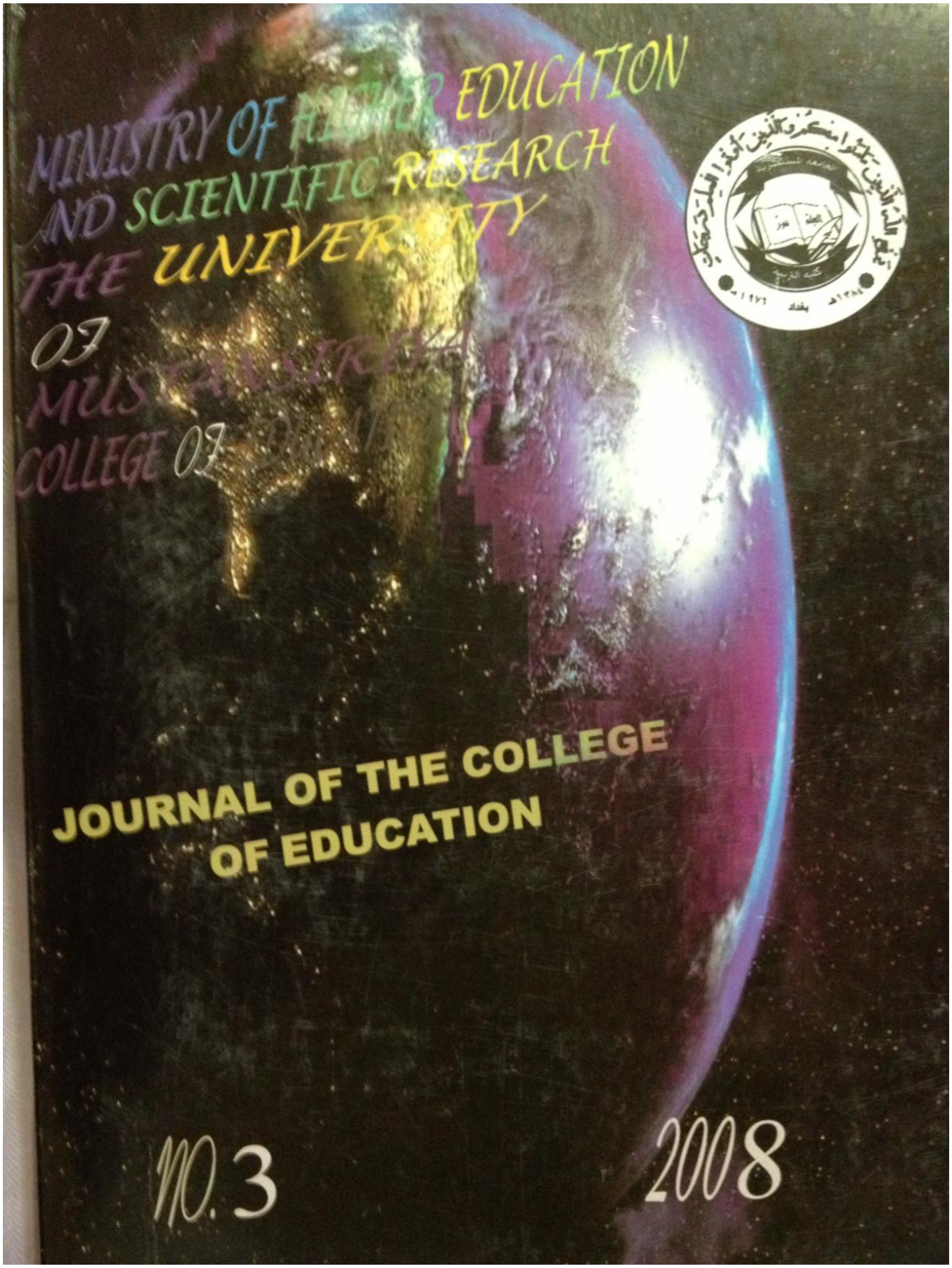
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Comparison of two numerical techniques for solving some boundary value problems

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Abstract:

In this work, we show how can we use the numerical Green's function obtained by the numerical method submitted by B.Fengsheng & L.Jiaqi (1987), for computing the general solution of the boundary value problems, which were covered by non-homogeneous linear ordinary differential equations of the second order with homogeneous boundary conditions.

K. Siu & Y.Billy (١٩٩٩) found the exact solution for some boundary value problems by the analytic fundamental solution (Green's function), and they found the numerical solution by the Finite Difference Method. We compare the results of the numerical solutions depending on numerical fundamental solution with those found by K. Siu & Y.Billy (١٩٩٩) and we find that they are identical.

المستخلص:

العديد من الكتب والبحوث تذكر الطريقة التحليلية التي تستخدم فيها الصيغة التحليلية للحل الأساس (دالة كرين) كإحدى الطرائق الجيدة لحساب الحل العام لمسائل القيم الحدودية التي يمكن إيجاد الحل الأساس (دالة كرين) لها.

العالمان B.Fengsheng & L. Jiaqi [1] عام (١٩٨٧) قدما طريقة عددية لحساب الحل الأساس(دالة كرين)،وطوّرها لتشمل المسائل ذات متغيرين الباحث حسين علي[11]عام (٢٠٠٣).

في هذا البحث تمّ توضيح كيفية حساب الحل العام عددياً لمسائل القيم الحدودية التي تحكمها معادلات تفاضلية اعتيادية خطية غير متجانسة من الرتبة الثانية مع شروط حدودية متجانسة باستخدام الحل الأساس (دالة كرين) الذي نستطيع حسابه بوساطة الطريقة العددية لحساب الحل الأساس (دالة كرين) التي تقدم بها العالمان المذكورين أعلاه وتم إعطاء مثال على ذلك.

