



## Solving Volterra integral equations via fourth-degree hat functions

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### ABSTRACT

The goal of this paper is to develop a novel operation matrix approach to solving Volterra integral equations by constructing fourth-degree hat functions and investigating their properties. This approach requires turning the problem under dissection into a set of algebraic equations and then solving it by any numerical method. In addition, we demonstrate that the convergence of this approach is of the order of  $O(h^5)$ , and numerical results show that it is practical and useful for dealing with such problems.

### 1. Introduction

A functional equation is a mathematical equation where the unknown is a function. In other words, a functional equation is always an equation that cannot be turned directly into an algebraic equation or a differential equation. Many years ago, this type of equation was an exciting topic in the minds of mathematicians. If the unknown function lies inside the integration symbol in a given equation, then this equation is called an integral equation, which is a common kind of functional equation. Such equations can be found in a wide range of fields in physics and applied mathematics.<sup>1–3</sup> However, the history of integral equations goes back to Vito Volterra (1860–1940), who was the first to introduce and realize the theory of integral equations in a significant and systematic manner. Indeed, the majority of references point to the population growth model of Volterra as the first real-world model to be described using integral equations. Also, the famous symmetric time problem using an integral equation was first formulated by Niels H. Abel (1802–1829). In fact, many great mathematicians made an effective contribution to the development of integral equations, most notably Vito Volterra, David Hilbert, Ivar Fredholm, and Erhard Schmidt.<sup>4</sup>

Indeed, many real-world problems are always described by using integral equations rather than differential equations because differential equations need boundary conditions or initial conditions. Therefore, integral equations provide a good mathematical tool for modelling many problems in a variety of fields, including mathematical biology and epidemiology,<sup>5,6</sup> diffusion problems,<sup>7</sup> heat transfer problems,<sup>8,9</sup> chemical reactor theory,<sup>10</sup> Dirichlet problems,<sup>11</sup> radiation,<sup>12</sup> mathematical economics,<sup>13</sup> ionospheric problems,<sup>14</sup> population genetics,<sup>15–18</sup> and fluid mechanics.<sup>19–21</sup>

Volterra integral equations (VIEs), Volterra–Fredholm equations, and integrodifferential equations are vital categories of integral equations. In general, there is no analytical method for finding the exact

solution to these equations. Meanwhile, many fractional differential equations (FDEs) can be converted to VIEs<sup>22</sup>; therefore, we believe many methods related to FDEs can be modified to be suitable for solving VIEs.<sup>23–25</sup> As a result, a number of efficient and accurate computational methods to solve these equations have been offered and developed, including the modified iterated projection method,<sup>26</sup> the Tau method,<sup>27</sup> rationalized Haar functions,<sup>28</sup> Lagrange polynomials,<sup>29</sup> Legendre polynomials,<sup>30</sup> Legendre wavelet approximation,<sup>31</sup> Chebyshev polynomials,<sup>32</sup> Laguerre polynomials,<sup>33</sup> block pulse functions,<sup>33,34</sup> Bell polynomials,<sup>35</sup> Chebyshev wavelets polynomials,<sup>36</sup> Euler matrix method,<sup>37</sup> collocation methods,<sup>15,38–40</sup> operational matrices,<sup>41,42</sup> collocation, and Galerkin methods,<sup>43</sup> Fibonacci collocation method,<sup>44</sup> and more. Many scholars, like Babolian and Mordad,<sup>45</sup> Li et al.,<sup>46</sup> Mirzaee et al.,<sup>35,44,47</sup> Khajehnasiri et al.,<sup>48</sup> Li,<sup>49</sup> have recently successfully employed the generalized hat basis functions (GHFs) to find numerical solutions to a variety of problems, especially integral equations. Modified hat functions (MHFs) have also been used to numerically solve some classes of integral equations by Mirzaee and Samadyar,<sup>50</sup> Mirzaee and Hadadian,<sup>51–58</sup> Nemati,<sup>59</sup> Nemati and Lima.<sup>60</sup> In addition, adjustment of hat functions (AHFs) has been employed by many authors to solve a class of third-kind Volterra integral equations<sup>61</sup> and nonlinear stochastic Volterra integral equations.<sup>62</sup>

On the other hand, it is well known that the GHFs are constructed by using first-degree polynomials (segment lines), the MHFs are constructed by using second-degree polynomials, and the AHFs are constructed by using third-degree polynomials. In fact, the rate of convergence of GHFs, MHFs, and AHFs are  $O(h^2)$ ,  $O(h^3)$ , and  $O(h^4)$ , respectively. We believe that as the degree of polynomials used to construct hat functions increases, we will be able to solve a variety of nonlinear equations more effectively. The motivation of this article is to construct

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novel fourth-degree hat functions (FDHFs) by using fourth-degree polynomials and studying their properties in order to build a new operations matrix for solving VIEs.

Now, consider following the general form of VIEs:

$$y(t) = g(t) + \lambda \int_a^t L(t, s) \sigma(y(s)) ds, \quad t \in \Omega = [0, T], \quad (1.1)$$

where  $\sigma(y(t))$ ,  $L(t, s)$ , and  $g(t)$  are any given functions, whereas our goal will be to find the unknown function  $y(t)$  that satisfies Eq. (1.1). Based on the linearity or nonlinearity of the function  $\sigma(\cdot)$ , Eq. (1.1) may be divided into a linear or nonlinear integral equation. This equation is known as the first (second) type of VIE, if  $\gamma \equiv 0$  ( $\gamma \equiv 1$ ).<sup>2</sup>

In this paper, we attempt to find the operations matrix corresponding to FDHFs to get an approximate solution for the nonlinear second kind of VIEs.

$$y(t) = g(t) + \int_0^t L(t, s) \sigma(y(s)) ds, \quad t \in \Omega = [0, T]. \quad (1.2)$$

There are different advantages to the proposed approach, which are listed as follows:

- ✓ Using FDHFs, the problem under consideration is converted to a system of algebraic equations that can be easily solved.
- ✓ The proposed approach is convergent and the rate of convergence is  $O(h^5)$ .
- ✓ It is simple to calculate the unknown coefficients of the function's approximation based on this approach without integrating anything. Consequently, the proposed approach has a low computational expense.
- ✓ Because of the simplicity of FDHFs, this approach is a powerful mathematical tool to solve various kinds of equations with little additional work.

The remainder of the paper is organized as follows: The definition and properties of FDHFs are given in Section 2. In Section 3, we will find the integration operational matrix. Section 4 proposes a new approach for finding a numerical solution for a nonlinear VIE based on FDHFs. The error analysis is proven in Section 5. The numerical examples and solutions are presented in Section 6 of this paper. And finally, some concluding remarks are given in the last section.

