

Finding the General Solution for the Two Dimensional Helmholtz Partial Differential Equation by Two Numerical methods

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الخلاصة

في هذا البحث قمنا بحساب الحل العام لمعادلة هيلمهولتز التفاضلية الجزئية ثنائية الابعاد بواسطة استخدام طريقتين عدديتين حيث قمنا بحساب الحل العام بواسطة صيغة تعتمد على المجموع لحاصل ضرب قيم دالة كرين في قيم الدالة $f(x, y)$ وقد ساعدتنا الطريقة العددية التي قدمها الباحثان B.Fengsheng & L. Jiaqi [1] عام (1987) في ايجاد القيم العددية لدالة كرين والتي استخدمناها في صيغة المجموع المذكورة اعلاه لحساب الحل العام.

كما قمنا بحساب الحل العام لمعادلة هيلمهولتز التفاضلية الجزئية ثنائية الابعاد بطريقة عددية اخرى وهي طريقة الفروقات المحددة وبذلك نكون قد ناقشنا طريقتين لحل هذه المعادلة المعروفة والمفيدة حيث كانت النتائج لكلا الطريقتين متقاربة بشكل ملحوظ.

ABSTRACT

The aim of this paper is to find the general solution for the two dimensional Helmholtz partial differential equation by two numerical techniques. first we show the summation formula to find the general solution by using numerical values of the Green's function, the numerical method submitted by B.Fengsheng & L.Jiaqi(1987) helps us to find the numerical values of the Green's function and we used this values in the summation formula to find the general solution, then we find the general solution by the Finite Difference Method and it is obvious that the numerical solutions strongly agreed in both techniques.

INTRODUCTION

The Helmholtz partial differential equation, named for Hermann von Helmholtz, is an elliptic partial differential equation, It is often arises in the study of physical problems involving partial differential equations, the inhomogeneous Helmholtz partial differential equation is the equation[2]:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \alpha^2 U = -f(x, y)$$

where α is a parameter, U is the function of the two independent variables x, y whose solution is sought for and $f(x, y)$ is forcing function.

This equation is very similar to the Poisson's equation which describes both electrostatics and Newtonian gravitation, and would be identical if $\alpha = 0$. Furthermore, if $f(x, y) = 0$ and $\alpha = 0$ then, we get the Laplace's equation.

In this paper, we found the numerical values of the Green's function for the two dimensional Helmholtz partial differential equation by the numerical method submitted by B.Fengsheng & L.Jiaqi(1987) [1] and we used those numerical values of the fundamental solution

(Green's function) to find the general solution for the two dimensional Helmholtz partial differential equation by using a suitable summation formula. We also found the general solution for the two dimensional Helmholtz partial differential equation by using the Finite Difference Method and it is obvious that the numerical solutions strongly agreed in both techniques.

SUMMATION FORMULA OF THE GENERAL SOLUTION BY GREEN'S FUNCTION

If the numerical values of the fundamental solution (Green's function) of the two dimensional Helmholtz partial differential equation has been found, then the general solution is easy to be expressed in terms of the fundamental solution by the following summation formula:

$$U(x, y) = \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} g(x, y; \xi_i, \eta_j) f(\xi_i, \eta_j) \quad (1)$$

Where $g(x, y; \xi_i, \eta_j)$ is taken from the matrix of the numerical solution of Green's function that we found by the numerical method for solving Green's function (as in the next section):

For example, the solution at the point (x_1, y_1) is:

$$U_{1,1} = \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} g(x_1, y_1; \xi_i, \eta_j) f(\xi_i, \eta_j)$$

In this paper we use $g(x, y; \xi, \eta)$ to denote the value of Green's function at each fixed point x, y with ξ, η as the corresponding dummy integration variables [7].

GREEN'S FUNCTION FOR THE TWO DIMENSIONAL HELMHOLTZ EQUATION

The inhomogeneous two dimensional Helmholtz partial differential equation is the equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \alpha^2 U = -f(x, y) \quad ; \quad a \leq x \leq b, \quad c \leq y \leq d \quad (2)$$

Where α is a parameter, U is the function of the two independent variables x, y whose solution is sought for and $f(x, y)$ is forcing function.

The Green's function of this equation is the solution of the following equation[3]:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \alpha^2 G = \delta(x - \xi) \delta(y - \eta) \quad ; \quad a \leq x, \xi \leq b, \quad c \leq y, \eta \leq d \quad (3)$$

Where δ is the two dimensional delta-dirac function[7].

