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A Class of Janowski-Type (p, q)-Convex Harmonic Functions Involving a Generalized q-Mittag–Leffler Function

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Abstract: This research aims to present a linear operator $\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f$ utilizing the q-Mittag–Leffler function. Then, we introduce the subclass of harmonic (p, q)-convex functions $\mathcal{HT}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ related to the Janowski function. For the harmonic p-valent functions f class, we investigate the harmonic geometric properties, such as coefficient estimates, convex linear combination, extreme points, and Hadamard product. Finally, the closure property is derived using the subclass $\mathcal{HT}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ under the q-Bernardi integral operator.

Keywords: harmonic p-valent functions; the q-Mittag–Leffler function; (p, q)-convex functions; extreme points; Hadamard product; closed convex hulls; q-Bernardi integral operator

MSC: 05A30; 30C45; 11B65; 47B38

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1. Main Concepts of Quantum Calculus

Quantum calculus, often known as q-calculus (or q-analysis), is a method for studying calculus that is similar to traditional calculus but focused on finding q-analogous conclusions without the need for limits. The q-differential equations are generally defined on the scale T_q, where T and q are the time and scale index, respectively. Euler and Jacobi devised the fundamental formulae of q-calculus in the eighteenth century. Jackson ([1,2]) introduced and developed the concepts of q-derivative and q-integral. Moreover, the geometries of q-analysis were found in many studies presented on quantum groups. It has also been identified that there is a relationship between q-integral and q-derivative. With the expansion of the q-calculus study, many relevant facts have also been explored, including the q-Gamma and q-Beta functions, the q-Laplace transform, and the q-Mittag–Leffler function. The theory of q-calculus operators has been recently applied in the areas of ordinary fractional calculus, optimal control problems, finding solutions to the q-difference and q-integral equations, and q-transform analysis (see [3,4]). Furthermore, certain classes of functions that are analytic in $\mathbb{U}$ using fractional q-calculus operators were investigated by numerous research (for example, see [5–12]).

This paper aims to further develop the theory of fractional q-calculus operators in geometric function theory. Initially, this study provides some essential definitions and concepts of q-calculus and symmetric q-calculus, which have been employed in this research.

This work begins with the basic concepts and, consequently, an in-depth analysis of our proposed applications of the q-calculus. Throughout this paper, assume that $0 < q < 1$. The following definitions provide an introduction to the q-calculus operators for a complex-valued function f:

Let $\mathcal{S}(p)$ be the class of analytic and multivalent functions f in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized form:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, \quad (p \in \mathbb{N}). \tag{1}$$

Definition 1. *For $0 < q < 1$, the q-number $[\kappa]_q$ is expressed by*

$$[\kappa]_q := \begin{cases} \dfrac{1-q^\kappa}{1-q} & (\kappa \in \mathbb{C}) \\ \sum\limits_{\kappa=0}^{n-1} q^j & (\kappa = n \in \mathbb{N}). \end{cases}$$

Definition 2 ([1]). *The q-derivative operator $\mathfrak{D}_q$ is given by*

$$\mathfrak{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \ (z \neq 0).$$

The q-derivative of the function f in (1) is given by

$$\mathfrak{D}_q f(z) = [p]_q z^{p-1} + \sum_{j=p+1}^{\infty} [j]_q a_j z^{j-1}.$$

The q-factorial indicated by $[j]_q!$ is defined by

$$[j]_q! = \begin{cases} [j]_q[j-1]_q \ldots [2]_q[1]_q, & j = 1, 2, 3, \ldots, \\ 1 & j = 0, \end{cases}$$

so that

$$\begin{aligned} f'(z) &:= \lim_{q \to 1^-} \mathfrak{D}_q \left\{ [p]_q z^{p-1} + \sum_{j=p+1}^{\infty} [j]_q a_j z^{j-1} \right\} \\ &= p z^{p-1} + \sum_{j=p+1}^{\infty} j a_j z^{j-1}. \end{aligned}$$

The q-Gamma function is defined by

$$\Gamma_q(\varrho) = (1-q)^{1-\varrho} \prod_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q^{k+\varrho}} = (1-q)^{1-\varrho} \frac{(q;q)_\infty}{(q^\varrho;q)_\infty},$$

where $(\varrho; k)_q$ the q-Pochhammer is given as

$$(\varrho; k)_q = (\varrho)_q(\varrho+1)_q(\varrho+2)_q \cdots (\varrho+k-1)_q = \frac{(\varrho;q)_n}{(1-q)^n}, \quad (\varrho \in \mathbb{R}, k \in \mathbb{N}).$$

