets method,^{41,42} the block-pulse functions,⁴³ the shifted Legendre–Laguerre method,⁴⁴ and others.

Hat basis functions consist of a set of piecewise continuous functions with the shape of hats when plotted in two-dimensional planes. Indeed, it is well known that the generalized hat functions (GHFs) are constructed by using first-degree polynomials (segment lines), the modified hat functions (MHFs) are constructed by using second-degree polynomials, and the adjustment hat functions (AHFs) are constructed by using third-degree polynomials. Hat basis functions have recently drawn a lot of interest in solving a variety of optimal control problems (for more information, see Refs. 45–52). We believe that as the degree of polynomials used to construct hat functions increases, we will be able to solve a variety of nonlinear optimal control problems. The motivation of the present work is to extend the HFs to fourth-degree polynomials for the establishment of FDHFs and study their properties in order to construct a new numerical method to solve the following nonlinear optimal control problem (NOCPs).

$$\min_u J[y, u] = \psi(y(T), T) + \int_0^T L(y(t), u(t), t) dt, \quad (1.1)$$

where $L(y(t), u(t), t)$ is positive definite function and $\psi(y(T), T)$ is any given function. subject to the dynamic system and the following initial conditions

$$\frac{dy(t)}{dt} = f(y(t), u(t), t), \quad y(0) = y_0, \quad (1.2)$$

which is known as the trajectory or equation of motion, on a fixed interval $[0, T]$, where f is a continuously differentiable function with respect to each of its arguments and $y(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}^m$ are the state and control vectors, respectively. The control functions $u(t)$ are assumed to be piecewise continuous functions belonging to the class of admissible controls, U . Every choice for the control $u(\cdot) \in U \subset \mathbb{R}^m$ produces a state variable $y(t) \in \mathbb{R}^n$, which is the unique solution to Eq. (1.2). Moreover, $\psi(y(t), t)$ is the terminal cost, and $L(y(t), u(t), t)$ is the running cost (lagrangian term). The goal of problems (1.1)–(1.2) is to identify the optimal state $y^*(t)$ and optimal control $u^*(t)$ that minimize the cost function (performance index).

The conditions required to determine optimal control were established by Pontryagin and his associates. This work is regarded as one of the most significant mathematical discoveries of the 20th century. Pontryagin introduced the notion of adjoint functions to enable the differential equation to be connected to the objective function. In fact, adjoint functions perform a similar turn to Lagrange multipliers in the unconstrained optimization process. The OCP is defined as the task of

identifying the control $u^*(t) \in \mathbb{R}^m$ that satisfies Eq. (1.2) and gives the minimum value for Eq. (1.1). Now, we can first create the well-known Hamiltonian function as follows:

$$H(y, u, \lambda, t) = L(y, u, t) + \lambda^T f(y, u, t), \quad (1.3)$$

where $\lambda(t) \in \mathbb{R}^m$ is the co-state vector. The second step is applying Pontryagin's maximum principle [52] which is a set of necessary conditions that make the objective function in the case of as utmost maximization as possible (which could also be referred to as the minimum principle since we are considering a minimization problem). These conditions are as follows:

$$\frac{\partial \lambda^*}{\partial t} = -H_{y^*}(y^*, u^*, \lambda^*, t), \quad (1.4)$$

$$\frac{\partial y^*}{\partial t} = H_{\lambda^*}(y^*, u^*, \lambda^*, t), \quad (1.5)$$

$$H_{u^*}(y^*, u^*, \lambda^*, t) = 0, \quad (1.6)$$

$$H(y^*, u^*, \lambda^*, t) \leq H(y^*, u, \lambda^*, t), \quad (1.7)$$

$$\lambda(T) = \left. \frac{\partial \psi}{\partial y} \right|_{t=T}. \quad (1.8)$$

Applying the necessary conditions will result in a set of differential equations, which are then solved using the suggested numerical method to obtain the optimal values. The remaining sections of the essay are structured as follows. The definitions and properties of FDHFs are given in Section 2. In Section 3, we will find the operational integration matrix. In Section 4, a novel method based on FDHFs is suggested for finding a numerical solution for NOCPs. The numerical examples and solutions are presented in Section 5 of this paper. The last section of the essay concludes with a few final thoughts.

2. Fourth-degree hat functions and their properties

The FDHFs are created by first dividing $\Omega = [0, T]$ into n equidistant subintervals and then further subdividing each of these subintervals into four equidistant subintervals, each of which must have a length equal to h , where $h = \frac{T}{4n}$ and $n \in \mathbb{N}$. Now, for any $n \in \mathbb{N}$, we define the following functions:

$$\xi_0(t) = \begin{cases} \frac{(t-h)(t-2h)(t-3h)(t-4h)}{24h^4}, & 0 \leq t \leq 4h, \\ 0, & \text{otherwise,} \end{cases} \quad (2.1)$$

if $k = 1, 2, \dots, n-1$,

$$\xi_{4k}(t) = \begin{cases} \frac{(t-(4k-1)h)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{24h^4}, & 4(k-1)h \leq t \leq 4kh, \\ \frac{(t-(4k+1)h)(t-(4k+2)h)(t-(4k+3)h)(t-(4k+4)h)}{24h^4}, & 4kh \leq t \leq 4(k+1)h, \\ 0, & \text{otherwise,} \end{cases} \quad (2.2)$$

if $k = 1, 2, \dots, n$,

$$\xi_{4k-1}(t) = \begin{cases} \frac{-(t-4kh)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{6h^4}, & (4k-4)h \leq t \leq (4k)h, \\ 0, & \text{otherwise,} \end{cases} \quad (2.3)$$

$$\xi_{4k-2}(t) = \begin{cases} \frac{(t-4kh)(t-(4k-1)h)(t-(4k-3)h)(t-(4k-4)h)}{4h^4}, & (4k-4)h \leq t \leq 4kh, \\ 0, & \text{otherwise,} \end{cases} \quad (2.4)$$

$$\xi_{4k-3}(t) = \begin{cases} \frac{(t-(4k-2)h)(t-(4k-1)h)(t-4kh)(t-(4k-4)h)}{6h^4}, & (4k-4)h \leq t \leq 4kh, \\ 0, & \text{otherwise,} \end{cases} \quad (2.5)$$

and

$$\xi_{4n}(t) = \begin{cases} \frac{(t-(T-h))(t-(T-2h))(t-(T-3h))(t-(T-4h))}{24h^4}, & T-4h \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

The following are the basic properties of FDHF:

- Using the definition of FDHF, there is a very important relationship as follows:

$$\xi_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad \forall i, j = 0, 1, 2, \dots, 4n. \quad (2.7)$$

- The total sum of FDHF is one, which means:

$$\sum_{i=0}^{4n} \xi_i(t) = 1. \quad (2.8)$$

- The functions $\xi_0(t), \xi_1(t), \dots, \xi_{4n}(t)$ are linearly independent for all $t \in \Omega$.