## 2. Fourth-degree hat functions and their properties

In order to construct the FDHFs,  $\Omega = [0, T]$  must first be divided into  $n$  equidistant subintervals, and then each of these subintervals must be divided again into four equidistant subintervals with a length equal to  $h$ , where  $h = \frac{T}{4n}$  and  $n \in \mathbb{N}$ . Now, for any  $n \in \mathbb{N}$ , we define the following functions:

$$\xi_0(t) = \begin{cases} \frac{(t-h)(t-2h)(t-3h)(t-4h)}{24h^4}, & 0 \leq t \leq 4h, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

If  $k = 1, 2, \dots, n-1$ ,

$$\xi_{4k}(t) = \begin{cases} \frac{(t-(4k-1)h)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{24h^4}, & 4(k-1)h \leq t \leq 4kh, \\ \frac{(t-(4k+1)h)(t-(4k+2)h)(t-(4k+3)h)(t-(4k+4)h)}{24h^4}, & 4kh \leq t \leq 4(k+1)h, \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$

If  $k = 1, 2, \dots, n$ ,

$$\xi_{4k-1}(t) = \begin{cases} \frac{-(t-4kh)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{6h^4}, & (4k-4)h \leq t \leq (4k)h, \\ 0, & \text{otherwise.} \end{cases} \quad (2.3)$$

$$\xi_{4k-2}(t) = \begin{cases} \frac{(t-4kh)(t-(4k-1)h)(t-(4k-3)h)(t-(4k-4)h)}{4h^4}, & (4k-4)h \leq t \leq 4kh, \\ 0, & \text{otherwise,} \end{cases} \quad (2.4)$$

$$\xi_{4k-3}(t) = \begin{cases} \frac{(t-(4k-2)h)(t-(4k-1)h)(t-4kh)(t-(4k-4)h)}{6h^4}, & (4k-4)h \leq t \leq 4kh, \\ 0, & \text{otherwise,} \end{cases} \quad (2.5)$$

and

$$\xi_{4n}(t) = \begin{cases} \frac{(t-(T-h))(t-(T-2h))(t-(T-3h))(t-(T-4h))}{24h^4}, & T-4h \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

To clarify the definition of FDHFs on the interval  $[0, 1]$  and  $n = 2$ , one can see Fig. 1, which shows the 9-set of FDHFs.

The following are the basic properties of FDHFs:

1. Using the definition of FDHFs, there is a very important relationship as follows:

$$\xi_i(jh) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad \forall i, j = 0, 1, 2, \dots, 4n. \quad (2.7)$$

2. The total sum of FDHFs is one, which means:

$$\sum_{i=0}^{4n} \xi_i(t) = 1. \quad (2.8)$$

3. The functions  $\xi_0(t), \xi_1(t), \dots, \xi_{4n}(t)$  are linearly independent for all  $t \in [0, T]$ .

4. Any function  $y(t) \in L^2[0, T]$  can be approximated in terms of FDHFs as:

$$y(t) \simeq y_{4n}(t) = \sum_{\kappa=0}^{4n} c_\kappa \xi_\kappa(t) = C^T \Xi(t) = \Xi^T(t) C, \quad (2.9)$$

where

$$\Xi(t) = [\xi_0(t), \xi_1(t), \xi_2(t), \dots, \xi_{4n}(t)]^T, \quad (2.10)$$

and

$$C = [c_0, c_1, c_2, \dots, c_{4n}]^T. \quad (2.11)$$

The coefficients in Eq. (2.9) are given by

$$c_\kappa = y(\kappa h), \quad \forall \kappa = 0, 1, \dots, 4n. \quad (2.12)$$

5. Any function  $L(t, s) \in L^2[0, T]$  can be approximated in terms of FDHFs as:

$$L(t, s) \simeq L_{4n}(t, s) = \sum_{\kappa=0}^{4n} \sum_{r=0}^{4n} L_{\kappa r} \xi_\kappa(t) \xi_r(s) = \Xi^T(t) D \Xi(s) = \Xi^T(s) D^T \Xi(t), \quad (2.13)$$

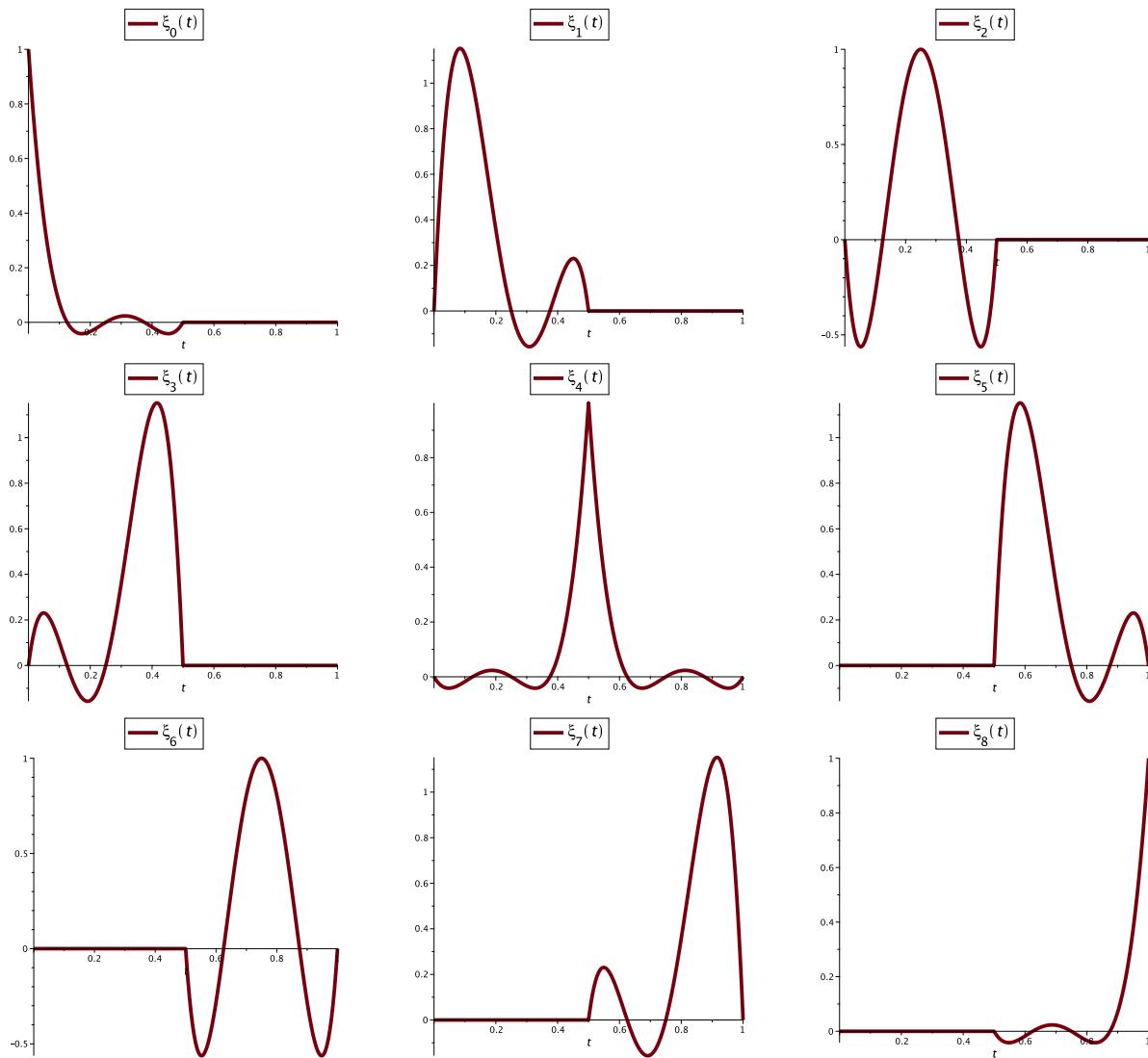
where

$$L_{\kappa r} = L(\kappa h, rh), \quad \forall \kappa, r = 0, 1, 2, \dots, 4n. \quad (2.14)$$