We divide the square domain $\Omega = [a, b] \times [c, d]$ (sides of this square domain are parallel to the coordinate axes and its boundary is $\partial\Omega$) into n^2 equal square subdomains Ω_{ij} with boundary $\partial\Omega_{ij} = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, letting the grid spacing be h , then the number of the internal grid points is $(n-1)^2$, the coordinates of a typical internal grid points are $x_i = ih, y_j = jh, i = 1, 2, 3, \dots, (n-1), j = 1, 2, 3, \dots, (n-1)$

Integrating equation (3) in each subregion $\Omega_{ij}, \forall i = 1, 2, 3, \dots, (n-1), j = 1, 2, 3, \dots, (n-1)$ gives [1]:

$$\iint_{\Omega_{ij}} \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \alpha^2 G \right) dx dy = \iint_{\Omega_{ij}} \delta(x - \xi) \delta(y - \eta) dx dy \tag{4}$$

Using Green's theorem [4] gives:

$$- \int_{\Gamma_1 + \Gamma_3} \frac{\partial G}{\partial x} dy + \int_{-\Gamma_2 + \Gamma_4} \frac{\partial G}{\partial y} dx + \iint_{\Omega_{ij}} \alpha^2 G dx dy = \iint_{\Omega_{ij}} \delta(x - \xi) \delta(y - \eta) dx dy \tag{5}$$

Applying the mean-value theorem and the Dirac-delta function properties will lead to:

$$- G(x_i, y_{j-1}, \xi_i, \eta_{j-1}) - G(x_{i-1}, y_j, \xi_{i-1}, \eta_j) + (4 + \alpha^2 h^2) G(x_i, y_j, \xi_i, \eta_j) - G(x_{i+1}, y_j, \xi_{i+1}, \eta_{j-1}) - G(x_i, y_{j+1}, \xi_i, \eta_{j+1}) = \begin{cases} 1 & ; x_i = \xi_i \text{ and } y_j = \eta_j \\ 0 & ; x_i \neq \xi_i \text{ or } y_j \neq \eta_j \end{cases} \tag{6}$$

$\forall i = 1, 2, 3, \dots, (n-1), j = 1, 2, 3, \dots, (n-1)$

In matrix form: $AG = I$ where I is the identity matrix of order $(n-1)^2$, A and G are square matrices of order $(n-1)^2$ given by [7]:

$$A = \begin{bmatrix} B & -I & \Phi & \Phi & \Phi & \dots & \Phi \\ -I & B & -I & \Phi & \Phi & \dots & \Phi \\ \Phi & -I & B & -I & \Phi & \dots & \Phi \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \Phi & \dots & \Phi & \Phi & -I & B & -I \\ \Phi & \dots & \Phi & \Phi & \Phi & -I & B \end{bmatrix} \tag{7}$$

In the matrix A every element I is the identity matrix of order $(n-1)$, Φ is the zero matrix of order $(n-1)$ and B is a square matrix of order $(n-1)$ given by [7]:

$$B = \begin{bmatrix} (4 + \alpha^2 h^2) & -1 & 0 & 0 & \dots & 0 \\ -1 & (4 + \alpha^2 h^2) & -1 & 0 & \dots & 0 \\ 0 & -1 & (4 + \alpha^2 h^2) & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & (4 + \alpha^2 h^2) & -1 \\ 0 & \dots & 0 & 0 & -1 & (4 + \alpha^2 h^2) \end{bmatrix}$$

$$G = \begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} & \cdots & G_{1,(n-1)} \\ G_{2,1} & G_{2,2} & G_{2,3} & \cdots & G_{2,(n-1)} \\ G_{3,1} & G_{3,2} & G_{3,3} & \cdots & G_{3,(n-1)} \\ \vdots & \vdots & & \ddots & \vdots \\ G_{(n-1),1} & G_{(n-1),2} & G_{(n-1),3} & \cdots & G_{(n-1),(n-1)} \end{bmatrix}$$