- Any function $y(t) \in L^2(\Omega)$ can be approximated in terms of FDHF as:

$$y(t) \simeq y_{4n}(t) = \sum_{\kappa=0}^{4n} y_\kappa \xi_\kappa(t) = Y^T \Xi(t) = \Xi^T(t) Y, \quad (2.9)$$

where

$$\Xi(t) = [\xi_0(t), \xi_1(t), \xi_2(t), \dots, \xi_{4n}(t)]^T, \quad (2.10)$$

and

$$Y = [y_0, y_1, y_2, \dots, y_{4n}]^T. \quad (2.11)$$

The coefficients in Eq. (2.9) are given by

$$y_\kappa = y(\kappa h), \quad \kappa = 0, 1, \dots, 4n. \quad (2.12)$$

- Any function $G(t, s) \in L^2(\Omega \times \Omega)$ can be approximated in terms of FDHF as:

$$G(t, s) \simeq G_{4n}(t, s) = \sum_{r=0}^{4n} \sum_{\kappa=0}^{4n} G_{\kappa r} \xi_\kappa(t) \xi_r(s) = \Xi^T(t) D \Xi(s) = \Xi^T(s) D^T \Xi(t), \quad (2.13)$$

where,

$$G_{\kappa r}(t, s) = G(\kappa h, rh), \quad \forall \kappa, r = 0, 1, 2, \dots, 4n. \quad (2.14)$$

Now, the power and benefit of using the approximation of any function by FDHF appear clearly in the next theorem. This theorem demonstrates that the rate of convergence of any function $g(t) \in C^5(\Omega)$ approximation with FDHF is $O(h^5)$. Here we use

$$\|y\| = \sup_{t \in \Omega} |y(t)|. \quad (2.15)$$

Theorem 2.1. For any function $g(t) \in C^5(\Omega)$, let $g_{4n}(t) = \sum_{k=0}^{4n} g(t_k) \xi_k(t)$ be the FDHF expansion of $g(t)$, then

$$\|g(t) - g_{4n}(t)\| \leq \omega_1 h^5, \quad (2.16)$$

where ω_1 is a constant numbers.

Proof. Suppose that

$$e_i(t) = \begin{cases} g(t) - g_{4n}(t), & t \in V_i, \\ 0, & t \in \Omega \setminus V_i, \end{cases}$$

where $V_i = \{t | ih \leq t \leq (i+4)h, h = \frac{T}{4n}\}, i = 0, 1, 2, \dots, 4n-4$. Then, we get

$$e_i(t) = g(t) - g_{4n}(t) = g(t) - \sum_{j=0}^{4n} g(jh) \xi_j(t),$$

Therefore

$$e_i(t) = g(t) - [g(ih) \xi_i(t) + g((i+1)h) \xi_{i+1}(t) + g((i+2)h) \xi_{i+2}(t) + g((i+3)h) \xi_{i+3}(t) + g((i+4)h) \xi_{i+4}(t)].$$

We have the following results when we interpolate using a fourth-degree error⁵³:

$$e_i(t) = \frac{(t-ih)(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)(t-(i+4)h)}{120} \cdot \frac{d^5 g(\chi_i)}{dt^5},$$

where $\chi_i \in (ih, (i+4)h)$.

Now consider $\phi(t) = (t-ih)(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)(t-(i+4)h)$. Since, $\phi(t)$ is a continuous function and V_i is compacted, so we have

$$\sup_{t \in V_i} |\phi(t)| = \max_{t \in V_i} |\phi(t)| = 3.6314 h^5.$$

Therefore, we have

$$|e_i(t)| \leq \frac{1}{120} |\phi(t)| \left| \frac{d^5 g(\chi_i)}{dt^5} \right|.$$

As a result, we have

$$\|e(t)\| = \max_{i=0, 1, \dots, 4n-4} \sup_{t \in V_i} |e_i(t)| \leq \max_{i=0, 1, \dots, 4n-4} 0.03026 h^5 \left| \frac{d^5 g(\chi_i)}{dt^5} \right|.$$

Then there is an $\varepsilon \in \{0, 1, \dots, 4n-4\}$, where

$$\|e(t)\| \leq \max_{i=0, 1, \dots, 4n-4} 0.03026 h^5 \left| \frac{d^5 g(\chi_i)}{dt^5} \right| = 0.03026 h^5 \left| \frac{d^5 g(\chi_\varepsilon)}{dt^5} \right|.$$

Lastly, by using the relation (4.12), we obtain

$$\|e(t)\| \leq 0.03026 h^5 \left| \frac{d^5 g(\chi_\varepsilon)}{dt^5} \right| \leq 0.03026 h^5 \left\| \frac{d^5 g(t)}{dt^5} \right\| \leq \omega_1 h^5. \quad (2.17)$$

According to the relation (2.17), we get

$$\|e(t)\| = O(h^5).$$

Finally, the proof was completed.

3. Operational matrix of FDHF

The purpose of the work in this section is to construct an operational matrix P that satisfies

$$\int_0^t \Xi(s) ds \simeq P \Xi(t), \quad (3.1)$$

where $\Xi(t)$ is the vector defined in relation (2.10). Now, we try to write the $\int_0^t \xi_\kappa(s) ds$ as a linear combination of the functions $\xi_0(t), \xi_1(t), \dots, \xi_{4n}(t)$ as follows:

$$\int_0^t \xi_\kappa(s) ds \simeq \sum_{r=0}^{4n} P_{\kappa r} \xi_r(t), \quad \forall \kappa = 0, 1, 2, \dots, 4n. \quad (3.2)$$

The coefficients $P_{\kappa r}$ can be calculated as follows:

$$P_{\kappa r} = \int_0^{rh} \xi_\kappa(s) ds, \quad \forall r, \kappa = 0, 1, 2, \dots, 4n. \quad (3.3)$$

As a direct consequence of this, we can state the following theorem:

Theorem 3.1. If $\Xi(t)$ is a FDHF vector defined in relation (2.10), then the integration of $\Xi(t)$ can be approximated as follows:

$$\int_0^t \Xi(s) ds \simeq P \Xi(t), \quad (3.4)$$

where P is the $(4n+1) \times (4n+1)$ operational matrix of integration for the FDHFs and is defined as follows:

$$P = \frac{h}{720} \begin{pmatrix} 0 & \rho_1 & \rho_2 & \rho_2 & \rho_2 & \rho_2 & \cdots & \rho_2 & \rho_2 \\ \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_6 & \rho_6 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_4 & \rho_5 & \rho_6 & \rho_6 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_7 & \rho_4 & \rho_5 & \rho_6 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_4 & \rho_5 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \rho_4 & \cdots & \rho_6 & \rho_6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \cdots & \rho_4 & \rho_5 \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \cdots & \rho_7 & \rho_4 \end{pmatrix}, \quad (3.5)$$

where

$$\rho_1 = (251, 232, 243, 224), \quad \rho_2 = (224, 224, 224, 224), \quad \rho_3 = (0, 0, 0, 0)^T,$$

$$\rho_4 = \begin{pmatrix} 646 & 992 & 918 & 1024 \\ -264 & 192 & 648 & 384 \\ 106 & 32 & 378 & 1024 \\ -19 & -8 & -27 & 224 \end{pmatrix}, \quad \rho_5 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 475 & 456 & 467 & 448 \end{pmatrix},$$

$$\rho_6 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 448 & 448 & 448 & 448 \end{pmatrix}, \quad \rho_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3.2. For any function $g(t) \in C^5(\Omega)$, let $g_{4n}(t) = \sum_{k=0}^{4n} g(t_k) \xi_k(t)$ be the FDHFs expansion of $g(t)$, then

$$\left| \int_0^t g(s) ds - \int_0^t g_{4n}(s) ds \right| \leq \omega_1 O(h^5) T, \quad \forall t \in V_i. \quad (3.6)$$

where, $V_i = \{t | ih < t < (i+4)h, h = \frac{T}{4n}\}$, $i = 0, 4, 8, \dots, 4n-4$. Now, we have

$$\begin{aligned} \left| \int_0^t g(s) ds - \int_0^t g_{4n}(s) ds \right| &\leq \int_0^t |g(s) - g_{4n}(s)| ds \\ &= \sum_{\kappa=0}^{i-1} \int_{\kappa h}^{(k+1)h} |g(s) - g_{4n}(s)| ds \\ &\quad + \int_{ih}^t |g(s) - g_{4n}(s)| ds. \end{aligned}$$