## 3. Operational matrix of FDHFs

The purpose of the work in this section is to construct an operational matrix  $P$  that satisfies

$$\int_0^t \Xi(s) ds \simeq P \Xi(t), \quad (3.1)$$

Fig. 1. Plot of FDHF with  $T = 1$  and  $n = 2$ .

where  $\Xi(t)$  is the vector defined in relation (2.10). Now, we try to write the  $\int_0^t \xi_\kappa(s) ds$  as a linear combination of the functions  $\xi_0(t), \xi_1(t), \dots, \xi_{4n}(t)$  as follows:

$$\int_0^t \xi_\kappa(s) ds \simeq \sum_{r=0}^{4n} P_{\kappa,r} \xi_r(t), \quad \forall \kappa = 0, 1, 2, \dots, 4n. \quad (3.2)$$

The coefficients  $P_{\kappa,r}$  can be calculated as follows:

$$P_{\kappa,r} = \int_0^{rh} \xi_\kappa(s) ds, \quad \forall r, \kappa = 0, 1, 2, \dots, 4n. \quad (3.3)$$

As a direct consequence of this, we can state the following theorem:

**Theorem 3.1.** If  $\Xi(t)$  be FDHF vectors, then the integration of  $\Xi(t)$  can be approximated as follows:

$$\int_0^t \Xi(s) ds \simeq P \Xi(t), \quad (3.4)$$

where  $P$  is the  $(4n+1) \times (4n+1)$  operational matrix of integration for the FDHFs and is defined as follows:

$$P = \frac{h}{720} \begin{pmatrix} 0 & \rho_1 & \rho_2 & \rho_2 & \rho_2 & \rho_2 & \cdots & \rho_2 & \rho_2 \\ \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_6 & \rho_6 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_4 & \rho_5 & \rho_6 & \rho_6 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_7 & \rho_4 & \rho_5 & \rho_6 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_4 & \rho_5 & \cdots & \rho_6 & \rho_6 \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \rho_4 & \cdots & \rho_6 & \rho_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \cdots & \rho_4 & \rho_5 \\ \rho_3 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \rho_7 & \cdots & \rho_7 & \rho_4 \end{pmatrix}, \quad (3.5)$$

where  $\rho_1 = (251, 232, 243, 224)^T$ ,  $\rho_2 = (224, 224, 224, 224)^T$ ,  $\rho_3 = (0, 0, 0, 0)^T$ ,

$$\rho_4 = \begin{pmatrix} 646 & 992 & 918 & 1024 \\ -264 & 192 & 648 & 384 \\ 106 & 32 & 378 & 1024 \\ -19 & -8 & -27 & 224 \end{pmatrix},$$

$$\rho_5 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 475 & 456 & 467 & 448 \end{pmatrix},$$

$$\rho_6 = \begin{pmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 448 & 448 & 448 & 448 \end{pmatrix},$$

$$\rho_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

#### 4. The suggested method's description

This section focuses on designing a computational method based on FDHFs to solve nonlinear VIE. In this procedure, the problem of the nonlinear VIE (1.2) will be converted to a problem of nonlinear algebraic equations in the following steps: First, we expand the given functions  $g(t)$  and  $L(t, s)\sigma(y(s))$  in terms of FDHFs as follows:

$$g(t) \simeq g_{4n}(t) = \sum_{\kappa=0}^{4n} g_\kappa \xi_\kappa(t), \quad (4.1)$$

where  $g_\kappa = g(\kappa h)$ ,  $\forall \kappa = 0, 1, \dots, 4n$ .

Also,

$$L(t, s)\sigma(y(s)) \simeq L_{4n}(t, s)\sigma(y_{4n}(s)) = \sum_{\tau=0}^{4n} \sum_{\kappa=0}^{4n} \psi_{\kappa, \tau} \xi_\kappa(t) \xi_\tau(s), \quad (4.2)$$

where  $\psi_{\kappa, \tau} = L(\kappa h, \tau h)\sigma(y(rh))$ ,  $\forall \kappa, \tau = 0, 1, 2, \dots, 4n$ .

Second, we assume the solution of Eq. (1.2) has the following form:

$$y(t) \simeq y_{4n}(t) = \sum_{\kappa=0}^{4n} y_\kappa \xi_\kappa(t), \quad (4.3)$$

where the coefficients  $y_\kappa$  are unknown, and it will be our task to figure out what these coefficients are.

Then, Eq. (1.2) can be rewritten as follows:

$$\sum_{\kappa=0}^{4n} y_\kappa \xi_\kappa(t) = \sum_{\kappa=0}^{4n} [g_\kappa \xi_\kappa(t) + \sum_{\tau=0}^{4n} L(\kappa h, \tau h) \sigma(y_\tau) \xi_\kappa(t) \int_0^t \xi_\tau(s) ds]. \quad (4.4)$$

Consequently, at the  $i$ th node point ( $ih$ ), Eq. (4.4) is

$$\sum_{\kappa=0}^{4n} y_\kappa \xi_\kappa(ih) = \sum_{\kappa=0}^{4n} [g_\kappa \xi_\kappa(ih) + \sum_{\tau=0}^{4n} L(\kappa h, \tau h) \sigma(y_\tau) \xi_\kappa(ih) \int_0^{ih} \xi_\tau(s) ds], \quad (4.5)$$

for all  $i = 0, 1, 2, \dots, 4n$ .

By employing the properties of FDHFs and simplifying the system given in Eq. (4.5), we obtain

$$y_i = g_i + \sum_{\tau=0}^{4n} [L(ih, \tau h) \sigma(y_\tau) \int_0^{ih} \xi_\tau(s) ds]. \quad (4.6)$$

Now, using Eq. (3.3), we have

$$y_i = g_i + \sum_{\tau=0}^{4n} L(ih, \tau h) \sigma(y_\tau) P_{\tau, i}, \quad i = 0, 1, 2, \dots, 4n, \quad (4.7)$$

where  $P$  is the operational matrix of integration given in Eq. (3.5).

Finally, the approximate solution of Eq. (1.2) can be obtained using Eq. (4.3) after solving the nonlinear system (4.7) using any numerical method and finding the unknown coefficients,  $y_\kappa$ ,  $\forall \kappa = 0, 1, 2, \dots, 4n$ .

#### 5. Error analysis

This section is covered to examine the suggested method to solve nonlinear VIEs from the standpoint of error analysis. Indeed, we find that the suggested method to solve nonlinear VIEs has a rate of convergence of  $O(h^5)$ . Here, we define

$$\|y\| = \sup_{t \in \Omega} |y(t)|. \quad (5.1)$$

**Theorem 5.1.** Suppose that  $t_k = kh$ ,  $k = 0, 1, \dots, 4n$ ,  $g(t) \in C^5(\Omega)$  and  $g_{4n}(t) = \sum_{k=0}^{4n} g(t_k) \xi_k(t)$  be the FDHFs expansion of  $g(t)$ . Also, let  $e(t) = g(t) - g_{4n}(t)$ ,  $t \in \Omega$ . Then

$$\|g(t) - g_{4n}(t)\| \leq \omega_1 h^5, \quad (5.2)$$

where  $\omega_1$  is a constant number.