$$G_{i,j} = \begin{bmatrix} g(x_1, y_i, \xi_1, \eta_j) & g(x_1, y_i, \xi_2, \eta_j) & g(x_1, y_i, \xi_3, \eta_j) & \cdots & g(x_1, y_i, \xi_{(n-1)}, \eta_j) \\ g(x_2, y_i, \xi_1, \eta_j) & g(x_2, y_i, \xi_2, \eta_j) & g(x_2, y_i, \xi_3, \eta_j) & \cdots & g(x_2, y_i, \xi_{(n-1)}, \eta_j) \\ g(x_3, y_i, \xi_1, \eta_j) & g(x_3, y_i, \xi_2, \eta_j) & g(x_3, y_i, \xi_3, \eta_j) & \cdots & g(x_3, y_i, \xi_{(n-1)}, \eta_j) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g(x_{(n-1)}, y_i, \xi_1, \eta_j) & g(x_{(n-1)}, y_i, \xi_2, \eta_j) & g(x_{(n-1)}, y_i, \xi_3, \eta_j) & \cdots & g(x_{(n-1)}, y_i, \xi_{(n-1)}, \eta_j) \end{bmatrix}$$

$$\forall i = 1, 2, 3, \dots, (n-1) \quad , \quad j = 1, 2, 3, \dots, (n-1)$$

SOLVING HELMHOLTZ EQUATION BY FINITE DIFFERENCE METHOD

In this section, we find the general solution of the two dimensional Helmholtz partial differential equation (2) by using the finite difference method.

Divid the square domain $\Omega = [a, b] \times [c, d]$ into n^2 equal square subdomains Ω_{ij} , letting the grid spacing be h , then the number of the internal grid points is $(n-1)^2$, the coordinates of a typical internal grid points are $x_i = ih, y_j = jh, i = 1, 2, 3, \dots, (n-1), j = 1, 2, 3, \dots, (n-1)$

And the function value of U at these grid points are denoted by $U_{i,j}$

Now, for the two dimensional Helmholtz partial differential equation (1) let us replace each second derivative with the usual finite difference approximation at the grid point $U_{i,j}$ except the point where the point source is acted then we get the following difference equation:

$$\begin{aligned} \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} + \alpha^2 U_{i,j} &= -f(x_i, y_j) \\ -U_{i+1,j} + 2U_{i,j} - U_{i-1,j} - U_{i,j+1} + 2U_{i,j} - U_{i,j-1} - \alpha^2 h^2 U_{i,j} &= h^2 f(x_i, y_j) \\ -U_{i,j-1} - U_{i-1,j} + (4 - \alpha^2 h^2) U_{i,j} - U_{i+1,j} - U_{i,j+1} &= h^2 f(x_i, y_j) \end{aligned} \quad (8)$$

$$\forall i = 1, 2, 3, \dots, (n-1) \quad , \quad j = 1, 2, 3, \dots, (n-1)$$

The above equations are equivalent to the following algebraic system:

$$AU = b$$

Where A is an order $(n-1)^2$ square matrix, U and b are $(n-1) \times 1$ column matrices, such that[7]:

$$A = \begin{bmatrix} B & -I & \Phi & \Phi & \Phi & \dots & \Phi \\ -I & B & -I & \Phi & \Phi & \dots & \Phi \\ \Phi & -I & B & -I & \Phi & \dots & \Phi \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \Phi & \dots & \Phi & \Phi & -I & B & -I \\ \Phi & \dots & \Phi & \Phi & \Phi & -I & B \end{bmatrix} \tag{9}$$

$$U^T = [U_{1,j} \quad U_{2,j} \quad \dots \quad U_{(n-2),j} \quad U_{(n-1),j}]$$

$$b^T = [h^2 f(x_1, y_j) \quad h^2 f(x_2, y_j) \quad \dots \quad h^2 f(x_{(n-2)}, y_j) \quad h^2 f(x_{(n-1)}, y_j)]$$

$\forall j = 1, 2, 3, \dots, (n-1)$

Where I the identity matrix of order $(n-1)$, Φ is the zero matrix of order $(n-1)$ and B is a square matrix of order $(n-1)$ given by [7]:

$$B = \begin{bmatrix} (4 - \alpha^2 h^2) & -1 & 0 & 0 & \dots & 0 \\ -1 & (4 - \alpha^2 h^2) & -1 & 0 & \dots & 0 \\ 0 & -1 & (4 - \alpha^2 h^2) & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & (4 - \alpha^2 h^2) & -1 \\ 0 & \dots & 0 & 0 & -1 & (4 - \alpha^2 h^2) \end{bmatrix}$$

ILLUSTRATIVE EXAMPLES

In this section, we give examples of the two dimensional Helmholtz partial differential equation, in each of them, we find the general solution by two different techniques. In the first example we will take $f(x, y) = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2})$ and $\alpha = 0$, then, we get an example on the Poisson's equation [5], in the second example we take $f(x, y) = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2})$ and $\alpha = 1$.