Obviously,

$$\Gamma_q(\varrho+1) = [\varrho]_q \Gamma_q(\varrho) \text{ and } \Gamma_q(1) = 1.$$

In the following section, we have introduced some concepts of harmonic p-valent functions and the Mittag–Leffler function. Then, we have derived a number interesting results regarding p-valent functions related to the operator $\mathcal{L}_{p,q}^{\rho,\sigma,\mu} f(z)$. Furthermore, this paper demonstrates some of the geometric results of the operator $\mathcal{L}_{p,q}^{\rho,\sigma,\mu} f(z)$.

2. Harmonic Functions, Definitions and Motivation

In the complex domain $\mathcal{D} \subset \mathbb{U}$, if the values u and v are real harmonic, then the continuous function $f = u + iv$ is called the harmonic function in $\mathcal{D}$. In any simply connected domain $\mathcal{D}$, the function f can be stated by

$$f = \mathcal{F} + \overline{\mathcal{G}}, \tag{2}$$

where both $\mathcal{F}$ and $\mathcal{G}$ are analytic functions in $\mathcal{D}$. The function $\mathcal{F}$ is called analytic of f, and $\mathcal{G}$ the conjugate-analytic (or co-analytic) of f. Clunie and Sheil-Small [13] discovered that $|\mathcal{F}'(z)| > |\mathcal{G}'(z)|$ is a necessary and sufficient condition for the harmonic functions (2) to be locally multivalent and sense-preserving in $\mathcal{D}$ (also, see [14]).

Let $\mathcal{H}(p,j)$ be the family of harmonic multivalent functions $f = \mathcal{F} + \overline{\mathcal{G}}$ that are orientation keeping the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. The analytic functions $\mathcal{F}$ and $\mathcal{G}$ are defined by

$$\mathcal{F} = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad \text{and} \quad \mathcal{G} = \sum_{j=p}^{\infty} d_j z^j$$

and

$$f = \mathcal{F} + \overline{\mathcal{G}} = z^p + \sum_{j=p+1}^{\infty} a_j z^j + \sum_{j=p}^{\infty} \overline{d_j z^j}, \tag{3}$$

where $p \geq 1$ and $|d_p| < 1$.

The family $\mathcal{H}(1,j) = \mathcal{H}(j)$ of harmonic univalent functions is presented by Jahangiri et al. [15] (also see [16–21]).

Furthermore, we consider the subclass $\widetilde{\mathcal{H}}(p,j)$ of the family $\mathcal{H}(p,j)$ that consists of functions $f = \mathcal{F} + \overline{\mathcal{G}}$, where the functions $\mathcal{F}$ and $\mathcal{G}$ are defined as below:

$$\mathcal{F}(z) = z^p - \sum_{j=p+1}^{\infty} |a_j| z^j \quad \text{and} \quad \mathcal{G}(z) = -\sum_{j=p}^{\infty} |d_j| z^j, \quad (|d_p| < 1). \tag{4}$$

Recently, many studies have emphasized the concept of p-valent harmonic functions and their applications (for example, see [22–26]).

If the analytic functions $f, h \in \mathcal{H}(p,j)$, then the function f is subordinate to the function h, denoted by ($f \prec h$), if there exists a Schwarz function Φ with

$$\Phi(0) = 0, \; |\Phi(z)| < 1, \quad (z \in \mathbb{U}),$$

such that

$$f(z) = h(\Phi(z)).$$

In addition, we get the following equivalence if the function h is univalent in $\mathbb{U}$:

$$f(z) \prec h(z) \Leftrightarrow f(0) = h(0) \text{ and } f(\mathbb{U}) \subset h(\mathbb{U}).$$

We now mention the well-known Mittag–Leffler function $E_\sigma(z)$ provided by Mittag–Leffler [27], which is defined by

$$E_\sigma(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\sigma j + 1)}, \quad (\sigma, z \in \mathbb{C}, \mathcal{R}(\sigma) > 0),$$

where $\mathcal{R}, \Gamma$ are the real part and the gamma function, respectively.