**Proof.** Suppose that

$$e_i(t) = \begin{cases} g(t) - g_{4n}(t), & t \in V_i, \\ 0, & t \in \Omega \setminus V_i, \end{cases}$$

where  $V_i = \{t | ih \leq t \leq (i+4)h, h = \frac{T}{4n}\}$ ,  $i = 0, 4, 8, \dots, 4n-4$ . Then, we get

$$e_i(t) = g(t) - g_{4n}(t) = g(t) - \sum_{j=0}^{4n} g(jh) \xi_j(t),$$

Therefore

$$e_i(t) = g(t) - [g(ih) \xi_i(t) + g((i+1)h) \xi_{i+1}(t) + g((i+2)h) \xi_{i+2}(t) + g((i+3)h) \xi_{i+3}(t) + g((i+4)h) \xi_{i+4}(t)].$$

We have the following results when we interpolate using a fourth-degree error<sup>63</sup>:

$$e_i(t) = \frac{(t - ih)(t - (i+1)h)(t - (i+2)h)(t - (i+3)h)(t - (i+4)h)}{120} \cdot \frac{d^5 g(\chi_i)}{dt^5},$$

where  $\chi_i \in (ih, (i+4)h)$ .

Now consider  $\phi(t) = (t - ih)(t - (i+1)h)(t - (i+2)h)(t - (i+3)h)(t - (i+4)h)$ . Since  $\phi(t)$  is a continuous function and  $V_i$  is compacted, we have

$$\sup_{t \in V_i} |\phi(t)| = \max_{t \in V_i} |\phi(t)| = 3.6314 h^5.$$

Therefore, we have

$$|e_i(t)| \leq \frac{1}{120} |\phi(t)| \left| \frac{d^5 g(\chi_i)}{dt^5} \right|.$$

As a result, we have

$$\|e(t)\| = \max_{i=0, 4, \dots, 4n-4} \sup_{t \in V_i} |e_i(t)| \leq \max_{i=0, 4, \dots, 4n-4} 0.03026 h^5 \left| \frac{d^5 g(\chi_i)}{dt^5} \right|.$$

Then there is an  $\varepsilon \in \{0, 4, \dots, 4n-4\}$ , where

$$\|e(t)\| \leq \max_{i=0, 4, \dots, 4n-4} 0.03026 h^5 \left| \frac{d^5 g(\chi_i)}{dt^5} \right| = 0.03026 h^5 \left| \frac{d^5 g(\chi_\varepsilon)}{dt^5} \right|.$$

Lastly, by using the relation (5.1), we obtain

$$\|e(t)\| \leq 0.03026 h^5 \left| \frac{d^5 g(\chi_\varepsilon)}{dt^5} \right| \leq 0.03026 h^5 \left\| \frac{d^5 g(t)}{dt^5} \right\| \leq \omega_1 h^5. \quad (5.3)$$

According to the relation (5.3), we get

$$\|e(t)\| = O(h^5).$$

Finally, the proof was completed.

**Theorem 5.2.** Assume that  $L(t, s) \in C^5(\Omega \times \Omega)$ , and  $e(t, s) = L(t, s) - L_{4n}(t, s)$ ,  $(t, s) \in \Lambda = (\Omega \times \Omega)$  be the truncation error where  $L_{4n}(t, s) = \sum_{i=0}^{4n} \sum_{j=0}^{4n} L(ih, jh) \xi_i(t) \xi_j(s)$  is the FDHFs of  $L(t, s)$ . Then, we have

$$\|L(t, s) - L_{4n}(t, s)\| \leq \omega_2 h^5, \quad (5.4)$$

where  $\omega_2$  is a constant number.

**Proof.** Suppose that

$$e_{qr}(t, s) = \begin{cases} L(t, s) - L_{4n}(t, s), & (t, s) \in V_{qr}, \\ 0, & (t, s) \in \Lambda \setminus V_{qr}, \end{cases}$$

where

$$V_{qr} = \{(t, s) | qh \leq t \leq (q+4)h, rh \leq s \leq (r+4)h, h = \frac{T}{4n}\},$$

$$q, r = 0, 4, 8, \dots, 4n-4.$$

Then, we get

$$e_{qr}(t, s) = L(t, s) - L_{4n}(t, s) = L(t, s) - \sum_{i=0}^{4n} \sum_{j=0}^{4n} L(ih, jh) \xi_i(t) \xi_j(s),$$

Therefore

$$\begin{aligned} e_{qr}(t, s) &= L(t, s) - [L(qh, rh) \xi_q(t) \xi_r(s) + L(qh, (r+1)h) \xi_q(t) \xi_{r+1}(s) + \dots \\ &\quad + L(qh, (r+4)h) \xi_q(t) \xi_{r+4}(s) + \dots \\ &\quad + L((q+4)h, (r+4)h) \xi_{q+4}(t) \xi_{r+4}(s)]. \end{aligned}$$

We have the following results when we interpolate using a fourth-degree error<sup>63</sup>:

$$\begin{aligned} e_{qr}(t, s) &= \frac{(t-qh)(t-(q+1)h)(t-(q+2)h)(t-(q+3)h)(t-(q+4)h)}{120} \cdot \left| \frac{\partial^5 L(\chi_q, s)}{\partial t^5} \right| \\ &\quad + \frac{(s-rh)(s-(r+1)h)(s-(r+2)h)(s-(r+3)h)(s-(r+4)h)}{120} \cdot \left| \frac{\partial^5 L(t, \eta_r)}{\partial s^5} \right| \\ &\quad - \frac{(t-qh)(t-(q+1)h) \dots (t-(q+4)h)(s-rh)(s-(r+1)h) \dots (s-(r+4)h)}{14400} \\ &\quad \times \left| \frac{\partial^{10} L(\tilde{\chi}_q, \tilde{\eta}_r)}{\partial t^5 \partial s^5} \right|, \end{aligned}$$

where,  $\chi_q, \tilde{\chi}_q \in (qh, (q+4)h)$  and  $\eta_r, \tilde{\eta}_r \in (rh, (r+4)h)$ .

Now, assume  $u(t) = (t-qh)(t-(q+1)h)(t-(q+2)h)(t-(q+3)h)(t-(q+4)h)$ , and  $v(s) = (s-rh)(s-(r+1)h)(s-(r+2)h)(s-(r+3)h)(s-(r+4)h)$ .

Therefore, we have

$$\begin{aligned} |e_{qr}(t, s)| &\leq \frac{1}{120} |u(t)| \left| \frac{\partial^5 L(\chi_q, s)}{\partial t^5} \right| + \frac{1}{120} |v(s)| \left| \frac{\partial^5 L(t, \eta_r)}{\partial s^5} \right| \\ &\quad + \frac{1}{14400} |u(t)| |v(s)| \left| \frac{\partial^{10} L(\tilde{\chi}_q, \tilde{\eta}_r)}{\partial t^5 \partial s^5} \right|. \end{aligned}$$

Since  $\sup_{t \in (qh, (q+4)h)} |u(t)| = 3.6314 h^5$ , and  $\sup_{s \in (rh, (r+4)h)} |v(s)| = 3.6314 h^5$ , we obtain

$$\begin{aligned} \|e(t, s)\| &= \max_{\substack{q=0,4,\dots,4n-4 \\ r=0,4,\dots,4n-4}} \sup_{(t,s) \in V_{qr}} |e_{qr}(t, s)| \\ &\leq 0.03026 h^5 \max_{\substack{q=0,4,\dots,4n-4 \\ r=0,4,\dots,4n-4}} \sup_{(t,s) \in V_{qr}} \left( \left| \frac{\partial^5 L(\chi_q, s)}{\partial t^5} \right| + \left| \frac{\partial^5 L(t, \eta_r)}{\partial s^5} \right| \right. \\ &\quad \left. + 0.03026 h^5 \left| \frac{\partial^{10} L(\tilde{\chi}_q, \tilde{\eta}_r)}{\partial t^5 \partial s^5} \right| \right). \end{aligned}$$