Therefore, we obtain:

$$\left| \int_0^t g(s) ds - \int_0^t g_{4n}(s) ds \right| \leq \sum_{\kappa=0}^{i-1} \int_{\kappa h}^{(k+1)h} \omega_1 O(h^5) ds + \int_{ih}^t \omega_1 O(h^5) ds. \quad (3.7)$$

Therefore

$$\left| \int_0^t g(s) ds - \int_0^t g_{4n}(s) ds \right| \leq \omega_1 O(h^5) \left(\sum_{\kappa=0}^{i-1} \int_{\kappa h}^{(k+1)h} ds + \int_{ih}^t ds \right) = \omega_1 O(h^5) t.$$

Because of the fact that $t \leq (i+1)h \leq T$, then we have

$$\left| \int_0^t g(s) ds - \int_0^t g_{4n}(s) ds \right| \leq \omega_1 O(h^5) T.$$

4. Suggested method's description

This section introduces a novel numerical method for solving the optimal control problem (1.1)–(1.2). Our approach is based on the properties of the Fourth-Degree Hat Functions. In order to solve this optimal control problem, we use the FDHFs to approximate the state variable, $y(t)$, the control variable, $u(t)$, and the costate variable $\lambda(t)$ as follows:

$$\frac{dy(t)}{dt} \simeq D^T \Xi(t) = \Xi^T(t) D, \quad (4.1)$$

$$u(t) \simeq u_{4n}(t) = U^T \Xi(t) = \Xi^T(t) U. \quad (4.2)$$

Now, we expand the variable t in terms of the FDHFs.

$$t \simeq H^T \Xi(t) = \Xi^T(t) H. \quad (4.3)$$

By taking the integral of both sides of the Eq. (4.1), we get

$$y(t) \simeq y_0 + D^T \int_0^t \Xi(s) ds = (D^T P + Q^T) \Xi(t) = Y^T \Xi(t), \quad (4.4)$$

where Q is the vector of coefficients for y_0 and $Y = D^T P + Q^T$.

Using Eq. (4.4), we have

$$\psi(y(T)) \simeq \psi(Y^T) \Xi(t). \quad (4.5)$$

Now, we have

$$f(y(t), u(t), t) \simeq f(Y^T, U^T, H^T) \Xi(t), \quad (4.6)$$

and

$$L(y(t), u(t), t) \simeq L(Y^T, U^T, H^T) \Xi(t). \quad (4.7)$$

The performance index J is then approximated using Eqs. (4.7) and (4.5) as follows:

$$J[y, u] \simeq J[Y, U] = \psi(Y^T) \Xi(t) + L(Y^T, U^T, H^T) M, \quad (4.8)$$

where M is a known column vector denoted by:

$$M = \left[\int_0^1 \xi_0(t) dt, \int_0^1 \xi_1(t) dt, \dots, \int_0^1 \xi_{4n}(t) dt \right]^T. \quad (4.9)$$

Similarly, Eqs. (4.1) and (4.6) can be used to approximate the dynamical system (1.2) as follows:

$$D^T \Xi(t) \simeq f(Y^T, U^T, H^T) \Xi(t).$$

Therefore

$$D^T - f(Y^T, U^T, H^T) \simeq 0. \quad (4.10)$$

Now, we suppose that

$$J^*[Y, U, \lambda] = J[Y, U] + (D^T - f(Y^T, U^T, H^T)) \lambda, \quad (4.11)$$

where $\lambda = [\lambda_0, \lambda_1, \dots, \lambda_{4n}]^T$ is a Lagrange multiplier that is unknown. Lastly, the following are the necessary conditions for the extremum given by the system of nonlinear algebraic equations shown below:

$$\frac{\partial J^*}{\partial Y} = 0, \quad \frac{\partial J^*}{\partial U} = 0, \quad \frac{\partial J^*}{\partial \lambda} = 0. \quad (4.12)$$

The aforementioned system can be solved for Y , U , and λ using the Maple software packages. We can find the approximate solutions of $u(t)$ and $y(t)$ from (4.2) and (4.4), respectively, by determining Y and U .

5. Numerical examples

The effectiveness and dependability of the suggested method are demonstrated in this section by a few numerical examples for which exact solutions are available. All computations in this paper have been performed using Maple 2020. The accuracy of the described method in Section 4 is evaluated by computing the absolute error values using the formula below for different values of n .

$$E(t) = |y(t) - y_{4n}(t)|, \quad t \in \Omega,$$

where $y(t)$ and $y_{4n}(t)$ stand for the exact and approximate solutions to Eq. (1.2), respectively.

Example 5.1 (Ref. 55). Take into account the following optimal control problem:

$$\min_u J[y, u] = \frac{1}{2} \int_0^1 (y^2(t) + u^2(t)) dt,$$

subject to

$$\frac{dy(t)}{dt} = y(t) + tu(t), \quad y(0) = 1.$$

Also, the exact state function is the following:

$$y(t) = \frac{\sqrt{\pi}(\text{erf}(1) - \text{erf}(t))(t+1)e^{\frac{t^2}{2}} - 2e^{-\frac{t^2}{2}}}{\text{erf}(1)\sqrt{\pi} - 2}.$$