كما يحتوي البحث على مقارنة للنتائج التي تم الحصول عليها مع تلك النتائج التي حصل عليها العالمان K. Siu & Y.Billy [2] عام (١٩٩٩) حيث قام هذان العالمان بحساب الحل العام بالاعتماد على الصيغة التحليلية للحل الأساس (دالة كرين) ومقارنته مع الحل بطريقة الفروقات المحددة، وعند المقارنة مع النتائج التي حصلوا عليها أتضح تطابق النتائج.

Key words:
the fundamental solution (Green's function), the exact solution of boundary value problems, the numerical solution depending on the numerical values of Green's function, Finite Difference Method, Dirac-Delta function.

1. Introduction:

Boundary value problems associated with either ordinary or partial differential equations are almost inescapable these days in mathematics and many of the applied sciences[6]. Green's function first introduced by George Green as early as 1828 [6], [4] which is associated with most boundary value problems and it is used to find the exact solution for those problems.

Green's function plays an important role in the solution of linear ordinary and partial differential equations, and are a key component to the development of the boundary integral equation methods[8]. And by Green's function many boundary value problems in theory of partial differential equations can be formulated in terms of integral equations[6].

It is no doubt that Green's function is a good method to find the exact solution for the differential problems with different input data[2].

Also, Green's function is used in the Boundary Element Method and the Generalized Boundary Element Method [10].

Therefore, many text books and papers devoted to Green's function and the analysis method to find this fundamental solution, some of those: [1], [5], [8], [7], [9].

In this paper, we will use two different methods, the numerical method for solving Green's function and the Finite Difference Method, to solve some boundary value problems, which were covered by non-homogeneous linear ordinary differential equations of the second order with homogeneous boundary conditions. Here we will use the numerical method for solving Green's function submitted by B.Fengsheng & L.Jiaqi (1987)[1] to solve the Green's function numerically and then we will use the results to find the solution of the boundary problems. We compare the results they are identical.

2. The numerical method for solving Green's function[1]:

In general, the analysis method for constructing Green's function (fundamental solution) is available only for one-dimensionary problem. It is more difficult and even impossible in the two-dimensions or higher-dimensional problems, especially, in complicated boundary condition.

It is therefore necessary to think of a numerical method. Here we will give the numerical method submitted by B. Fengsheng & L. Jiaqi (1987)[1]. The numerical method proposed here is the method first integration, and then discretization. To explain our statement, Consider Strum-Liouville equation:

$$L[U(x)] = -\frac{d}{dx}\left(p(x)\frac{dU(x)}{dx}\right) + q(x) = f(x) \qquad ; \quad a \le x \le b \quad \text{.....(1)}$$

$U(a) = 0$

$U(b) = 0$

The two point chains below shows the discrete interval $[a,b]$:

$\{x_i\}; \; i = 0,1,2,\ldots,n$

$\left\{x_{i-\frac{1}{2}}\right\}$ here $x_{i-\frac{1}{2}} = \frac{x_i + x_{i-1}}{2}$ .....(2)

that is a point chain of the mid-points in each subregion $[x_i, x_{i+1}]$; $i = 0,1,2,\ldots,(n-1)$.

$$a = x_0 < x_{1-\frac{1}{2}} < x_1 < x_{2-\frac{1}{2}} < x_2 < \ldots < x_{i-1} < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < x_{i+1} < \ldots < x_{n-\frac{1}{2}} < x_n = b$$

Equation (1) is integrated in each subregion $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $\forall\; i = 0,1,2,\ldots,n$ gives:

$$-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{d}{dx}\left(p(x)\frac{dU(x)}{dx}\right)dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x)U(x)dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)dx \quad \text{.....(3)}$$

Let:

$$W(x) = p(x)\frac{dU(x)}{dx} \quad \text{.....(4)}$$

Then (3) is reduced to:

$$W_{i-\frac{1}{2}} - W_{i+\frac{1}{2}} + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x)U(x)dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)dx \quad \text{.....(5)}$$

According to the mean-value theorem:

$$W_{i-\frac{1}{2}} - W_{i+\frac{1}{2}} + q_i U_i \frac{h_i + h_{i+1}}{2} = f_i \frac{h_i + h_{i+1}}{2} \quad \text{.....(6)}$$

Where: $q_i = q(x_i)$ is the mean-value of $q(x)$.

$U_i = U(x_i)$ is the mean-value of $U(x)$.

$f_i = f(x_i)$ is the mean-value of $f(x)$.

$h_i = x_i - x_{i-1}$ is the mean-value of $[x_{i-1}, x_i]$.

$h_{i+1} = x_{i+1} - x_i$ is the mean-value of $[x_i, x_{i+1}]$.

$$W_{i-\frac{1}{2}} = W(x_{i-\frac{1}{2}}).$$

$$W_{i+\frac{1}{2}} = W(x_{i+\frac{1}{2}}).$$

In terms of (4), we have:

$$\frac{W(x)}{p(x)} = \frac{dU(x)}{dx}$$

Integrating over $[x_{i-1}, x_i]$ for it at both sides:

$$\int_{x_{i-1}}^{x_i} \frac{dU(x)}{dx} dx = \int_{x_{i-1}}^{x_i} \frac{W(x)}{p(x)} dx$$

then:

$$U_i - U_{i-1} = \frac{W_{i-\frac{1}{2}} h_i}{p_{i-\frac{1}{2}}}$$

Therefore:

$$W_{i-\frac{1}{2}} = \frac{U_i - U_{i-1}}{h_i} P_{i-\frac{1}{2}} \quad \text{.......(7)}$$