Then, there are  $\alpha, \epsilon \in \{0, 4, \dots, 4n-4\}$ , where

$$\begin{aligned} \|e(t, s)\| &\leq 0.03026 h^5 \sup_{(t,s) \in V_{qr}} \left( \left| \frac{\partial^5 L(\chi_\alpha, s)}{\partial t^5} \right| + \left| \frac{\partial^5 L(t, \eta_\epsilon)}{\partial s^5} \right| \right. \\ &\quad \left. + 0.03026 h^5 \left| \frac{\partial^{10} L(\tilde{\chi}_\alpha, \tilde{\eta}_\epsilon)}{\partial t^5 \partial s^5} \right| \right). \end{aligned}$$

Lastly, by using the relation (5.1), we obtain

$$\begin{aligned} \|e(t, s)\| &\leq 0.03026 h^5 \left( \left\| \frac{\partial^5 L(t, s)}{\partial t^5} \right\| + \left\| \frac{\partial^5 L(t, s)}{\partial s^5} \right\| \right. \\ &\quad \left. + 0.03026 h^5 \left\| \frac{\partial^{10} L(t, s)}{\partial t^5 \partial s^5} \right\| \right) \leq \omega_2 h^5. \end{aligned} \quad (5.5)$$

According to the relation (5.5), we get

$$\|e(t, s)\| = O(h^5).$$

Finally, the proof is complete.

**Theorem 5.3.** Assume that  $y(t)$  is the exact solution and  $y_{4n}(t)$  is the approximate solution of Eq. (1.2). Also, let

$$(i) \quad \|\sigma(y(t))\| \leq M, \quad t \in \Omega,$$

$$(ii) \quad \|L(t, s)\| \leq \omega_3, \quad (t, s) \in \Omega \times \Omega,$$

(iii) The nonlinear term  $\sigma(y(t))$  satisfies the Lipschitz condition, i.e.,

$$\|\sigma(y(t)) - \sigma(y_{4n}(t))\| \leq Y \|y(t) - y_{4n}(t)\|, \quad (5.6)$$

$$(iv) \quad 1 - tY\omega_3 > 0, \forall t \in \dot{\Omega} = [0, \mu], \text{ where } \mu = \frac{1}{Y\omega_3}.$$

where  $Y$  is a positive constant. Then

$$\|y(t) - y_{4n}(t)\| \leq \frac{(\omega_1 + tM\omega_2)h^5}{1 - tY\omega_3}, \quad \forall t \in \dot{\Omega}, \quad (5.7)$$

and  $\|y(t) - y_{4n}(t)\| = O(h^5)$ ,  $\forall t \in \dot{\Omega}$ .

**Proof.** We can approximate Eq. (1.2) using the FDHF as follows:

$$y_{4n}(t) = g_{4n}(t) + \int_0^t L_{4n}(t, s) \sigma(y_{4n}(s)) ds. \quad (5.8)$$

From Eqs. (1.2), (5.8), and norm properties, we have

$$\|y(t) - y_{4n}(t)\| \leq \|g(t) - g_{4n}(t)\| + \int_0^t \|L(t, s) \sigma(y(s)) - L_{4n}(t, s) \sigma(y_{4n}(s))\| ds. \quad (5.9)$$

Now, Theorem 5.2, Eq. (5.6), and the hypotheses (i) and (ii) all lead us to the conclusion that

$$\begin{aligned} \|L(t, s) \sigma(y(s)) - L_{4n}(t, s) \sigma(y_{4n}(s))\| &\leq \|L(t, s)\| \|\sigma(y(s)) - \sigma(y_{4n}(s))\| \\ &\quad + \|\sigma(y_{4n}(s))\| \|L(t, s) - L_{4n}(t, s)\| \\ &\leq \omega_3 Y \|y(s) - y_{4n}(s)\| + M\omega_2 h^5. \end{aligned} \quad (5.10)$$

From Eqs. (5.9), (5.10), and Theorem 5.1, we can deduce the following:

$$\begin{aligned} \|y(t) - y_{4n}(t)\| &\leq \|g(t) - g_{4n}(t)\| + \int_0^t [\omega_3 Y \|y(s) - y_{4n}(s)\| + M\omega_2 h^5] ds \\ &\leq \omega_1 h^5 + tM\omega_2 h^5 + \omega_3 Y \int_0^t \|y(s) - y_{4n}(s)\| ds \\ &\leq \omega_1 h^5 + tM\omega_2 h^5 + t\omega_3 Y \max_{0 \leq \tau \leq t} \|y(\tau) - y_{4n}(\tau)\|. \end{aligned} \quad (5.11)$$

Clearly, Eq. (5.11) represent linear relationship between  $\|y(t) - y_{4n}(t)\|$  and the time  $t$ , so we have

$$\|y(t) - y_{4n}(t)\| = \max_{0 \leq \tau \leq t} \|y(\tau) - y_{4n}(\tau)\|. \quad (5.12)$$

According to Eqs. (5.12), (5.11), and assumption (iii), we get:

$$\|y(t) - y_{4n}(t)\| \leq \frac{(\omega_1 + tM\omega_2)h^5}{1 - tY\omega_3}, \quad \forall t \in \dot{\Omega}. \quad (5.13)$$

Moreover, we can deduce from Eq. (5.13) that

$$\|y(t) - y_{4n}(t)\| = O(h^5), \quad \forall t \in \dot{\Omega}.$$

## 6. Illustrated examples

This section tests the suggested method on several examples to ensure its applicability, efficiency, and accuracy. Indeed, five examples have been solved by using four base functions, including GHFs, MHFs, AHFs, and FDHF. We use the same length of subintervals (same  $h = \frac{T}{mn}$ , where  $m$  is the degree of polynomials used in the definition of the basis function in each method) to ensure a fair comparison between these base functions; that is, we use the same number of basis functions in each method. Also, all the calculations were done using Maple 2020 on a laptop Windows and 11th Gen Intel(R) Core(TM) i9-11900H 2.5HzG with RAM 40Gb.

**Example 6.1** (64). Let the nonlinear VIE that follows:

$$y(t) = te^{-t^2} + \int_0^t 2ts e^{-y^2(s)} ds, \quad 0 \leq t \leq 1. \quad (6.1)$$

Here, the exact solution is  $y(t) = t$ .