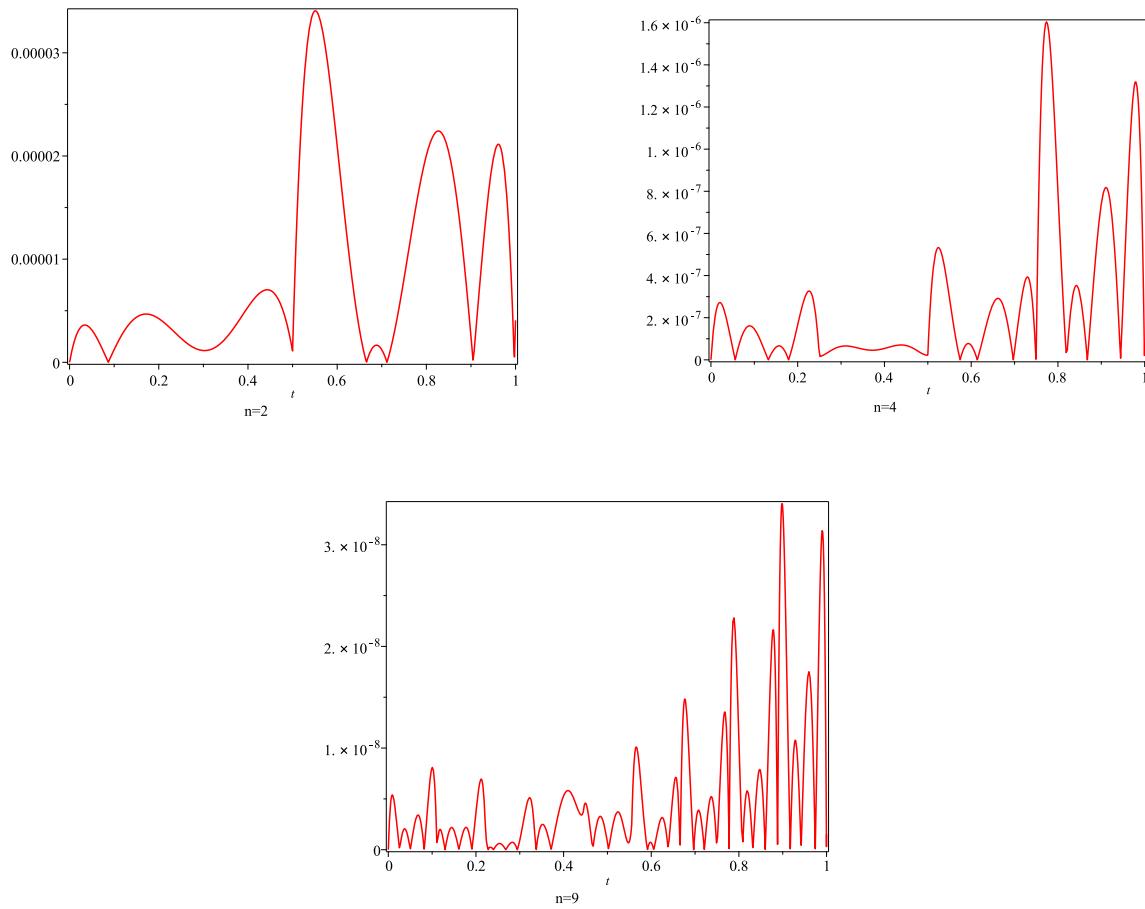


Fig. 1. The error for the state function $|y^*(t) - y(t)|$ for $m = 4$ and different values of n in Example 5.1.

Table 1
The absolute errors for Example 5.1.

Time	GHFs ⁴⁶ $m = 1, n = 24$	MHFs ⁴⁸ $m = 2, n = 12$	AHFs ⁵⁴ $m = 3, n = 8$	Present method $m = 4, n = 6$
0.1	6.296×10^{-5}	1.126×10^{-5}	1.777×10^{-6}	1.200×10^{-8}
0.2	4.880×10^{-5}	6.590×10^{-7}	8.090×10^{-7}	2.000×10^{-9}
0.3	8.182×10^{-5}	2.181×10^{-6}	8.260×10^{-7}	1.300×10^{-8}
0.4	4.368×10^{-5}	5.055×10^{-6}	1.471×10^{-6}	3.000×10^{-9}
0.5	1.139×10^{-4}	8.640×10^{-7}	1.700×10^{-7}	5.000×10^{-9}
0.6	1.618×10^{-4}	2.755×10^{-5}	1.526×10^{-6}	2.500×10^{-8}
0.7	2.272×10^{-4}	1.983×10^{-5}	9.590×10^{-7}	5.900×10^{-8}
0.8	4.621×10^{-4}	1.630×10^{-5}	9.520×10^{-7}	2.670×10^{-8}
0.9	9.839×10^{-4}	4.663×10^{-5}	3.186×10^{-6}	5.840×10^{-8}
1.0	4.748×10^{-4}	2.143×10^{-6}	4.200×10^{-7}	6.000×10^{-9}

The absolute error of the state function y obtained by the suggested method for $n = 6$ is compared with the absolute error of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$ in Table 1.

Example 5.2 (Ref. 56). Take into account the following optimal control problem:

$$\min_u J[y, u] = \frac{1}{2} \int_0^1 (y^4(t) + u^2(t)) dt,$$

subject to

$$\frac{dy(t)}{dt} = y(t) + u^2(t), \quad y(0) = 1.$$

Also, the exact state function is the following:

$$y(t) = e^t.$$

Table 2
The absolute errors for Example 5.2.

Time	GHFs ⁴⁶ $m = 1, n = 36$	MHFs ⁴⁸ $m = 2, n = 18$	AHFs ⁵⁴ $m = 3, n = 12$	Present method $m = 4, n = 9$
0.1	1.092×10^{-4}	1.478×10^{-6}	2.408×10^{-8}	6.000×10^{-10}
0.2	9.153×10^{-5}	8.020×10^{-7}	8.180×10^{-9}	6.300×10^{-10}
0.3	1.089×10^{-4}	9.674×10^{-7}	9.000×10^{-9}	6.800×10^{-10}
0.4	1.767×10^{-4}	2.104×10^{-6}	3.340×10^{-8}	1.540×10^{-9}
0.5	5.301×10^{-5}	1.100×10^{-8}	6.000×10^{-9}	0.000
0.6	2.387×10^{-4}	2.425×10^{-6}	4.670×10^{-8}	6.200×10^{-10}
0.7	2.156×10^{-4}	1.309×10^{-6}	6.050×10^{-9}	9.000×10^{-11}
0.8	2.511×10^{-4}	1.609×10^{-6}	5.300×10^{-9}	6.290×10^{-9}
0.9	3.705×10^{-4}	3.486×10^{-6}	6.460×10^{-8}	1.320×10^{-8}
1.0	1.748×10^{-4}	3.500×10^{-8}	2.100×10^{-8}	0.000

The absolute error of the state function y obtained by the suggested method for $n = 9$ is compared with the absolute error of GHFs for $n = 36$, MHFs for $n = 18$, and AHFs for $n = 12$ in Table 2.

Example 5.3 (Ref. 56). Take into account the following optimal control problem:

$$\min_u J[y, u] = \int_0^1 (y^6(t) + u^2(t)) dt,$$

subject to

$$\frac{dy(t)}{dt} = y(t) + 2u^2(t), \quad y(0) = 1.$$

Also, the exact state function is the following:

$$y(t) = e^t.$$

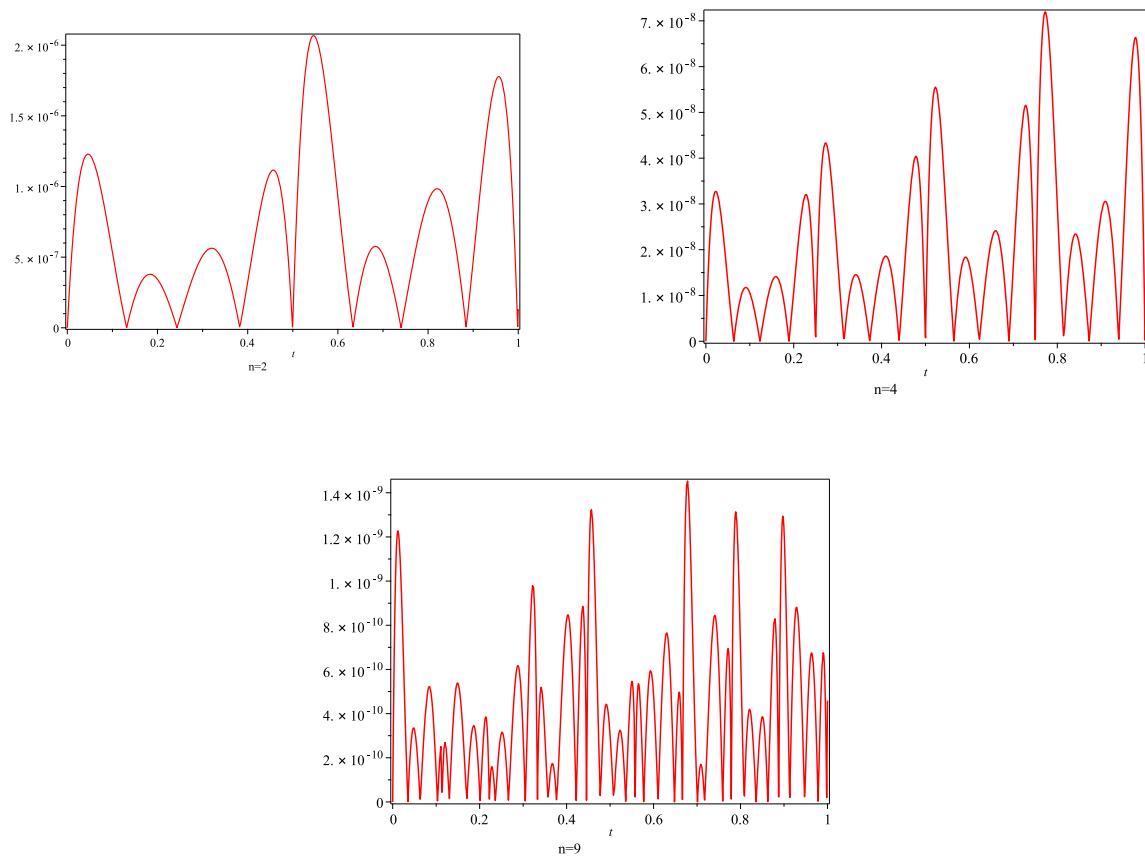


Fig. 2. The error for the state function $|y^*(t) - y(t)|$ for $m = 4$ and different values of n in Example 5.2.

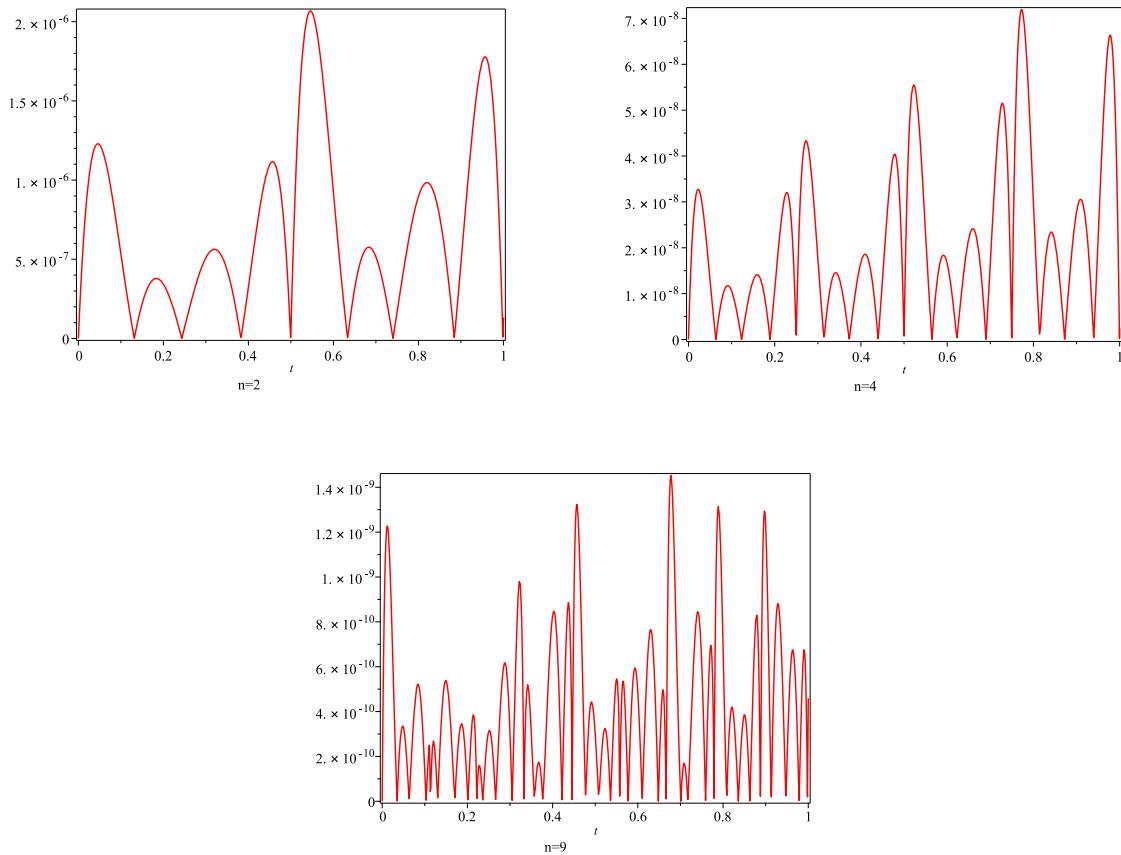


Fig. 3. The error for the state function $|y^*(t) - y(t)|$ for $m = 4$ and different values of n in Example 5.3.

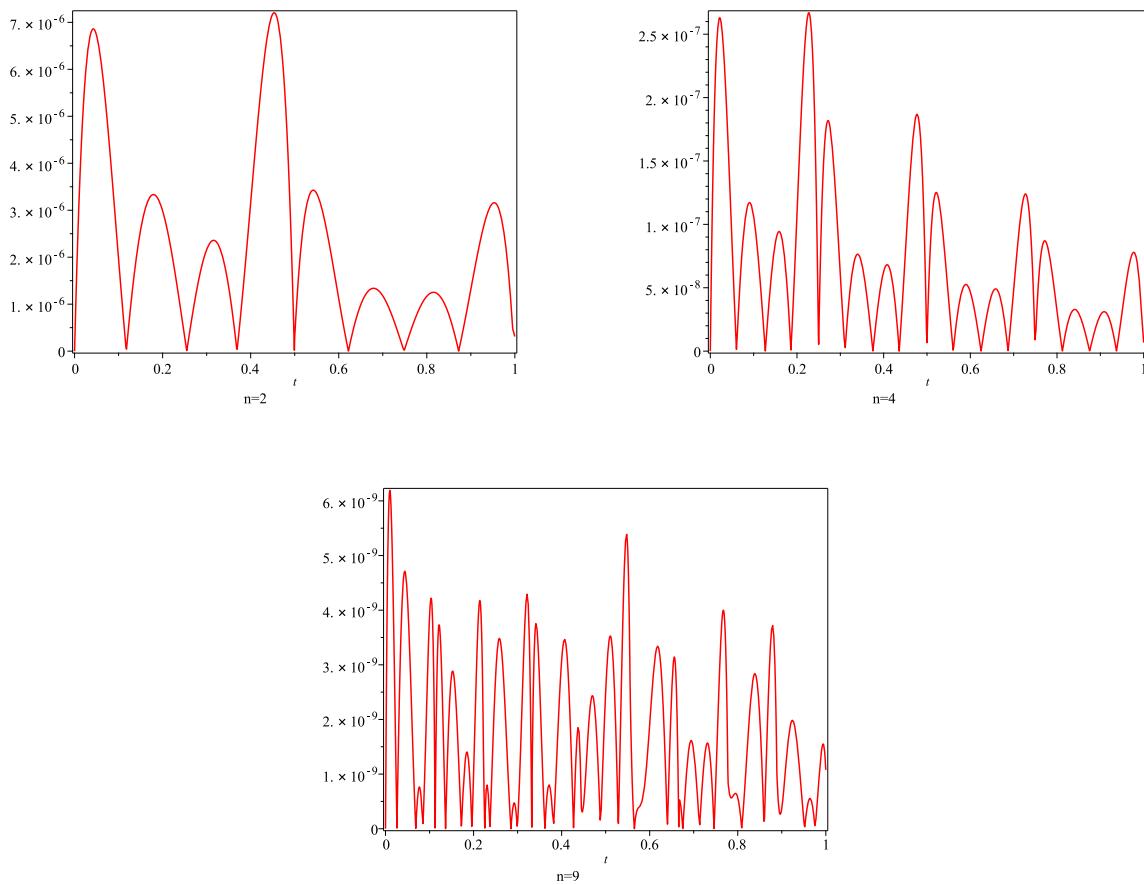
Fig. 4. The error for the state function $|y^*(t) - y(t)|$ for $m = 4$ and different values of n in Example 5.4.