In the same way:

$$W_{i+\frac{1}{2}} = \frac{U_{i+1} - U_i}{h_{i+1}} P_{i+\frac{1}{2}} \quad \text{.......(8)}$$

Where:

$$P_{i-\frac{1}{2}} = p(x_{i-\frac{1}{2}})$$

$$P_{i+\frac{1}{2}} = p(x_{i+\frac{1}{2}})$$

Substituting (7) and (8) into (6):

$$p_{i-\frac{1}{2}} \frac{U_i - U_{i-1}}{h_i} - p_{i+\frac{1}{2}} \frac{U_{i+1} - U_i}{h_{i+1}} + q_i U_i \frac{h_i + h_{i+1}}{2} = f_i \frac{h_i + h_{i+1}}{2}$$

Or equivalently:

$$\left(\frac{p_{i-\frac{1}{2}}}{h_i} - \frac{p_{i+\frac{1}{2}}}{h_{i+1}} + q_i \frac{h_i + h_{i+1}}{2} \right) U_i - \left(\frac{p_{i-\frac{1}{2}}}{h_i} \right) U_{i-1} - \left(\frac{p_{i+\frac{1}{2}}}{h_{i+1}} \right) U_{i+1} = f_i \frac{h_i + h_{i+1}}{2} \quad \text{.......(9)}$$

Where $i = 1,2,3,\ldots,(n-1)$, with (n-1)-order equations in which U_i are unknown is obtained and written in matrix form:

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$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & \cdots & 0 \\ . & & . & . & & & . \\ . & & & . & . & & . \\ . & & & . & . & . & . \\ . & & & & 0 & a_{n-1} & b_{n-1} \\ 0 & . & . & . & . & & \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ . \\ . \\ . \\ U_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ . \\ . \\ d_{n-1} \end{bmatrix}$$

Here:

$$a_i = -\frac{p_{i-\frac{1}{2}}}{h_i}$$

$$b_i = -a_i - a_{i+1} + q_i \frac{h_i + h_{i+1}}{2}$$

$$c_i = -\frac{p_{i+\frac{1}{2}}}{h_{i+1}}$$

$$d_i = f_i \frac{h_i + h_{i+1}}{2}$$

Green's function $G(x;\xi)$ is the solution of the equation:

$$-\frac{d}{dx}\left(p(x)\frac{dG(x;\xi)}{dx}\right) + q(x)G(x;\xi) = \delta\,(x-\xi) \qquad \text{.........(10)}$$

where $\delta(x-\xi)$ is the Dirac-delta function. Therefore $G(x;\xi)$ satisfying the following equation:

$$\left(\frac{p_{i-\frac{1}{2}}}{h_i} - \frac{p_{i+\frac{1}{2}}}{h_{i+1}} + q_i \frac{h_i + h_{i+1}}{2}\right)G(x_i;\xi_i) - \left(\frac{p_{i-\frac{1}{2}}}{h_i}\right)G(x_i;\xi_i) - \left(\frac{p_{i+\frac{1}{2}}}{h_{i+1}}\right)G(x_i;\xi_i) \qquad \text{.........(11)}$$

$$= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \delta\,(x_i - \xi_j)dx = \begin{cases} 0 & ;\ x_i \neq \xi_i \\ 1 & ;\ x_i = \xi_i \end{cases} \qquad \begin{matrix} i = 1,2,3,\dots,(n-1) \\ j = 1,2,3,\dots,(n-1) \end{matrix}$$

Or, in matrix form $A.G = I$ where:

$$A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & \cdots & 0 \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 0 & . & . & . & 0 & a_{n-1} & b_{n-1} \end{bmatrix}, \quad G = \begin{bmatrix} g(x_1,\xi_1) & g(x_1,\xi_2) & g(x_1,\xi_3) & \cdots & g(x_1,\xi_{n-1}) \\ g(x_2,\xi_1) & g(x_2,\xi_2) & g(x_2,\xi_3) & \cdots & g(x_2,\xi_{n-1}) \\ g(x_3,\xi_1) & g(x_3,\xi_2) & g(x_3,\xi_3) & \cdots & g(x_3,\xi_{n-1}) \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ g(x_{n-1},\xi_1) & g(x_{n-1},\xi_2) & g(x_{n-1},\xi_3) & \cdots & g(x_{n-1},\xi_{n-1}) \end{bmatrix}$$

and I is an order (9) identity matrix.

hence $G = A^{-1}$ ....(12)

The numerical solution of Green's function $G(x;\xi)$ is given by the following matrix:

$$G_0(x;\xi)=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & g(x_1,\xi_1) & g(x_1,\xi_2) & g(x_1,\xi_3) & \cdots & g(x_1,\xi_{n-1}) & 0 \\ 0 & g(x_2,\xi_1) & g(x_2,\xi_2) & g(x_2,\xi_3) & \cdots & g(x_2,\xi_{n-1}) & 0 \\ 0 & g(x_3,\xi_1) & g(x_3,\xi_2) & g(x_3,\xi_3) & \cdots & g(x_3,\xi_{n-1}) & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & g(x_{n-1},\xi_1) & g(x_{n-1},\xi_2) & g(x_{n-1},\xi_3) & \cdots & g(x_{n-1},\xi_{n-1}) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{.........(13)}$$