**Table 1**  
The absolute errors for Example 6.1.

Time	GHFs	MHFs	AHFs	Present method
	$m = 1, n = 24$	$m = 2, n = 12$	$m = 3, n = 8$	$m = 4, n = 6$
0.1	$9.6922000 \times 10^{-7}$	$1.1230000 \times 10^{-7}$	$1.3200000 \times 10^{-9}$	$8.2800000 \times 10^{-10}$
0.2	$6.8354000 \times 10^{-6}$	$2.4703000 \times 10^{-7}$	$1.3230000 \times 10^{-8}$	$6.4800000 \times 10^{-10}$
0.3	$2.1802550 \times 10^{-5}$	$2.8280000 \times 10^{-7}$	$6.5590000 \times 10^{-8}$	$1.9700000 \times 10^{-9}$
0.4	$4.8149900 \times 10^{-5}$	$1.9320000 \times 10^{-7}$	$1.5119000 \times 10^{-7}$	$1.0900000 \times 10^{-9}$
0.5	$8.5169400 \times 10^{-5}$	$8.0900000 \times 10^{-8}$	$2.0475000 \times 10^{-7}$	$2.6000000 \times 10^{-9}$
0.6	$1.3012130 \times 10^{-4}$	$1.9220000 \times 10^{-8}$	$2.6190000 \times 10^{-7}$	$2.1600000 \times 10^{-9}$
0.7	$1.7683950 \times 10^{-4}$	$2.3639000 \times 10^{-7}$	$3.6902000 \times 10^{-7}$	$8.5100000 \times 10^{-9}$
0.8	$2.1867070 \times 10^{-4}$	$3.5840000 \times 10^{-7}$	$4.6088000 \times 10^{-7}$	$3.1800000 \times 10^{-9}$
0.9	$2.4958760 \times 10^{-4}$	$1.8930000 \times 10^{-7}$	$4.8200000 \times 10^{-7}$	$1.7900000 \times 10^{-9}$
1.0	$2.6642710 \times 10^{-4}$	$3.8300000 \times 10^{-7}$	$4.6700000 \times 10^{-7}$	$6.8000000 \times 10^{-9}$

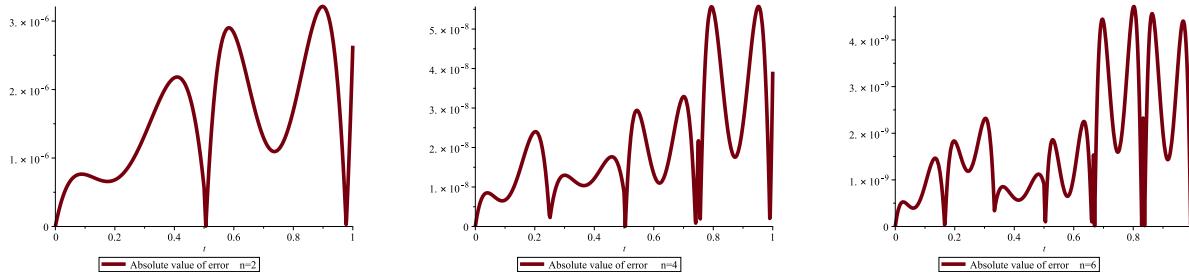


Fig. 2. Absolute value of error, Example 6.1 with  $n = 2, 4, 6$ .

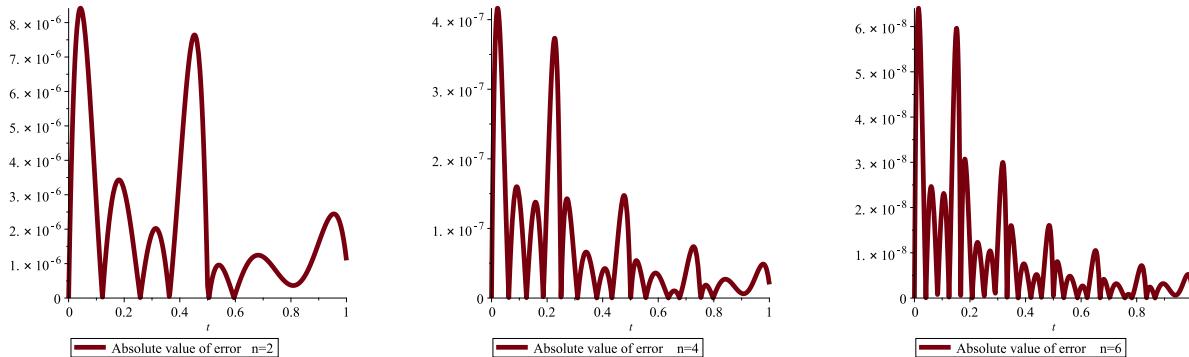


Fig. 3. Absolute value of error, Example 6.2 with  $n = 2, 4, 6$ .

In Table 1, the absolute error of GHFs for  $n = 24$ , MHFs for  $n = 12$ , and AHFs for  $n = 8$  are compared with the absolute error obtained by the suggested method for  $n = 6$ . As shown in the table, the results obtained using our suggested method are better than those obtained using the other hat basis functions. Additionally, Fig. 2 displays the absolute error obtained by the suggested method for different values of  $n$ . The results demonstrate that as  $n$  is increased, the absolute error values decrease, and thus the approximate solution converges to the exact solution.

**Example 6.2** (65). Let the nonlinear VIE that follows:

$$y(t) = g(t) + \int_0^t ts^2 y^2(s) ds, \quad 0 \leq t \leq 1, \quad (6.2)$$

where

$$g(t) = \left(1 + \frac{11t}{9} + \frac{2t^2}{3} - \frac{t^3}{3} + \frac{2t^4}{9}\right) \log(t+1) - \frac{1}{3}(t+t^4) \log(t+1)^2 - \frac{11t^2}{9} + \frac{5t^3}{18} - \frac{2t^4}{27}.$$

Here, the exact solution is  $y(t) = \log(t+1)$ .

In Table 2, the absolute error of GHFs for  $n = 24$ , MHFs for  $n = 12$ , and AHFs for  $n = 8$  are compared with the absolute error obtained

by the suggested method for  $n = 6$ . As shown in the table, the results obtained using our suggested method are better than those obtained using the other hat basis functions. Additionally, Fig. 3 displays the absolute error obtained by the suggested method for different values of  $n$ . The results demonstrate that as  $n$  is increased, the absolute error values decrease, and thus the approximate solution converges to the exact solution.

**Example 6.3** (66). Let the nonlinear VIE that follows:

$$y(t) = g(t) + \int_0^t \sin(t-s)(s+y^2(s)) ds, \quad t \in [0, 1], \quad (6.3)$$

where

$$g(t) = \frac{1}{8}t^3 - \frac{29}{4} \cos t + \sin t + \frac{29}{4} - \frac{1}{64}t^6 + \frac{7}{32}t^4 - \frac{29}{8}t^2,$$

and the exact solution is  $y(t) = t + \frac{1}{8}t^3$ .

In Table 3, the absolute error of GHFs for  $n = 24$ , MHFs for  $n = 12$ , and AHFs for  $n = 8$  are compared with the absolute error obtained by the suggested method for  $n = 6$ . As shown in the table, the results obtained using our suggested method are better than those obtained using the other hat basis functions. Additionally, Fig. 4 displays the absolute error obtained by the suggested method for different values of  $n$ . The results demonstrate that as  $n$  is increased, the absolute error