Within chaotic, stochastic, and dynamic systems, partial differential equations, and statistical distributions, many considerations can be seen in applying this function. Wiman [28] defined the Mittag–Leffler function with two parameters

$$E_{\sigma,\mu}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\sigma j + \mu)}, \quad (\sigma, \mu, z \in \mathbb{C}, \mathcal{R}(\sigma) > 0, \mathcal{R}(\mu) > 0).$$

Shukla and Prajapati [29] provided the Mittag–Leffler function with three parameters $E^{\rho,k}_{\sigma,\mu}(z)$ as follows:

$$E^{\rho,k}_{\sigma,\mu}(z) = \sum_{j=0}^{\infty} \frac{(\rho)_{kj}}{\Gamma(\mu + \sigma j)} \frac{z^j}{j!}, \quad (\sigma, \mu, \rho, z \in \mathbb{C}, \mathcal{R}(\sigma) > 0, \mathcal{R}(\mu) > 0, \mathcal{R}(\rho) > 0),$$

where $k \in (0,1) \bigcup \mathbb{N}$ and $(\rho)_{kj} = \frac{\Gamma(\rho+kj)}{\Gamma(\rho)}$ is the generalized Pochhammer symbol.

The Mittag–Leffler function plays a vital role in solving fractional order differential and integral equations. It has recently become a subject of rich interest in the field of fractional calculus and its applications. Numerous research has been conducted on the theory of the Mittag–Leffler function. For more review, Bansal and Prajapat [30] (also Srivastava and Bansal [31]) investigated geometric properties of the Mittag–Leffler function $E_{\sigma,\mu}(z)$. In addition, many other researchers studied properties of the Mittag–Leffler function, including starlikeness, convexity, and differential subordination (see [32–35]). In the fact, the generalized Mittag–Leffler function $E_{\sigma,\mu}(z)$ is still vastly used in geometric function theory and a variety of applications (see [36]).

Hadi et al. [37] defined a generalized q-Mittag–Leffler function with three parameters as below:

$$\mathcal{E}^{\rho}_{\sigma,\mu}(q;z) = z + \sum_{j=2}^{\infty} \frac{(\rho)_{kj}}{\Gamma_q(\mu + \sigma j)} \frac{z^j}{j!}, \quad (\sigma, \mu, \rho \in \mathbb{C}, \mathcal{R}(\sigma) > 0, \mathcal{R}(\mu) > 0, \mathcal{R}(\rho) > 0). \tag{5}$$

We note that if $q \to 1^-$, we have the Mittag–Leffler function defined by Shukla and Prajapati [29].

Motivated by the importance of studying the applications of quantum calculus and the Mittag–Leffler function in the physical and mathematical sciences, we first present a new linear operator $\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f$, which is defined by the q-Mittag–Leffler function with harmonic p-valent functions. Then, we use this operator to introduce a subclass of Janowski (p,q)-convex harmonic functions. For the harmonic p-valent functions f, we investigate some harmonic geometric properties, including coefficient estimates, convex hulls, convex linear combination, extreme point, and Hadamard product. Furthermore, we derive the closure property under the q-Bernardi integral operator.

Now, we introduce the function $\mathcal{M}^{\rho}_{\sigma,\mu}(p,q;z) \in \mathcal{S}(p)$ related to the q-Mittag–Leffler function in (5) as follows:

$$\begin{aligned} \mathcal{M}^{\rho}_{\sigma,\mu}(p,q;z) &= \frac{\Gamma_q(\sigma+\mu) z^{p-1}}{(\rho)_k} \left(\mathcal{E}^{\rho}_{\sigma,\mu}(q;z) - \frac{1}{\Gamma_q(\mu)} \right) \\ &= z^p + \sum_{j=p+1}^{\infty} \frac{\Gamma_q(\sigma+\mu)\Gamma(\rho+jk)}{\Gamma_q(\sigma j+\mu)\Gamma(\rho+k) j!} z^j, \quad (p \geq 1; z \in \mathbb{U}), \end{aligned} \tag{6}$$

where $\sigma, \mu, \rho \in \mathbb{C}, \mathcal{R}(\sigma) > 0, \mathcal{R}(\mu) > 0, \mathcal{R}(\rho) > 0,$ and $(\rho)_{kj} = \frac{\Gamma(\rho+kj)}{\Gamma(\rho)}$.