EXAMPLE

Consider the two dimensional Helmholtz partial differential equation (2) with forcing function $f(x, y) = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2})$ (two dimensional delta-dirac) and $\alpha = 0$:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\delta(x - \frac{1}{2})\delta(y - \frac{1}{2}) \quad ; \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Now, take $n = 4$ so $h = 0.25$

To find the general solution for this equation by the summation formula (1) we need to find the values of Green's function, The Green's function of this equation is the solution of the following equation:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x - \xi)\delta(y - \eta) \quad ; \quad 0 \leq x, \xi \leq 1, \quad 0 \leq y, \eta \leq 1$$

Then:

$$-G(x_i, y_{j-1}, \xi_i, \eta_{j-1}) - G(x_{i-1}, y_j, \xi_{i-1}, \eta_j) + 4G(x_i, y_j, \xi_i, \eta_j) \\ - G(x_{i+1}, y_j, \xi_{i+1}, \eta_{j-1}) - G(x_i, y_{j+1}, \xi_i, \eta_{j+1}) = \begin{cases} 1 & ; x_i = \xi_i \text{ and } y_j = \eta_j \\ 0 & ; x_i \neq \xi_i \text{ or } y_j \neq \eta_j \end{cases}$$

$$\forall i = 1, 2, 3, \quad j = 1, 2, 3$$

By matrix form: $AG = I$ where I is the identity matrix of order 9, A and G are square matrices of order 9 given by:

$$G = \begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} \\ G_{2,1} & G_{2,2} & G_{2,3} \\ G_{3,1} & G_{3,2} & G_{3,3} \end{bmatrix}$$

Where:

$$G_{i,j} = \begin{bmatrix} g(x_1, y_i, \xi_1, \eta_j) & g(x_1, y_i, \xi_2, \eta_j) & g(x_1, y_i, \xi_3, \eta_j) \\ g(x_2, y_i, \xi_1, \eta_j) & g(x_2, y_i, \xi_2, \eta_j) & g(x_2, y_i, \xi_3, \eta_j) \\ g(x_3, y_i, \xi_1, \eta_j) & g(x_3, y_i, \xi_2, \eta_j) & g(x_3, y_i, \xi_3, \eta_j) \end{bmatrix}$$

$$\forall i = 1, 2, 3, \quad j = 1, 2, 3$$

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}$$

By using the MATLAB programming we get:

$$G = A^{-1} = \begin{bmatrix} 0.2991 & 0.0982 & 0.0313 & 0.0982 & 0.0625 & 0.0268 & 0.0313 & 0.0268 & 0.0134 \\ 0.0982 & 0.3304 & 0.0982 & 0.0625 & 0.1250 & 0.0625 & 0.0268 & 0.0446 & 0.0268 \\ 0.0312 & 0.0982 & 0.2991 & 0.0268 & 0.0625 & 0.0982 & 0.0134 & 0.0268 & 0.0312 \\ 0.0982 & 0.0625 & 0.0268 & 0.3304 & 0.1250 & 0.0446 & 0.0982 & 0.0625 & 0.0268 \\ 0.0625 & 0.1250 & 0.0625 & 0.1250 & 0.3750 & 0.1250 & 0.0625 & 0.1250 & 0.0625 \\ 0.0268 & 0.0625 & 0.0982 & 0.0446 & 0.1250 & 0.3304 & 0.0268 & 0.0625 & 0.0982 \\ 0.0313 & 0.0268 & 0.0134 & 0.0982 & 0.0625 & 0.0268 & 0.2991 & 0.0982 & 0.0313 \\ 0.0268 & 0.0446 & 0.0268 & 0.0625 & 0.1250 & 0.0625 & 0.0982 & 0.3304 & 0.0982 \\ 0.0134 & 0.0268 & 0.0313 & 0.0268 & 0.0625 & 0.0982 & 0.0313 & 0.0982 & 0.2991 \end{bmatrix}$$

Now, we can find the general solution by the summation formula:

$$U(x, y) = \sum_{j=1}^3 \sum_{i=1}^3 g(x, y; \xi_i, \eta_j) f(\xi_i, \eta_j)$$

But $f(x, y) = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2})$, then:

$$U(x, y) = \sum_{j=1}^3 \sum_{i=1}^3 g(x, y; \xi_i, \eta_j) \delta(\xi_i - \frac{1}{2}) \delta(\eta_j - \frac{1}{2})$$