3. Computation of the solution by finite difference method:

We are going to use the method of finite difference to investigate the problem:

$$L[U(x)] = -\frac{d}{dx}\left(p(x)\frac{dU(x)}{dx}\right) + q(x) = f(x) \qquad ; \quad a \le x \le b$$

$$U(a) = 0$$

$$U(b) = 0$$

We divide the interval $[a,b]$ in to n equal parts, then the step size is $h=\frac{b-a}{n}$ and we have (n+1) node points. That is for each node point $x_j = jh$ where $j = 0,1,2,\ldots,n$ and let the corresponding functions value be:

$$U_j = U(x_j)$$

$$p_j = p(x_j)$$

$$p'_j = p'(x_j)$$

$$q_j = q(x_j) \qquad where\ j = 0,1,2,\ldots,n$$

Therefore, The problem can be approximated by:

$$-p_j\left(\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2}\right) - p'_j\left(\frac{U_{j-1} - U_{j+1}}{2h}\right) + q_j U_j = f_j \quad , j = 1,2,\ldots,n-1$$

$$U_0 = U(a) = 0$$

$$U_n = U(b) = 0 \qquad \text{.........(14)}$$

Then, We have the following difference equation:

$$\left(-p_j-\frac{h}{2}p'_j\right)U_{j-1}+\left(2p_j-h^2q_j\right)U_j+\left(-p_j+\frac{h}{2}p'_j\right)U_{j+1}=h^2 f_j \quad , j=1,2,\ldots,n-1 \quad \ldots.(15)$$

In matrix form, we have: $AU=b$

Where A is an order (n-1) square matrix, U and b are (n-1)x 1 column matrices, such that:

$$A=\begin{bmatrix} \beta_1 & \gamma_1 & 0 & \cdots & & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 \\ 0 & \alpha_3 & \beta_3 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & & 0 & \alpha_{n-1} & \beta_{n-1} \end{bmatrix}$$

$$where: \alpha_j=\left(-p_j-\frac{h}{2}p'_j\right), \beta_j=\left(2p_j-h^2q_j\right), \gamma_j=\left(-p_j+\frac{h}{2}p'_j\right), \quad j=1,2,\ldots,n-1$$

$$U^T=\left[U_1 \;\; U_2 \;\; \cdots \;\; U_{n-2} \;\; U_{n-1}\right]$$

$$b^T=\left[h^2 f_1 \;\; h^2 f_2 \;\; \cdots \;\; h^2 f_{n-2} \;\; h^2 f_{n-1}\right].$$

4. Computation of the solution by the numerical Green's function:

Let us consider the non-homogeneous boundary value problem with homogeneous boundary conditions, for example Strum-Liouville equation:

$$L[U(x)]=-\frac{d}{dx}\left(p(x)\frac{dU(x)}{dx}\right)+q(x)=f(x) \qquad ; \quad a\le x\le b$$

$$U(a)=0$$

$$U(b)=0$$

If the fundamental solution (Green's function $G(x;\xi)$) of the differential equation above is found, then the exact solution then is easy to be expressed in terms of fundamental solution, that is:

$$U(x)=\int_a^b G(x;\xi)\, f(\xi)\, d\xi \quad \ldots\ldots(16)$$

Where $G(x;\xi)$ in the analysis expression, and defined on the interval $a<\xi<b$

Now, by using the numerical method for solving Green's function, we will have the numerical values of Green's function defined at the points:

$$\xi_1,\xi_2,\xi_3,\ldots,\xi_{n-1}\in(a,b)$$

Therefore the solution of that boundary value problem can be expressed in terms of those numerical values of the fundamental solution(Green's function $G(x;\xi)$) by using the summation instead of the integral; that is:

$$\hat{U}(x)=\sum_{i=1}^{n-1} G(x;\xi_i)\, f(\xi_i) \quad \ldots\ldots(17)$$

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For example, if the matrix of the numerical solution of Green's function $G(x;\xi)$ that we found by the numerical method for solving Green's function (as in previous section) is:

$$G=\begin{bmatrix} g(x_1,\xi_1) & g(x_1,\xi_2) & g(x_1,\xi_3) & \cdots & g(x_1,\xi_{n-1}) \\ g(x_2,\xi_1) & g(x_2,\xi_2) & g(x_2,\xi_3) & \cdots & g(x_2,\xi_{n-1}) \\ g(x_3,\xi_1) & g(x_3,\xi_2) & g(x_3,\xi_3) & \cdots & g(x_3,\xi_{n-1}) \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ g(x_{n-1},\xi_1) & g(x_{n-1},\xi_2) & g(x_{n-1},\xi_3) & \cdots & g(x_{n-1},\xi_{n-1}) \end{bmatrix}$$

Then the solution of our boundary value problem at the point x_1 is:

$$\hat{U}(x_1)=\sum_{i=1}^{n-1} G(x_1;\xi_i)\, f(\xi_i)$$

When we use (17) it is important to note that if we take n large enough, then the solution converge to the exact solution given by (16).

In (17) the Green's function is used to find the numerical solution, which is compared with the numerical solution obtained by the other numerical method.

In this paper we compare the results of (17) with those found by the Finite Difference Method, which are found to be identical.

5. Illustrative examples:

In this section, we give examples of a boundary value problems, which were covered by non-homogeneous linear ordinary differential equation of the second order with homogeneous boundary conditions and use two different methods the numerical method for solving Green's function and the Finite Difference Method. Here we will use the numerical method for solving Green's function submitted by B.Fengsheng & L.Jiaqi (1987)[1] to solve the Green's function numerically and then we will use the results to find the solution of those examples.