Table 3

The absolute errors for Example 5.3.

Time	GHFs ⁴⁶ $m = 1, n = 24$	MHFs ⁴⁸ $m = 2, n = 12$	AHFs ⁵⁴ $m = 3, n = 8$	Present method $m = 4, n = 6$
0.1	2.469×10^{-4}	5.312×10^{-6}	1.096×10^{-7}	1.180×10^{-9}
0.2	2.036×10^{-4}	3.010×10^{-6}	4.385×10^{-8}	2.470×10^{-9}
0.3	2.477×10^{-4}	2.916×10^{-6}	4.251×10^{-8}	1.300×10^{-9}
0.4	3.964×10^{-4}	6.621×10^{-6}	1.818×10^{-7}	1.360×10^{-9}
0.5	1.193×10^{-4}	5.500×10^{-8}	3.100×10^{-8}	0.000
0.6	5.390×10^{-4}	8.818×10^{-6}	2.163×10^{-7}	2.350×10^{-9}
0.7	4.814×10^{-4}	5.030×10^{-6}	3.480×10^{-8}	2.890×10^{-9}
0.8	5.694×10^{-4}	4.734×10^{-6}	2.624×10^{-8}	3.400×10^{-9}
0.9	8.315×10^{-4}	1.083×10^{-5}	3.466×10^{-7}	2.420×10^{-9}
1.0	3.934×10^{-4}	1.810×10^{-7}	1.030×10^{-7}	0.000

The absolute error of the state function y obtained by the suggested method for $n = 6$ is compared with the absolute error of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$ in Table 3.

Example 5.4 (Ref. 55). Take into account the following optimal control problem:

$$\min_u J[y, u] = \int_0^1 (y^2(t) + u^2(t)) dt,$$

subject to

$$\frac{dy(t)}{dt} = -y(t) + u(t), \quad y(0) = 2.$$

Also, the exact state function is the following:

$$y(t) = \frac{2\sqrt{2}e^{\sqrt{2}t} + 2\sqrt{2}(e^{\sqrt{2}})^2 e^{-\sqrt{2}t} - 2e^{\sqrt{2}t} + 2(e^{\sqrt{2}})^2 e^{-\sqrt{2}t}}{\sqrt{2}(e^{\sqrt{2}})^2 + (e^{\sqrt{2}})^2 + \sqrt{2} - 1}.$$

Table 4

The absolute errors for Example 5.4.

Time	GHFs ⁴⁶ $m = 1, n = 24$	MHFs ⁴⁸ $m = 2, n = 12$	AHFs ⁵⁴ $m = 3, n = 8$	Present method $m = 4, n = 6$
0.1	6.494×10^{-4}	2.113×10^{-5}	7.190×10^{-7}	1.270×10^{-8}
0.2	2.976×10^{-4}	8.903×10^{-6}	3.230×10^{-7}	1.000×10^{-8}
0.3	1.948×10^{-4}	8.851×10^{-6}	3.230×10^{-7}	1.350×10^{-8}
0.4	2.875×10^{-4}	1.487×10^{-5}	3.330×10^{-7}	8.250×10^{-9}
0.5	2.164×10^{-4}	1.992×10^{-7}	1.138×10^{-7}	0.000
0.6	1.438×10^{-4}	1.011×10^{-5}	2.677×10^{-7}	6.550×10^{-9}
0.7	1.138×10^{-5}	4.255×10^{-6}	2.555×10^{-7}	7.580×10^{-9}
0.8	4.108×10^{-5}	3.941×10^{-6}	2.452×10^{-7}	5.680×10^{-9}
0.9	3.036×10^{-5}	6.536×10^{-6}	1.193×10^{-7}	4.250×10^{-9}
1.0	2.262×10^{-4}	2.080×10^{-7}	1.195×10^{-7}	8.600×10^{-10}

The absolute error of the state function y obtained by the suggested method for $n = 6$ is compared with the absolute error of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$ in Table 4.

Example 5.5 (Ref. 57). Take into account the following nonlinear optimal control problem:

$$\min_u J[y, u] = \int_0^1 (y^2(t) + u^2(t)) dt,$$

subject to

$$\frac{dy(t)}{dt} = -y^2(t) - tu^2(t), \quad y(0) = 2.$$

Also, the exact state function is the following:

$$y(t) = \frac{1}{(t + 1/2)}.$$

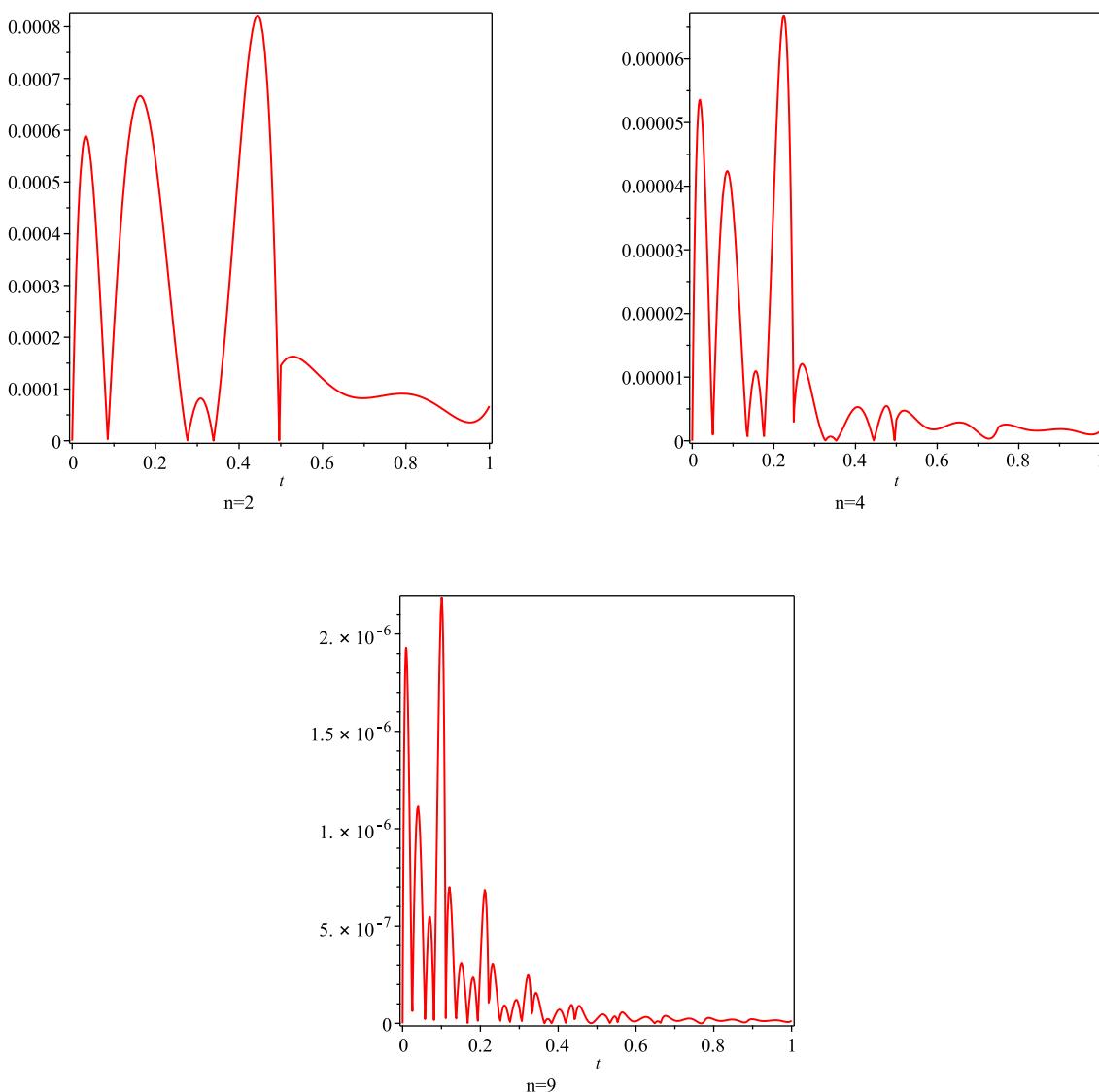


Fig. 5. The error for the state function $|y^*(t) - y(t)|$ for $m = 4$ and different values of n in Example 5.5.

Table 5

The absolute errors for Example 5.5.

Time	GHFs ⁴⁶ $m = 1, n = 24$	MHFs ⁴⁸ $m = 2, n = 12$	AHFs ⁵⁴ $m = 3, n = 8$	Present method $m = 4, n = 6$
0.1	1.108×10^{-3}	1.761×10^{-4}	3.736×10^{-5}	2.430×10^{-6}
0.2	1.172×10^{-4}	4.648×10^{-5}	2.023×10^{-5}	9.53×10^{-7}
0.3	4.930×10^{-4}	3.820×10^{-5}	1.734×10^{-5}	1.006×10^{-6}
0.4	3.772×10^{-4}	4.451×10^{-5}	9.025×10^{-6}	3.70×10^{-9}
0.5	8.698×10^{-4}	4.667×10^{-6}	1.0432×10^{-6}	3.160×10^{-7}
0.6	4.734×10^{-4}	2.068×10^{-5}	6.981×10^{-6}	3.362×10^{-7}
0.7	5.403×10^{-4}	8.876×10^{-6}	8.059×10^{-6}	2.538×10^{-7}
0.8	5.092×10^{-4}	6.947×10^{-6}	2.794×10^{-6}	1.330×10^{-7}
0.9	4.176×10^{-4}	5.435×10^{-6}	5.106×10^{-6}	1.408×10^{-7}
1.0	5.151×10^{-4}	2.284×10^{-6}	1.015×10^{-6}	1.444×10^{-7}

The absolute error of the state function y obtained by the suggested method for $n = 6$ is compared with the absolute error of GHFs for $n = 24$, MHFs for $n = 12$, and AHFs for $n = 8$ in Table 5.

6. Conclusion

In this paper, we constructed the base functions, FDHF, using fourth-order polynomials. Meanwhile, we implement FDHF to provide an effective

computational method for solving a general class of optimal control problems. The effort of this method is summarized by the operations matrices used to convert the considered problem into nonlinear algebraic equations. In fact, the advantage of this method is that it is easy to figure out the unknown coefficients of the approximation of the function without having to integrate anything. Consequently, the proposed method has a low computational expense. Several examples have been considered to make a comparison between the suggested method and three other methods: GHFs, MHFs, and AHFs. The comparison proved the superiority of our method over its counterparts. However, we have demonstrated in Figs. 1–5 that the absolute errors for the state function of the suggested method for different values of n . The results demonstrate that as n is increased, the absolute error values decrease, and thus the approximation results converge to the exact values for the state function. The tables and figures' reported numerical results demonstrate the method's high level of accuracy.

Data availability

No data was used for the research described in the article.