The term $g(x, y; \xi_i, \eta_j) \delta(\xi_i - \frac{1}{2}) \delta(\eta_j - \frac{1}{2})$ is called a weighted delta function and it is zero everywhere except at $\xi_i = \frac{1}{2} \& \eta_j = \frac{1}{2}$, where it has value $g(x, y; \frac{1}{2}, \frac{1}{2})$, so by using the properties of delta-dirac function, we get:

$$U(x, y) = g(x, y; \frac{1}{2}, \frac{1}{2})$$

$$U = \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = \begin{bmatrix} 0.0625 \\ 0.1250 \\ 0.0625 \\ 0.1250 \\ 0.3750 \\ 0.1250 \\ 0.0625 \\ 0.1250 \\ 0.0625 \end{bmatrix}$$

On the other hand, we are going to finding the general solution by another technique which is the finite difference method, again take $n = 4$ so $h = 0.25$.

In order to obtain the difference equation, we need to determine how to replace the forcing function:

$$\iint_{\Omega_{ij}} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dx dy = - \iint_{\Omega_{ij}} \delta(x - \frac{1}{2}) \delta(y - \frac{1}{2}) dx dy$$

By using Green's theorem we have:

$$- \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial U}{\partial y} dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial U}{\partial y} dx = \iint_{\Omega_{ij}} \delta(x - \frac{1}{2}) \delta(y - \frac{1}{2}) dx dy$$

$$- \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial U}{\partial y} dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial U}{\partial y} dx = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

Using the usual finite difference approximation to replace each derivative[8], at the nodal point $(x, y) = (\frac{1}{2}, \frac{1}{2})$:

$$h \left[-\frac{U_{i+1,j} - U_{i,j}}{h} + \frac{U_{i,j} - U_{i-1,j}}{h} + \frac{U_{i,j} - U_{i,j-1}}{h} - \frac{U_{i,j+1} - U_{i,j}}{h} \right] = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

$\forall i = 1,2,3, j = 1,2,3$

$$-U_{i+1,j} + U_{i,j} + U_{i,j} - U_{i-1,j} + U_{i,j} - U_{i,j-1} - U_{i,j+1} + U_{i,j} = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

$\forall i = 1,2,3, j = 1,2,3$

Then, we have the following difference equation:

$$-U_{i,j-1} - U_{i-1,j} + 4U_{i,j} - U_{i+1,j} - U_{i,j+1} = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

$$\forall i = 1, 2, 3, j = 1, 2, 3$$

In matrix form, we have: $A\hat{U} = b$

Where A is an order 9 square matrix and \hat{U} , b are 9 x1 column matrices, such that:

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}$$

$$\hat{U}^T = [\hat{U}_{1,1} \quad \hat{U}_{2,1} \quad \hat{U}_{3,1} \quad \hat{U}_{1,2} \quad \hat{U}_{2,2} \quad \hat{U}_{3,2} \quad \hat{U}_{1,3} \quad \hat{U}_{2,3} \quad \hat{U}_{3,3}]$$

$$b^T = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

So, by using the MATLAB programming we get:

$$\hat{U} = A^{-1}b$$

$$\hat{U} = \begin{bmatrix} 0.2991 & 0.0982 & 0.0313 & 0.0982 & 0.0625 & 0.0268 & 0.0313 & 0.0268 & 0.0134 \\ 0.0982 & 0.3304 & 0.0982 & 0.0625 & 0.1250 & 0.0625 & 0.0268 & 0.0446 & 0.0268 \\ 0.0312 & 0.0982 & 0.2991 & 0.0268 & 0.0625 & 0.0982 & 0.0134 & 0.0268 & 0.0312 \\ 0.0982 & 0.0625 & 0.0268 & 0.3304 & 0.1250 & 0.0446 & 0.0982 & 0.0625 & 0.0268 \\ 0.0625 & 0.1250 & 0.0625 & 0.1250 & 0.3750 & 0.1250 & 0.0625 & 0.1250 & 0.0625 \\ 0.0268 & 0.0625 & 0.0982 & 0.0446 & 0.1250 & 0.3304 & 0.0268 & 0.0625 & 0.0982 \\ 0.0313 & 0.0268 & 0.0134 & 0.0982 & 0.0625 & 0.0268 & 0.2991 & 0.0982 & 0.0313 \\ 0.0268 & 0.0446 & 0.0268 & 0.0625 & 0.1250 & 0.0625 & 0.0982 & 0.3304 & 0.0982 \\ 0.0134 & 0.0268 & 0.0313 & 0.0268 & 0.0625 & 0.0982 & 0.0313 & 0.0982 & 0.2991 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{U} = \begin{bmatrix} \hat{U}_{1,1} \\ \hat{U}_{2,1} \\ \hat{U}_{3,1} \\ \hat{U}_{1,2} \\ \hat{U}_{2,2} \\ \hat{U}_{3,2} \\ \hat{U}_{1,3} \\ \hat{U}_{2,3} \\ \hat{U}_{3,3} \end{bmatrix} = \begin{bmatrix} 0.0625 \\ 0.1250 \\ 0.0625 \\ 0.1250 \\ 0.3750 \\ 0.1250 \\ 0.0625 \\ 0.1250 \\ 0.0625 \end{bmatrix}$$