(5-1) Example:

For the problem:

$$\frac{d^2U(x)}{dx^2}-U(x)=Sin(x) \qquad ; \quad 0\le x\le 1$$

$$U(0)=0 \qquad \text{.......................................(18)}$$

$$U(1)=0$$

The handbook [٣] shows that the analysis expression for Green's function to this problem is:

$$G(x;\xi)=\begin{cases}\dfrac{\sinh(\xi-1)\sinh(x)}{\sinh(1)} & ;\ 0\le\xi\le x\\[2ex] \dfrac{\sinh(x-1)\sinh(\xi)}{\sinh(1)} & ;\ x\le\xi\le 1\end{cases} \quad \text{.......(19)}$$

And by: $U(x)=\int_a^b G(x;\xi)\, f(\xi)\, d\xi$

We get the exact solution:

$$U(x)=\frac{Sinh(x)Sin(1)}{2Sinh(1)}-\frac{Sin(x)}{2}\ ;0\le x\le 1 \quad \text{.......(20)}$$

If we divide the interval [0,1] into n=10 equal subintervals, then the step size equals to $h=\frac{1}{n}=\frac{1}{10}=0.1$ and we have (n+1=11) node points.

That is, for each node point: $x_j=jh \ \ where\ j=0,1,2,...,10$

And let the corresponding function value be:

$U_j=U(x_j)\ \ where\ j=0,1,2,...,10$ Then from (20) we have:

$$U=\begin{bmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\\U_7\\U_8\\U_9\end{bmatrix}=\begin{bmatrix}-0.000128898\\-0.000250349\\-0.000356832\\-0.000440677\\-0.000493987\\-0.000508559\\-0.000475801\\-0.000386651\\-0.00023148\end{bmatrix}$$

Note that: $U(0)=U(10)=0$

Now, we will find the solution of (18) numerically by using (17):

$$\hat{U}(x)=\sum_{i=1}^{n-1}G(x;\xi_i)\, f(\xi_i)$$

Then we need only to find the numerical values of Green's function, so we will apply the numerical method for solving Green's function as bellow:

The interval [0,1] is divided into n=10 equal subintervals:

$[x_i, x_{i+1}] \quad ;\ i=1,2,3,...,9$

Consider the midpoints in each subinterval $[x_i,x_{i+1}]$; $i=0,1,2,...,9$ are:

$x_{i-\frac{1}{2}}=\frac{x_i+x_{i-1}}{2}\ ;\ i=1,2,...,9$

Now integrate the following equation on the subregion $[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]$:

$$\frac{d^2U(x)}{dx^2} - U(x) = \delta(x-\xi)$$

We have:

$$[U_x(x)]\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} - \frac{h_i + h_{i+1}}{2} U(x_i) = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \delta\,(x-\xi)dx$$

And suppose that: $U_x(x) = W(x)$

so, we have:

$$W(x_{i+\frac{1}{2}}) - W(x_{i-\frac{1}{2}}) - \frac{h_i + h_{i+1}}{2} U(x_i) = 1 \qquad ; \quad x = \xi$$

$$W(x_{i+\frac{1}{2}}) - W(x_{i-\frac{1}{2}}) - \frac{h_i + h_{i+1}}{2} U(x_i) = 0 \qquad ; \quad x \neq \xi \qquad \text{.......(21)}$$

We can find $W(x_{i+\frac{1}{2}})$ *and* $W(x_{i-\frac{1}{2}})$ to be:

$$W(x_{i-\frac{1}{2}}) = \frac{U_i - U_{i-1}}{h_i}$$

$$W(x_{i+\frac{1}{2}}) = \frac{U_{i+1} - U_i}{h_{i+1}}$$

And substitute them in (21) to give:

$$\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} - \frac{h_i + h_{i+1}}{2} U_i = 1 \qquad ; \quad x = \xi$$

$$\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} - \frac{h_i + h_{i+1}}{2} U_i = 0 \qquad ; \quad x \neq \xi \qquad \text{.......(22)}$$

And since $h_{i+1} = h_i = h = 0.1$; $i = 1, 2, 3, \ldots, 9$. So, we get:

$$(10)U_{i+1} - (20.1)\,U_i + (10)\,U_{i-1} = 1 \qquad ; \quad x = \xi$$

$$U_{i+1} - (2.01)\,U_i + U_{i-1} = 0 \qquad ; \quad x \neq \xi \qquad \text{.......(23)}$$

Or, in matrix form we have: $A\,G = I$

Where A is an order (9) square matrix, such that:

$$A = \begin{bmatrix} -20.1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & -20.1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & -20.1 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & -20.1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & -20.1 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & -20.1 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & -20.1 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & -20.1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -20.1 \end{bmatrix}$$

I is an order (9) identity matrix and G is matrix of the fundamental solutions. Then $G = A^{-1}$, which reads as follows:

$$G=\begin{bmatrix} -0.0874 & -0.0756 & -0.0646 & -0.0542 & -0.0443 & -0.0350 & -0.0259 & -0.0171 & -0.0085 \\ -0.0756 & -0.1519 & -0.1298 & -0.1089 & -0.0891 & -0.0703 & -0.0521 & -0.0344 & -0.0171 \\ -0.0646 & -0.1298 & -0.1962 & -0.1647 & -0.1348 & -0.1063 & -0.0788 & -0.0521 & -0.0259 \\ -0.0542 & -0.1089 & -0.1647 & -0.2222 & -0.1818 & -0.1433 & -0.1063 & -0.0703 & -0.0350 \\ -0.0443 & -0.0891 & -0.1348 & -0.1818 & -0.2307 & -0.1818 & -0.1348 & -0.0891 & -0.0443 \\ -0.0350 & -0.0703 & -0.1063 & -0.1433 & -0.1818 & -0.2222 & -0.1647 & -0.1089 & -0.0542 \\ -0.0259 & -0.0521 & -0.0788 & -0.1063 & -0.1348 & -0.1647 & -0.1962 & -0.1298 & -0.0646 \\ -0.0171 & -0.0344 & -0.0521 & -0.0703 & -0.0891 & -0.1089 & -0.1298 & -0.1519 & -0.0756 \\ -0.0085 & -0.0171 & -0.0259 & -0.0350 & -0.0443 & -0.0542 & -0.0646 & -0.0756 & -0.0874 \end{bmatrix}$$