**Table 2**  
The absolute errors for Example 6.2.

Time	GHFs	MHFs	AHFs	Present method
	$m = 1, n = 24$	$m = 2, n = 12$	$m = 3, n = 8$	$m = 4, n = 6$
0.1	$1.7127799 \times 10^{-4}$	$6.6383800 \times 10^{-6}$	$4.6132000 \times 10^{-7}$	$2.1185000 \times 10^{-8}$
0.2	$9.7062400 \times 10^{-5}$	$2.7081700 \times 10^{-6}$	$1.5033000 \times 10^{-7}$	$9.9550000 \times 10^{-9}$
0.3	$7.7729950 \times 10^{-5}$	$2.0277200 \times 10^{-6}$	$1.3334000 \times 10^{-7}$	$1.2950000 \times 10^{-8}$
0.4	$9.6900900 \times 10^{-5}$	$3.4188000 \times 10^{-6}$	$7.7950000 \times 10^{-8}$	$6.6140000 \times 10^{-9}$
0.5	$2.1811900 \times 10^{-5}$	$3.5200000 \times 10^{-8}$	$8.4771015 \times 10^{-8}$	$9.0000000 \times 10^{-10}$
0.6	$3.8811400 \times 10^{-5}$	$2.3761700 \times 10^{-6}$	$5.3000000 \times 10^{-9}$	$1.8900000 \times 10^{-9}$
0.7	$2.5114500 \times 10^{-5}$	$1.2926400 \times 10^{-6}$	$1.6172000 \times 10^{-7}$	$1.8600000 \times 10^{-9}$
0.8	$7.8039700 \times 10^{-5}$	$3.8628000 \times 10^{-7}$	$1.7951000 \times 10^{-7}$	$2.3100000 \times 10^{-9}$
0.9	$1.3191560 \times 10^{-4}$	$1.0748000 \times 10^{-6}$	$1.4400000 \times 10^{-7}$	$2.2100000 \times 10^{-9}$
1.0	$2.9210500 \times 10^{-4}$	$2.7500000 \times 10^{-8}$	$2.3210000 \times 10^{-7}$	$1.1443464 \times 10^{-10}$

**Table 3**  
The absolute errors for Example 6.3.

Time	GHFs	MHFs	AHFs	Present method
	$m = 1, n = 24$	$m = 2, n = 12$	$m = 3, n = 8$	$m = 4, n = 6$
0.1	$1.436731 \times 10^{-5}$	$2.984910 \times 10^{-6}$	$3.820000 \times 10^{-9}$	$5.240000 \times 10^{-10}$
0.2	$4.367760 \times 10^{-5}$	$9.867300 \times 10^{-7}$	$1.318000 \times 10^{-8}$	$3.697000 \times 10^{-9}$
0.3	$6.782194 \times 10^{-5}$	$2.515280 \times 10^{-6}$	$6.570000 \times 10^{-9}$	$9.800000 \times 10^{-10}$
0.4	$7.768290 \times 10^{-5}$	$4.023800 \times 10^{-6}$	$1.403000 \times 10^{-8}$	$1.700000 \times 10^{-10}$
0.5	$1.836241 \times 10^{-4}$	$7.000000 \times 10^{-9}$	$3.348896 \times 10^{-8}$	$2.300000 \times 10^{-9}$
0.6	$1.3875050 \times 10^{-4}$	$2.764090 \times 10^{-6}$	$5.360000 \times 10^{-8}$	$9.800000 \times 10^{-10}$
0.7	$2.166350 \times 10^{-4}$	$5.624400 \times 10^{-7}$	$1.033800 \times 10^{-7}$	$5.360000 \times 10^{-9}$
0.8	$2.687260 \times 10^{-4}$	$3.051400 \times 10^{-6}$	$1.832300 \times 10^{-7}$	$1.450000 \times 10^{-9}$
0.9	$2.878721 \times 10^{-4}$	$4.481700 \times 10^{-6}$	$2.760000 \times 10^{-7}$	$8.500000 \times 10^{-10}$
1.0	$5.166140 \times 10^{-4}$	$9.000000 \times 10^{-8}$	$3.370000 \times 10^{-7}$	$3.834960 \times 10^{-9}$

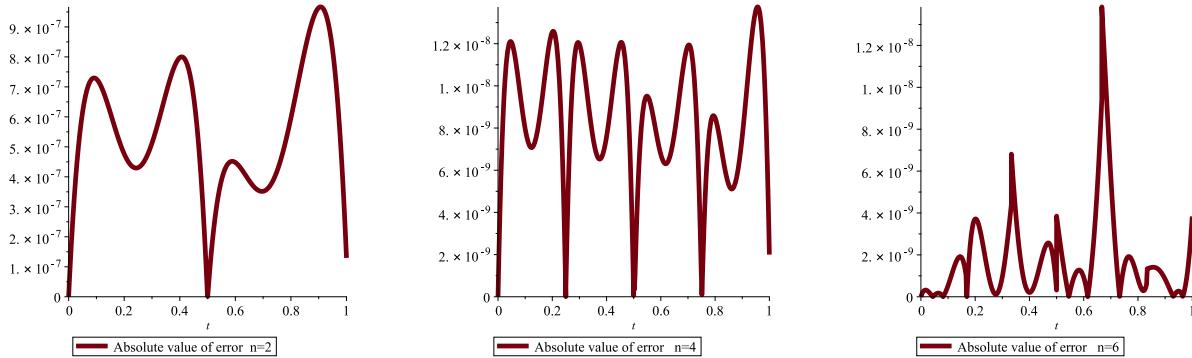


Fig. 4. Absolute value of error, Example 6.3 with  $n = 2, 4, 6$ .

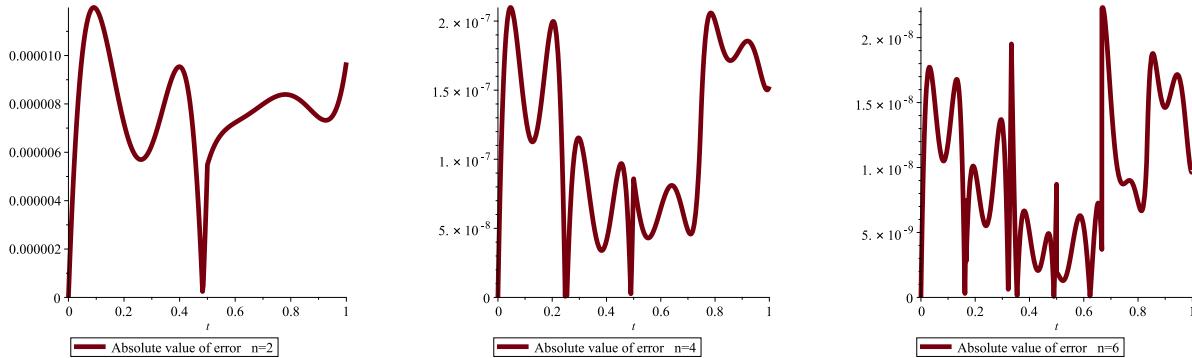


Fig. 5. Absolute value of error, Example 6.4 with  $n = 2, 4, 6$ .

values decrease, and thus the approximate solution converges to the exact solution.

**Example 6.4 (67).** Let the nonlinear VIE that follows:

$$y(t) = 1 + \sin^2 t - 3 \int_0^t \sin(t-s) y^2(s) ds, \quad 0 \leq t \leq 1. \quad (6.4)$$

Here, and the exact solution is  $y(t) = \cos t$ .