EXAMPLE

Again, consider the two dimensional Helmholtz partial differential equation (2) with forcing function $f(x, y) = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2})$ (two dimensional delta-dirac) but in this time with $\alpha = 1$:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + U = -\delta(x - \frac{1}{2})\delta(y - \frac{1}{2}) \quad ; \quad 0 \leq x \leq 1, 0 \leq y \leq 1$$

Take $n = 4$ so $h = 0.25$

To find the general solution for this equation by the summation formula (1) we need to find the values of Green's function, The Green's function of this equation is the solution of the following equation:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + G = \delta(x - \xi)\delta(y - \eta) \quad ; \quad 0 \leq x, \xi \leq 1, 0 \leq y, \eta \leq 1$$

Then:

$$-G(x_i, y_{j-1}, \xi_i, \eta_{j-1}) - G(x_{i-1}, y_j, \xi_{i-1}, \eta_j) + (4.0625)G(x_i, y_j, \xi_i, \eta_j)$$

$$-G(x_{i+1}, y_j, \xi_{i+1}, \eta_{j-1}) - G(x_i, y_{j+1}, \xi_i, \eta_{j+1}) = \begin{cases} 1 & ; x_i = \xi_i \text{ and } y_j = \eta_j \\ 0 & ; x_i \neq \xi_i \text{ or } y_j \neq \eta_j \end{cases}$$

$$\forall i = 1, 2, 3, \quad j = 1, 2, 3$$

And by matrix form: $AG = I$ where I is the identity matrix of order 9, A and G are square matrices of order 9 given by:

$$A = \begin{bmatrix} 4.0625 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4.0625 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4.0625 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4.0625 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4.0625 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4.0625 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4.0625 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4.0625 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4.0625 \end{bmatrix}$$

By using the MATLAB programming we get:

$$G = A^{-1} = \begin{bmatrix} 0.2921 & 0.0933 & 0.0289 & 0.0933 & 0.0579 & 0.0243 & 0.0289 & 0.0243 & 0.0120 \\ 0.0933 & 0.3210 & 0.0933 & 0.0579 & 0.1176 & 0.0579 & 0.0243 & 0.0409 & 0.0243 \\ 0.0289 & 0.0933 & 0.2921 & 0.0243 & 0.0579 & 0.0933 & 0.0120 & 0.0243 & 0.0289 \\ 0.0933 & 0.0579 & 0.0243 & 0.3210 & 0.1176 & 0.0409 & 0.0933 & 0.0579 & 0.0243 \\ 0.0579 & 0.1176 & 0.0579 & 0.1176 & 0.3619 & 0.1176 & 0.0579 & 0.1176 & 0.0579 \\ 0.0243 & 0.0579 & 0.0933 & 0.0409 & 0.1176 & 0.3210 & 0.0243 & 0.0579 & 0.0933 \\ 0.0289 & 0.0243 & 0.0120 & 0.0933 & 0.0579 & 0.0243 & 0.2921 & 0.0933 & 0.0289 \\ 0.0243 & 0.0409 & 0.0243 & 0.0579 & 0.1176 & 0.0579 & 0.0933 & 0.3210 & 0.0933 \\ 0.0120 & 0.0243 & 0.0289 & 0.0243 & 0.0579 & 0.0933 & 0.0289 & 0.0933 & 0.2921 \end{bmatrix}$$