Now, to find the general solution using these numerical values of Green's function we can use the equation:

$$\hat{U}(x)=\sum_{i=1}^{n-1} G(x;\xi_i)\, f(\xi_i)$$

For example to find the value of the general solution at the point x_1 we take the first raw of the matrix of the numerical values of Green's function G, and to find the value of the general solution at the point x_2 we take the second raw, exe… There for we have:

$$\hat{U}=\begin{bmatrix}\hat{U}_1\\ \hat{U}_2\\ \hat{U}_3\\ \hat{U}_4\\ \hat{U}_5\\ \hat{U}_6\\ \hat{U}_7\\ \hat{U}_8\\ \hat{U}_9\end{bmatrix}=\begin{bmatrix}-0.00258\\ -0.00500\\ -0.00713\\ -0.0088\\ -0.00987\\ -0.01016\\ -0.00951\\ -0.00772\\ -0.00462\end{bmatrix}$$

We are going to use the method of finite difference to investigate problem (18).

Under the consideration of problem (13), we divide the interval [0,1] into n equal parts, then the step size equal to $h=\frac{1}{n}$ and we have (n+1) node points.

That is, for each node point:

$$x_j = jh \quad where\ j = 0,1,2,\dots,n$$

And let the corresponding function value be:

$$U_i = U(x_j) \quad where\ j = 0,1,2,\dots,n$$

Therefore, problem (18) can be approximated by:

$$\frac{U_{j-1}-2U_j+U_{j+1}}{h^2}-U_j=Sin(x_j) \quad ; j=1,2,3,\dots,(n-1)$$

$$U_0=0$$

$$U_n=0$$

Then we could have:

$$U_{j-1}-(2+h^2)U_j+U_{j+1}=h^2 Sin(x_j) \quad ; j=1,2,3,\dots,(n-1)$$

So that we can form a matrix equation:

$$AU=b$$

Where A is an order (n-1) square matrix, U and b are (n-1)x 1 column matrices, such that:

$$A=\begin{bmatrix} -(2+h^2) & 1 & 0 & \cdots & & 0 \\ 1 & -(2+h^2) & 1 & 0 & \cdots & 0 \\ 0 & 1 & -(2+h^2) & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & 1 \\ 0 & & \cdots & 0 & 1 & -(2+h^2) \end{bmatrix}$$

$$U^T=[U_1 \quad U_2 \quad \cdots \quad U_{n-2} \quad U_{n-1}]$$

$$b^T=[h^2 Sin(x_1) \quad h^2 Sin(x_2) \quad \cdots \quad h^2 Sin(x_{n-2}) \quad h^2 Sin(x_{n-1})]$$

if we take n=10 then we get:

$$\tilde{U}=\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \\ \tilde{U}_4 \\ \tilde{U}_5 \\ \tilde{U}_6 \\ \tilde{U}_7 \\ \tilde{U}_8 \\ \tilde{U}_9 \end{bmatrix}=\begin{bmatrix} -0.0003 \\ -0.0005 \\ -0.0007 \\ -0.0009 \\ -0.0010 \\ -0.0010 \\ -0.0010 \\ -0.0008 \\ -0.0005 \end{bmatrix}$$

(5-2) <u>Example</u>:

By the same way for the following problem:

$$\frac{d^2U(x)}{dx^2}-U(x)=\begin{cases} 1 & ; \quad 0<x<\frac{1}{2} \\ -1 & ; \quad \frac{1}{2}<x<1 \end{cases}$$

$$U(0)=0$$

$$U(1)=0 \qquad \text{.......(24)}$$

The exact solution will be:

$$U(x)=\begin{cases}\dfrac{2\sinh(x)\cosh(\frac{1}{2})}{\sinh(1)}-\dfrac{\sinh(x)+\sinh(x-1)}{\sinh(1)}-1 & ; \quad 0<x<\frac{1}{2}\\[2ex] \dfrac{2\sinh(x-1)\cosh(\frac{1}{2})}{\sinh(1)}-\dfrac{\sinh(x)+\sinh(x-1)}{\sinh(1)}+1 & ; \quad \frac{1}{2}<x<1\end{cases} \qquad \text{..........(25)}$$

Therefore:

$$U=\begin{bmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\\U_7\\U_8\\U_9\end{bmatrix}=\begin{bmatrix}-0.019528539\\-0.029244188\\-0.029244189\\-0.01952854\\0\\0.01952854\\0.029244194\\0.029244191\\0.019528539\end{bmatrix}$$

To find the general solution using the numerical values of Green's function we can use the equation:

$$\hat{U}(x)=\sum_{i=1}^{n-1}G(x;\xi_i)\,f(\xi_i)$$

And because of:

$$f(x)=\begin{cases}1 & ; \quad 0<x<\frac{1}{2}\\ -1 & ; \quad \frac{1}{2}<x<1\end{cases}$$

Then we have:

$$\hat{U}(x)=\begin{cases}\displaystyle\sum_{i=1}^{9}G(x;\xi_i) & ; \quad 0<x<\frac{1}{2}\\[2ex] \displaystyle\sum_{i=1}^{9}G(x;\xi_i) & ; \quad \frac{1}{2}<x<1\end{cases}$$