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References

- Heydari MH, Razzaghi M, Avazzadeh Z. Orthonormal piecewise bernoulli functions: Application for optimal control problems generated using fractional integro-differential equations. *J Vib Control.* 2022;29(5–6):1164–1175.
- Dehestani H, Ordokhani Y. A modified numerical algorithm based on fractional euler functions for solving time-fractional partial differential equations. *Int J Comput Math.* 2021;98(10):2078–2096.
- Dehestani H, Ordokhani Y, Razzaghi M. Modified wavelet method for solving multitype variable-order fractional partial differential equations generated from the modeling of phenomena. *Math Sci.* 2021;16(4):343–359.
- Zhao B, Liu R, Shi D, Li S, Cai Q, Shen W. Optimal control strategy of path tracking and braking energy recovery for new energy vehicles. *Processes.* 2022;10(7):1292.
- Zhi Y, Weiqing W, Jing C, Razmjoo N. Interval linear quadratic regulator and its application for speed control of DC motor in the presence of uncertainties. *ISA Trans.* 2022;125:252–259.
- Fang F, Godlewski B, Cho RY, Savitz SI, Selvaraj S, Zhang Y. Personalizing repetitive transcranial magnetic stimulation for precision depression treatment based on functional brain network controllability and optimal control analysis. *Neuroimage.* 2022;260:119465.
- Köhler J, Schwenkel L, Koch A, Berberich J, Pauli P, Allgöwer F. Robust and optimal predictive control of the COVID-19 outbreak. *Annu Rev Control.* 2021;51:525–539.
- Rawlings JB, Patel NR, Risbeck MJ, Maravelias CT, Wenzel M.J. Turney RD. Economic MPC and real-time decision making with application to large-scale HVAC energy systems. *Comput Chem Eng.* 2018;114:89–98.
- Mahdy A, Lofti K, El-Bary A. Use of optimal control in studying the dynamical behaviors of fractional financial awareness models. *Soft Comput.* 2022;26(7):3401–3409.
- Sabermahani S, Ordokhani Y, Hassani H. General lagrange scaling functions: application in general model of variable order fractional partial differential equations. *Comput Appl.* 2021;40:269.
- Marzban HR. Optimal control of linear multi-delay systems based on a multi-interval decomposition scheme. *Optim Control Appl Methods.* 2015;37(1):190–211.
- Marzban H, Pirmoradian H. A direct approach for the solution of nonlinear optimal control problems with multiple delays subject to mixed state-control constraints. *Appl Math Model.* 2018;53:189–213.
- Marzban HR, Ashani MR. A class of nonlinear optimal control problems governed by fredholm integro-differential equations with delay. *Internat J Control.* 2019;93(9):2199–2211.
- Marzban HR. An accurate method for fractional optimal control problems governed by nonlinear multi-delay systems. *J Vib Control.* 2022;29(3–4):820–843.
- Mahdi NK, Khudair AR. Stability of nonlinear q-fractional dynamical systems on time scale. *Partial Differ Equ Appl Math.* 2023;7:100496.
- Lazima ZA, Khalaf SL. Optimal control design of the in-vivo HIV fractional model. *Iraqi J Sci.* 2022;63(9):3877–3888.
- Sanaa LK, Mohammed SK. Design of optimal control for the in-host tuberculosis fractional model. *Iraqi J Sci.* 2023;64(12). (in press).
- Ahmad I, Ahmad H, Inc M, Rezaizadeh H, Akbar MA, Khater MM, Akinyemi L, Jhangeer A. *Solution of Fractional-Order Korteweg-de Vries and Burgers' Equations Utilizing Local Meshless Method.* JOES; 2021 (in press).
- Abazari R, Rezazadeh H, Akinyemi L, Inc M. Numerical simulation of a binary alloy of 2d cahn-hilliard model for phase separation. *J Comput Appl Math.* 2022;41(8):1–21.
- Khalaf SL, Kadhim MS, Khudair AR. Studying of COVID-19 fractional model: Stability analysis. *Partial Differ Equations Appl Math.* 2023;7:100470.
- Khalaf SL, Flayyih HS. Analysis, predicting, and controlling the COVID-19 pandemic in iraq through. *SIR Model.* 2023;RICO10:100214.
- Yaghobipour S, Yarahmadi M. Optimal control design for a class of quantum stochastic systems with financial applications. *Physica A.* 2018;512:507–522.
- Robuschi N, Zeile C, Sager S, Braghin F. Multiphase mixed-integer nonlinear optimal control of hybrid electric vehicles. *Automatica J IFAC.* 2021;123:109325.
- Zhao J. Data-driven adaptive dynamic programming for optimal control of continuous-time multicontroller systems with unknown dynamics. *IEEE Access.* 2022;10:41503–41511.
- Fleming WH, Rishel RW. *Deterministic and Stochastic Optimal Control*; Vol. 1. Springer Science & Business Media; 2012.
- Saerens B, Diehl M, EVd Bulck. Optimal control using pontryagin's maximum principle and dynamic programming. In: del RL, Allgöwer F, Glielmo L, Guardiola C, Kolmanovsky I, eds. *Automotive Model Predictive Control*. London: Springer; 2010. In: Lecture Notes in Control and Information Sciences; 402.
- Xie F, Li H, Xu Z. An approximate dynamic programming approach to project scheduling with uncertain resource availabilities. *Appl Math Model.* 2021;97:226–243.
- Escobar-Anel M. A dynamic programming approach to path-dependent constrained portfolios. *Ann Oper Res.* 2022;315:1–17.
- Lin Z, Song C, Zhao J, Yin H. Improved approximate dynamic programming for real-time economic dispatch of integrated microgrids. *Energy.* 2022;255:124513.
- Khudair AR. On solving non-homogeneous fractional differential equations of euler type. *J Comput Appl Math.* 2013;32(3):577–584.
- Kompas T, Chu L. Comparing approximation techniques to continuous-time stochastic dynamic programming problems: Applications to natural resource modelling. *Environ Model Softw.* 2012;38:1–12.
- Vichitkunakorn P, Vo TN, Razzaghi M. A numerical method for fractional pantograph differential equations based on taylor wavelets. *Trans Inst Meas Control.* 2019;42(7):1334–1344.
- Khalaf SL, Khudair AR. Particular solution of linear sequential fractional differential equation with constant coefficients by inverse fractional differential operators. *Differ Equ Dyn Syst.* 2017;25(3):373–383.
- Haq IU, Ali N, Nisar KS. An optimal control strategy and grünwald-letnikov finite-difference numerical scheme for the fractional-order COVID-19 model. *Math Models Comput Simul.* 2022;2(2):108–116.
- Yoo H, Kim B, Kim JW, Lee JH. Reinforcement learning based optimal control of batch processes using Monte-Carlo deep deterministic policy gradient with phase segmentation. *Comput Chem Eng.* 2021;144:107133.
- Nakazawa T. Shape optimization problem for transient non-newtonian fluid in hybridized discontinuous galerkin method. *J Fluid Sci Technol.* 2021;16(3):JFST0019.
- Gong W, Hu W, Mateos M, Singler JR, Zhang Y. Analysis of a hybridizable discontinuous galerkin scheme for the tangential control of the stokes system. *Esaim Math Model Numer Anal.* 2020;54(6):2229–2264.
- Pager ER, Rao AV. Method for solving bang-bang and singular optimal control problems using adaptive radau collocation. *Comput Optim Appl.* 2022;81(3):857–887.
- Kumar N, Mehra M. Legendre wavelet collocation method for fractional optimal control problems with fractional bolza cost. *Numer Methods Partial Differ Equ.* 2021;37(2):1693–1724.
- Edrisi-Tabrizi Y, Lakestani M, Razzaghi M. Study of b-spline collocation method for solving fractional optimal control problems. *Trans Inst Meas Control.* 2021;43(11):2425–2437.
- Ghanbari G, Razzaghi M. Numerical solutions for distributed-order fractional optimal control problems by using generalized fractional-order chebyshev wavelets. *Nonlinear Dynam.* 2022;108(1):265–277.
- Xu X, Xiong L, Zhou F. Solving fractional optimal control problems with inequality constraints by a new kind of chebyshev wavelets method. *J Comput Sci.* 2021;54:101412.
- Keighobadi J, Faraji J, Janabi-Sharifi F, Hamed MA. Design and experimental evaluation of block-pulse functions and legendre polynomials observer for attitude-heading reference system. *ISA Trans.* 2021;116:232–244.
- Farhood AK, Mohammed OH, Taha BA. Solving fractional time-delay diffusion equation with variable-order derivative based on shifted legendre-laguerre operational matrices.. *Arab J Math.* 2023. (in press).
- Mohammed OH. A direct method for solving fractional order variational problems by hat basis functions. *Ain Shams Eng J.* 2018;9(4):1513–1518.
- Heydari M, Hooshmandasl M, Shakiba A, Cattani C. An efficient computational method based on the hat functions for solving fractional optimal control problems. *Tbilisi Math J.* 2016;9(1):143–157.
- Mirzaee F, Hadadiyan E. Numerical solution of optimal control problem of the non-linear volterra integral equations via generalized hat functions. *IMA J Math Control Inform.* 2016;34(3):889–904.
- Nemati S, Lima PM, Torres DF. A numerical approach for solving fractional optimal control problems using modified hat functions. *Commun Nonlinear Sci Numer Simul.* 2019;78:104849.
- Nemati S, Lima PM, Torres DF. A numerical approach for solving fractional optimal control problems using modified hat functions. *Commun Nonlinear Sci Numer Simul.* 2019;78:104849.
- Nemati S, Ordokhani Y. Numerical solution of delay fractional optimal control problems using modification of hat functions. *J Math Res.* 2019;4(2):241–258.
- Mohammed JK, Khudair AR. Solving volterra integral equations via fourth-degree hat functions. *Partial Differ Equ Appl Math.* 2023;7:100494.
- Jalil AFA, Khudair AR. Toward solving fractional differential equations via solving ordinary differential equations. *J Comput Appl Math.* 2022;41:1–12.
- Nemati S, Lima P, Ordokhani Y. Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using legendre polynomials. *J Comput Appl Math.* 2013;242:53–69.
- Mirzaee F, Hamzeh A. A computational method for solving nonlinear stochastic volterra integral equations. *J Comput Appl Math.* 2016;306:166–178.
- Gamkrelidze RV. *Principles of Optimal Control Theory*. US: Springer; 1978.
- Hull DG. *Optimal Control Theory for Applications*. New York: Springer; 2003.
- Ma Z, Zou S. *Optimal Control Theory*. Singapore: Springer; 2021.