Now, we can find the general solution by the summation formula:

$$U(x, y) = \sum_{j=1}^3 \sum_{i=1}^3 g(x, y; \xi_i, \eta_j) f(\xi_i, \eta_j)$$

But here $f(x, y) = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2})$ then:

$$U(x, y) = \sum_{j=1}^3 \sum_{i=1}^3 g(x, y; \xi_i, \eta_j) \delta(\xi_i - \frac{1}{2})\delta(\eta_j - \frac{1}{2})$$

The term $g(x, y; \xi_i, \eta_j) \delta(\xi_i - \frac{1}{2})\delta(\eta_j - \frac{1}{2})$ is called a weighted delta function and it is zero everywhere except at $\xi_i = \frac{1}{2}, \eta_j = \frac{1}{2}$, where it has value $g(x, y; \frac{1}{2}, \frac{1}{2})$, so by using the properties of delta-dirac function we get:

$$U(x, y) = g(x, y; \frac{1}{2}, \frac{1}{2})$$

$$U = \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = \begin{bmatrix} 0.0579 \\ 0.1176 \\ 0.0579 \\ 0.1176 \\ 0.3619 \\ 0.1176 \\ 0.0579 \\ 0.1176 \\ 0.0579 \end{bmatrix}$$

On the other hand, we are going to finding the general solution by another technique which is the finite difference method, again take $n = 4$ so $h = 0.25$.

As in the example (5-1) in order to obtain the difference equation, we need to determine how to replace the forcing function:

$$\iint_{\Omega_{ij}} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + U \right) dx dy = - \iint_{\Omega_{ij}} \delta(x - \frac{1}{2}) \delta(y - \frac{1}{2}) dx dy$$

By using Green's theorem we have:

$$- \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{x_{i+\frac{1}{2}}}^{x_{i-\frac{1}{2}}} \frac{\partial U}{\partial y} dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial U}{\partial y} dx - \iint_{\Omega_{ij}} U dx dy = \iint_{\Omega_{ij}} \delta(x - \frac{1}{2}) \delta(y - \frac{1}{2}) dx dy$$

$$- \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{y_{i+\frac{1}{2}}}^{y_{i-\frac{1}{2}}} \frac{\partial U}{\partial x} dy + \int_{x_{i+\frac{1}{2}}}^{x_{i-\frac{1}{2}}} \frac{\partial U}{\partial y} dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial U}{\partial y} dx - h^2 U_{i,j} = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

Using the usual finite difference approximation to replace each derivative, at the nodal point $(x, y) = (\frac{1}{2}, \frac{1}{2})$:

$$h \left[- \frac{U_{i+1,j} - U_{i,j}}{h} + \frac{U_{i,j} - U_{i-1,j}}{h} + \frac{U_{i,j} - U_{i,j-1}}{h} - \frac{U_{i,j+1} - U_{i,j}}{h} \right] - h^2 U_{i,j} = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

$$\forall i = 1,2,3, \quad j = 1,2,3$$

$$-U_{i+1,j} + U_{i,j} + U_{i,j} - U_{i-1,j} + U_{i,j} - U_{i,j-1} - U_{i,j+1} + U_{i,j} - h^2 U_{i,j} = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

$$\forall i = 1,2,3, \quad j = 1,2,3$$

Then we have the following difference equation:

$$-U_{i,j-1} - U_{i-1,j} + (4 - h^2)U_{i,j} - U_{i+1,j} - U_{i,j+1} = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

$$\forall i = 1,2,3, \quad j = 1,2,3$$

$$-U_{i,j-1} - U_{i-1,j} + (3.9375)U_{i,j} - U_{i+1,j} - U_{i,j+1} = \begin{cases} 1 & x = \frac{1}{2} \text{ and } y = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \text{ or } y \neq \frac{1}{2} \end{cases}$$

$$\forall i = 1,2,3, \quad j = 1,2,3$$

In matrix form, we have: $A\hat{U} = b$

Where A is an order 9 square matrix and \hat{U} , b are 9 x1 column matrices, such that:

$$A = \begin{bmatrix} 3.9375 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3.9375 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3.9375 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3.9375 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 3.9375 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3.9375 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 3.9375 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3.9375 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3.9375 \end{bmatrix}$$