So at n=10:

$$\hat{U} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \\ \hat{U}_3 \\ \hat{U}_4 \\ \hat{U}_5 \\ \hat{U}_6 \\ \hat{U}_7 \\ \hat{U}_8 \\ \hat{U}_9 \end{bmatrix} = \begin{bmatrix} -0.4126 \\ -0.7292 \\ -0.9532 \\ -1.0867 \\ 0 \\ 1.0867 \\ 0.9532 \\ 0.7292 \\ 0.4126 \end{bmatrix}$$

And by using the method of finite difference we can form a matrix equation:

$$AU = b$$

Where A is an order (n-1) square matrix, U and b are (n-1)x 1 column matrices, such that:

$$A = \begin{bmatrix} -(2+h^2) & 1 & 0 & \cdots & & 0 \\ 1 & -(2+h^2) & 1 & 0 & \cdots & 0 \\ 0 & 1 & -(2+h^2) & 0 & & \vdots \\ \vdots & & 0 & 0 & & 1 \\ 0 & \cdots & & 0 & 1 & -(2+h^2) \end{bmatrix}$$

$$U^T = [U_1 \;\; U_2 \;\; \cdots \;\; U_{n-2} \;\; U_{n-1}]$$

$$b^T = [h^2 \;\; h^2 \;\; \ldots \;\; h^2 \;\; 0 \;\; -h^2 \;\; \ldots \;\; -h^2 \;\; -h^2]$$

So at n=10 we get:

$$\tilde{U} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \\ \tilde{U}_4 \\ \tilde{U}_5 \\ \tilde{U}_6 \\ \tilde{U}_7 \\ \tilde{U}_8 \\ \tilde{U}_9 \end{bmatrix} = \begin{bmatrix} -0.004126 \\ -0.007292 \\ -0.009532 \\ -0.010867 \\ 0 \\ 0.010867 \\ 0.009532 \\ 0.007292 \\ 0.004126 \end{bmatrix}$$

(5-3) Example:

In the same way for the following problem:

$$\frac{d^2U(x)}{dx^2} - U(x) = \delta\left(x - \tfrac{1}{2}\right)$$

$$U(0) = 0$$

$$U(1) = 0$$

The exact solution is:

$$U(x)=\begin{cases}\dfrac{\sinh(-\frac{1}{2})\sinh(x)}{\sinh(1)} & ;\ 0<x<\frac{1}{2}\\[2ex] \dfrac{\sinh(x-1)\sinh(\frac{1}{2})}{\sinh(1)} & ;\ \frac{1}{2}<x<1\end{cases} \qquad \text{.........(27)}$$

$$\text{So: } U=\begin{bmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\\U_7\\U_8\\U_9\end{bmatrix}=\begin{bmatrix}-0.0444\\-0.0892\\-0.1350\\-0.1821\\-0.2310\\-0.1821\\-0.1350\\-0.0892\\-0.0444\end{bmatrix}$$

We can use the following equation to find the general solution by using the numerical values of Green's function:

$$\hat{U}(x)=\sum_{i=1}^{n-1} G(x;\xi_i)\,f(\xi_i)$$

And because of: $f(x)=\delta(x-\frac{1}{2})$

Then: $\hat{U}(x_i)=G(x_i;\frac{1}{2}) \quad ; \quad i=1,2,3,\ldots,9$

$$\text{So: } \hat{U}=\begin{bmatrix}\hat{U}_1\\\hat{U}_2\\\hat{U}_3\\\hat{U}_4\\\hat{U}_5\\\hat{U}_6\\\hat{U}_7\\\hat{U}_8\\\hat{U}_9\end{bmatrix}=\begin{bmatrix}-0.0443\\-0.0891\\-0.1348\\-0.1818\\-0.2307\\-0.1818\\-0.1348\\-0.0891\\-0.0443\end{bmatrix}$$

And by using the method of finite difference we can form a matrix equation : $AU=b$

Where A is an order (n-1) square matrix, U and b are (n-1)x 1 column matrices, such that:

$$A=\begin{bmatrix} -2(4+h^2) & (4-h^2) & 0 & 0 & \cdots & & 0 \\ (4-h^2) & -2(4+h^2) & (4-h^2) & 0 & \cdots & & 0 \\ 0 & (4-h^2) & -2(4+h^2) & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & & (4-h^2) \\ 0 & & \cdots & & 0 & (4-h^2) & -2(4+h^2) \end{bmatrix}$$

$$U^T=[U_1 \ \ U_2 \ \ \cdots \ \ U_{n-2} \ \ U_{n-1}]$$

$$b^T=[0 \ \ 0 \ \ \ldots \ \ 0 \ \ 4h \ \ 0 \ \ \ldots \ \ 0 \ \ 0]$$

At n=10 we have:

$$\tilde{U}=\begin{bmatrix}\tilde{U}_1\\ \tilde{U}_2\\ \tilde{U}_3\\ \tilde{U}_4\\ \tilde{U}_5\\ \tilde{U}_6\\ \tilde{U}_7\\ \tilde{U}_8\\ \tilde{U}_9\end{bmatrix}=\begin{bmatrix}-0.0444\\ -0.0893\\ -0.1351\\ -0.1823\\ -0.2312\\ -0.1823\\ -0.1351\\ -0.0893\\ -0.0444\end{bmatrix}$$

6. Conclusions:

For all previous examples we have $\hat{U}(x)$ and $\tilde{U}(x)$ are identical as shown in the tables below:

For the problem in example (5-1) we have						
i	x_i	$U(x_i)$	$\hat{U}(x_i)$	$\tilde{U}(x_i)$	$U(x_i)-\hat{U}(x_i)$	$U(x_i)-\tilde{U}(x_i)$
1	0.1	-0.000128898	-0.000258	-0.0003	0.129102×10^{-3}	0.1291×10^{-3}
2	0.2	-0.000250349	-0.000500	-0.0005	0.249651×10^{-3}	0.2497×10^{-3}
3	0.3	-0.000356832	-0.000713	-0.0007	0.356168×10^{-3}	0.3562×10^{-3}
4	0.4	-0.000440677	-0.00088	-0.0009	0.439323×10^{-3}	0.4393×10^{-3}
5	0.5	-0.000493987	-0.000987	-0.0010	0.493013×10^{-3}	0.4930×10^{-3}
6	0.6	-0.000508559	-0.001016	-0.0010	0.507441×10^{-3}	0.5074×10^{-3}
7	0.7	-0.000475801	-0.000951	-0.0010	0.475199×10^{-3}	0.4752×10^{-3}
8	0.8	-0.000386651	-0.000772	-0.0008	0.385349×10^{-3}	0.3853×10^{-3}
9	0.9	-0.00023148	-0.000462	-0.0005	0.23052×10^{-3}	0.2305×10^{-3}

For the problem in example (5-2) we have						
i	x_i	$U(x_i)$	$\hat{U}(x_i)$	$\tilde{U}(x_i)$	$U(x_i)-\hat{U}(x_i)$	$U(x_i)-\tilde{U}(x_i)$
1	0.1	−0.019528539	−0.004126	−0.004126	0.15402539×10^{-1}	0.15402539×10^{-1}
2	0.2	−0.029244188	−0.007292	−0.007292	0.21952188×10^{-1}	0.21952188×10^{-1}
3	0.3	−0.029244189	−0.009532	−0.009532	0.19712189×10^{-1}	0.19712189×10^{-1}
4	0.4	−0.01952854	−0.010867	−0.010867	0.0866154×10^{-1}	0.0866154×10^{-1}
5	0.5	0	0	0	.	.
6	0.6	0.01952854	0.010867	0.010867	0.0866154×10^{-1}	0.0866154×10^{-1}
7	0.7	0.029244194	0.009532	0.009532	0.019712194×10^{-1}	0.019712194×10^{-1}
8	0.8	0.029244191	0.007292	0.007292	0.021952191×10^{-1}	0.021952191×10^{-1}
9	0.9	0.019528539	0.004126	0.004126	0.015402539×10^{-1}	0.015402539×10^{-1}

For the problem in example (5-3) we have						
i	x_i	$U(x_i)$	$\hat{U}(x_i)$	$\tilde{U}(x_i)$	$U(x_i)-\hat{U}(x_i)$	$U(x_i)-\tilde{U}(x_i)$
1	0.1	-0.0444	-0.0443	-0.0444	-0.1×10^{-3}	0
2	0.2	-0.0892	-0.0891	-0.0893	-0.1×10^{-3}	0.1×10^{-3}
3	0.3	-0.1350	-0.1348	-0.1351	-0.2×10^{-3}	0.1×10^{-3}
4	0.4	-0.1821	-0.1818	-0.1823	-0.3×10^{-3}	0.2×10^{-3}
5	0.5	-0.2310	-0.2307	-0.2312	-0.3×10^{-3}	0.2×10^{-3}
6	0.6	-0.1821	-0.1818	-0.1823	-0.3×10^{-3}	0.2×10^{-3}
7	0.7	-0.1350	-0.1348	-0.1351	-0.2×10^{-3}	0.1×10^{-3}
8	0.8	-0.0892	-0.0891	-0.0893	-0.1×10^{-3}	0.1×10^{-3}
9	0.9	-0.0444	-0.0443	-0.0444	-0.1×10^{-3}	0

Observe that:

1. U(x) is the exact solution at the point x.
2. $\hat{U}(x)$ is the numerical solution depending on the numerical values of Green's function .
3. $\tilde{U}(x)$ is the numerical solution by the Finite Difference Method.
4. $U(x_i)-\hat{U}(x_i)$ is the error between U(x) and $\hat{U}(x)$.
5. $U(x_i)-\tilde{U}(x_i)$ is the error between U(x) and $\tilde{U}(x)$.

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References:

[1] B. Fengsheng , L. Jiaqi; **A new numerical algorithm for solving fundamental solutions,** Springer-Verlay,Berlin (1987),321-335.

[2] K.Sui,Y.Billy;**Elliptic problems with concentrated loading,**Internet (1999).

[3] A. G. Butkovskiy ; **Green's functions and transfer functions handbook,** John Wiley & Sons (1982).

[4] A.Haurd, B.Irl,D.Stephen;**Calculus,**John Wiley & Sons,Inc.,(2002).

[5] E. M. Harrell , J. V. Herod; **Linear methods of applied mathematics,** Internet (1996),1-26.

[6] G. F. Roach; **Green's functions,** Cambridge University Press(1982).

[7] M. D. Greenberg ; **Application of Green's functions in science and engineering ,** Prentice-Hall , Inc. Englewood Cliffs , New Jersey(1971).

[8] N. I. S. T. ; **Introduction to Green's functions,** National institute of standards and technology, (2001),1-8.

[9] P. Duchateau , D. W. Zachmann; **Partial differential equations,** Schaum's outline series in mathematics , McGraw-Hill book company (1986).

[10] Huda Abdul-Satar Al-Kenany; **Solution of non linear partial differential equations by Generalized Boundary Element method,** Al-Mustansiriya University, college of Science, M.Sc. thesis(2002).

[11] Hussien Ali Hussien; **Numerical method for computation of the fundamental solution for some boundary value problems,** Al-Mustansiriya University, college of Science, M.Sc. thesis(2003).