In Table 4, the absolute error of GHFs for  $n = 24$ , MHFs for  $n = 12$ , and AHFs for  $n = 8$  are compared with the absolute error obtained by the suggested method for  $n = 6$ . As shown in the table, the results obtained using our suggested method are better than those obtained using the other hat basis functions. Additionally, Fig. 5 displays the

**Table 4**  
The absolute errors for Example 6.4.

Time	GHFs	MHFs	AHFs	Present method
	$m = 1, n = 24$	$m = 2, n = 12$	$m = 3, n = 8$	$m = 4, n = 6$
0.1	$2.094607 \times 10^{-4}$	$1.048900 \times 10^{-6}$	$1.009000 \times 10^{-7}$	$1.220000 \times 10^{-8}$
0.2	$1.447069 \times 10^{-4}$	$1.681980 \times 10^{-6}$	$1.566000 \times 10^{-8}$	$9.700000 \times 10^{-9}$
0.3	$1.500745 \times 10^{-4}$	$2.426800 \times 10^{-6}$	$9.575000 \times 10^{-8}$	$1.377000 \times 10^{-8}$
0.4	$2.212138 \times 10^{-4}$	$2.336000 \times 10^{-6}$	$3.466800 \times 10^{-7}$	$5.040000 \times 10^{-9}$
0.5	$4.093670 \times 10^{-5}$	$9.405000 \times 10^{-7}$	$2.054219 \times 10^{-7}$	$5.600000 \times 10^{-9}$
0.6	$2.231093 \times 10^{-4}$	$3.276780 \times 10^{-6}$	$2.334000 \times 10^{-7}$	$5.470000 \times 10^{-9}$
0.7	$1.662196 \times 10^{-4}$	$2.399650 \times 10^{-6}$	$2.207500 \times 10^{-7}$	$1.484000 \times 10^{-8}$
0.8	$1.585570 \times 10^{-4}$	$1.917200 \times 10^{-7}$	$3.942100 \times 10^{-7}$	$7.160000 \times 10^{-9}$
0.9	$1.903342 \times 10^{-4}$	$1.711900 \times 10^{-6}$	$5.626000 \times 10^{-7}$	$1.461000 \times 10^{-8}$
1.0	$5.184350 \times 10^{-5}$	$1.704700 \times 10^{-6}$	$3.682000 \times 10^{-7}$	$9.636012 \times 10^{-9}$

**Table 5**  
The absolute errors for Example 6.5.

Time	GHFs	MHFs	AHFs	Present method
	$m = 1, n = 24$	$m = 2, n = 12$	$m = 3, n = 8$	$m = 4, n = 6$
0.1	$4.286791 \times 10^{-4}$	$1.828800 \times 10^{-6}$	$1.260000 \times 10^{-8}$	$1.625000 \times 10^{-8}$
0.2	$3.236294 \times 10^{-4}$	$2.457900 \times 10^{-6}$	$1.140500 \times 10^{-7}$	$2.072000 \times 10^{-8}$
0.3	$3.780862 \times 10^{-4}$	$2.037400 \times 10^{-6}$	$4.261300 \times 10^{-7}$	$2.446000 \times 10^{-8}$
0.4	$5.881722 \times 10^{-4}$	$1.034800 \times 10^{-6}$	$8.309000 \times 10^{-7}$	$1.720000 \times 10^{-8}$
0.5	$2.527798 \times 10^{-4}$	$2.464000 \times 10^{-7}$	$9.648000 \times 10^{-7}$	$1.000000 \times 10^{-10}$
0.6	$7.511171 \times 10^{-4}$	$8.126000 \times 10^{-7}$	$1.156900 \times 10^{-6}$	$1.500000 \times 10^{-8}$
0.7	$6.804226 \times 10^{-4}$	$2.647000 \times 10^{-6}$	$1.813000 \times 10^{-6}$	$2.130000 \times 10^{-8}$
0.8	$7.162285 \times 10^{-4}$	$4.108800 \times 10^{-6}$	$2.795300 \times 10^{-6}$	$2.330000 \times 10^{-8}$
0.9	$8.364223 \times 10^{-4}$	$3.706100 \times 10^{-6}$	$3.704000 \times 10^{-6}$	$1.240000 \times 10^{-8}$
1.0	$3.273170 \times 10^{-4}$	$4.090000 \times 10^{-7}$	$3.884700 \times 10^{-6}$	$5.200000 \times 10^{-9}$

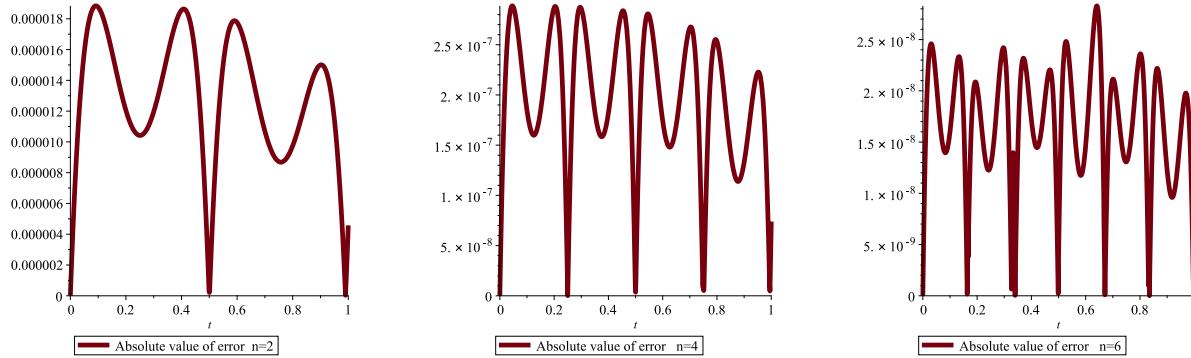


Fig. 6. Absolute value of error, Example 6.5 with  $n = 2, 4, 6$ .

absolute error obtained by the suggested method for different values of  $n$ . The results demonstrate that as  $n$  is increased, the absolute error values decrease, and thus the approximate solution converges to the exact solution.

**Example 6.5 (64).** Let the nonlinear VIE that follows:

$$y(t) = \frac{2}{15}t^6 - \frac{1}{3}t^4 + t^2 - 1 + \int_0^t (t-2s)y^2(s) ds, \quad 0 \leq t \leq 1. \quad (6.5)$$

Here, the exact solution is  $y(t) = t^2 - 1$ .

In Table 5, the absolute error of GHFs for  $n = 24$ , MHFs for  $n = 12$ , and AHFs for  $n = 8$  are compared with the absolute error obtained by the suggested method for  $n = 6$ . As shown in the table, the results obtained using our suggested method are better than those obtained using the other hat basis functions. Additionally, Fig. 6 displays the absolute error obtained by the suggested method for different values of  $n$ . The results demonstrate that as  $n$  is increased, the absolute error values decrease, and thus the approximate solution converges to the exact solution (see Table 6).

**Table 6**  
Comparing methods according to CPU time in seconds.

Example no.	GHFs	MHFs	AHFs	Present method
1	3.531	3.093	2.985	2.828
2	3.625	3.047	2.875	2.781
3	1.969	1.797	1.797	1.687
4	3.578	3.078	2.875	2.797
5	2.015	1.781	1.750	1.700

## 7. Conclusion

This paper focuses on constructing a novel hat functions by using fourth-degree polynomials. The main properties of FDHF have been discussed and used to construct some new operational matrices. Meanwhile, we implement FDHF to provide an effective computational method for solving nonlinear VIEs. The effort of this method is summarized by the operations matrices used to convert the considered problem into nonlinear algebraic equations. In fact, the advantage of this method is that it is easy to figure out the unknown coefficients of

the approximation of the function without having to integrate anything. Consequently, the proposed method has a low computational expense. Some error analysis results associated with the suggested method were studied, and we discovered that the suggested method to solve nonlinear VIEs has a rate of convergence of  $O(h^5)$ . To verify the convergence order in **Theorem 5.3**, we give a comparison of the absolute error for different values of the results in Figs. 2–6. This comparison confirms the validity of this approach, as we obtain a clear convergence of all the illustrated examples when the value of  $n$  is increased. Also, the resulting tables and CPU time test confirm that the suggested method is better than GHFs, MHFs, and AHFs.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

The data that has been used is confidential.

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