$$\hat{U}^T = [\hat{U}_{1,1} \quad \hat{U}_{2,1} \quad \hat{U}_{3,1} \quad \hat{U}_{1,2} \quad \hat{U}_{2,2} \quad \hat{U}_{3,2} \quad \hat{U}_{1,3} \quad \hat{U}_{2,3} \quad \hat{U}_{3,3}]$$

$$b^T = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

So, by using the MATLAB programming we get:

$$\hat{U} = A^{-1}b$$

$$\Rightarrow \hat{U} = \begin{bmatrix} 0.3066 & 0.1037 & 0.0338 & 0.1037 & 0.0677 & 0.0296 & 0.0338 & 0.0296 & 0.0150 \\ 0.1037 & 0.3405 & 0.1037 & 0.0677 & 0.1333 & 0.0677 & 0.0296 & 0.0489 & 0.0296 \\ 0.0338 & 0.1037 & 0.3066 & 0.0296 & 0.0677 & 0.1037 & 0.0150 & 0.0296 & 0.0338 \\ 0.1037 & 0.0677 & 0.0296 & 0.3405 & 0.1333 & 0.0489 & 0.1037 & 0.0677 & 0.0296 \\ 0.0677 & 0.1333 & 0.0677 & 0.1333 & 0.3893 & 0.1333 & 0.0677 & 0.1333 & 0.0677 \\ 0.0296 & 0.0677 & 0.1037 & 0.0489 & 0.1333 & 0.3405 & 0.0296 & 0.0677 & 0.1037 \\ 0.0338 & 0.0296 & 0.0150 & 0.1037 & 0.0677 & 0.0296 & 0.3066 & 0.1037 & 0.0338 \\ 0.0296 & 0.0489 & 0.0296 & 0.0677 & 0.1333 & 0.0677 & 0.1037 & 0.3405 & 0.1037 \\ 0.0150 & 0.0296 & 0.0338 & 0.0296 & 0.0677 & 0.1037 & 0.0338 & 0.1037 & 0.3066 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \hat{U} = \begin{bmatrix} \hat{U}_{1,1} \\ \hat{U}_{2,1} \\ \hat{U}_{3,1} \\ \hat{U}_{1,2} \\ \hat{U}_{2,2} \\ \hat{U}_{3,2} \\ \hat{U}_{1,3} \\ \hat{U}_{2,3} \\ \hat{U}_{3,3} \end{bmatrix} = \begin{bmatrix} 0.0677 \\ 0.1333 \\ 0.0677 \\ 0.1333 \\ 0.3893 \\ 0.1333 \\ 0.0677 \\ 0.1333 \\ 0.0677 \end{bmatrix}$$

For all previous examples it is obvious that the numerical solutions strongly agreed in both techniques as shown in the tables below:

For the problem in the first example we have						
i	j	x_i	y_j	$U(x_i, y_j)$	$\widehat{U}(x_i, y_j)$	$U(x_i, y_j) - \widehat{U}(x_i, y_j)$
1	1	0.25	0.25	0.0625	0.0625	0
2	1	0.5	0.25	0.1250	0.1250	0
3	1	0.75	0.25	0.0625	0.0625	0
1	2	0.25	0.5	0.1250	0.1250	0
2	2	0.5	0.5	0.3750	0.3750	0
3	2	0.75	0.5	0.1250	0.1250	0
1	3	0.25	0.75	0.0625	0.0625	0
2	3	0.5	0.75	0.1250	0.1250	0
3	3	0.75	0.75	0.0625	0.0625	0
For the problem in the second example we have						
i	j	x_i	y_j	$U(x_i, y_j)$	$\widehat{U}(x_i, y_j)$	$U(x_i, y_j) - \widehat{U}(x_i, y_j)$
1	1	0.25	0.25	0.0579	0.0677	-0.0098
2	1	0.5	0.25	0.1176	0.1333	-0.0157
3	1	0.75	0.25	0.0579	0.0677	-0.0098
1	2	0.25	0.5	0.1176	0.1333	-0.0157
2	2	0.5	0.5	0.3619	0.3893	-0.0274
3	2	0.75	0.5	0.1176	0.1333	-0.0157
1	3	0.25	0.75	0.0579	0.0677	-0.0098
2	3	0.5	0.75	0.1176	0.1333	-0.0157
3	3	0.75	0.75	0.0579	0.0677	-0.0098

Observe that:

1. $U(x_i, y_j)$ is the numerical solution by the summation formula.
2. $\widehat{U}(x_i, y_j)$ is the numerical solution by the Finite Difference Method.

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