From the function $\mathcal{M}^{\rho}_{\sigma,\mu}(p,q;z)$, we define a linear operator $\mathcal{L}^{\rho,\sigma,\mu}_{p,q} : \mathcal{S}(p) \to \mathcal{S}(p)$ as follows:

$$\begin{aligned}\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z) &= \mathcal{M}^{\rho}_{\sigma,\mu}(p,q;z) * f(z) \\ &= z^p + \sum_{j=p+1}^{\infty} \frac{\Gamma_q(\sigma+\mu)\Gamma(\rho+jk)}{\Gamma_q(\sigma j+\mu)\Gamma(\rho+k)j!} a_j z^j.\end{aligned} \tag{7}$$

When $q \to 1^-, k = 1, \sigma = 0,$ and $\rho = 1$, then $\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z) = f(z)$.

Remark 1. *If $\mathcal{R}(\sigma) > \max\{0, \mathcal{R}(k) - 1\}$ and $\mathcal{R}(k) > 0$, the following operators are obtained*

1. *If $q \to 1^-$, we find the operator $\mathcal{L}^{\rho,\sigma,\mu}_{p} f(z)$ investigated by Xu and Liu [38].*
2. *When $q \to 1^-$ and $p = 1$, we find the operator $\mathcal{L}^{\rho,\mu}_{\sigma} f(z)$ investigated by Attiya [39].*

For $f \in \mathcal{H}(p,j)$ in the form (3), we define the operator $\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f$ as follows:

$$\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z) = \mathcal{L}^{\rho,\sigma,\mu}_{p,q} \mathcal{F}(z) + \overline{\mathcal{L}^{\rho,\sigma,\mu}_{p,q} \mathcal{G}(z)}, \tag{8}$$

where

$$\mathcal{L}^{\rho,\sigma,\mu}_{p,q} \mathcal{F}(z) = z^p + \sum_{j=p+1}^{\infty} \psi_q a_j z^j,$$

$$\mathcal{L}^{\rho,\sigma,\mu}_{p,q} \mathcal{G}(z) = \sum_{j=p}^{\infty} \psi_q d_j z^j$$

and $\psi_q = \frac{\Gamma_q(\sigma+\mu)\Gamma(\rho+jk)}{\Gamma_q(\sigma j+\mu)\Gamma(\rho+k)j!}$.

We define the class $\mathcal{HT}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ using the operator $\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f$ in (8) as follows:

Definition 3. *A multivalent functions $f = \mathcal{F} + \overline{\mathcal{G}} \in \mathcal{H}(p,j)$ is said to be in the class $\mathcal{HT}_{p,q}$ $(\vartheta,\mathcal{W},\mathcal{V})$ if*

$$\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))}{\mathfrak{D}_q(\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))} \prec ([p]_q - \vartheta)\frac{1+\mathcal{W}z}{1+\mathcal{V}z} + \vartheta, \tag{9}$$

or equivalently

$$\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))}{\mathfrak{D}_q(\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))} \prec \frac{[p]_q + \{(\mathcal{W}-\mathcal{V})([p]_q - \vartheta) + \mathcal{V}[p]_q\}z}{1+\mathcal{V}z}.$$

Utilizing of the subordination principle, $f \in \mathcal{HT}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ if and only if there is a Schwarz function φ such that

$$\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))}{\mathfrak{D}_q(\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))} = \frac{[p]_q + \{(\mathcal{W}-\mathcal{V})([p]_q - \vartheta) + \mathcal{V}[p]_q\}\varphi(z)}{1+\mathcal{V}\varphi(z)},$$

that is

$$\left| \frac{\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))}{\mathfrak{D}_q(\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))} - [p]_q}{\mathcal{V}\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))}{\mathfrak{D}_q(\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z))} - \{(\mathcal{W}-\mathcal{V})([p]_q - \vartheta) + \mathcal{V}[p]_q\}} \right| < 1, \tag{10}$$

where $0 \le \vartheta < [p]_q, -1 \le \mathcal{V} < \mathcal{W} \le 1$, and $\mathcal{L}^{\rho,\sigma,\mu}_{p,q} f(z)$ is defined in (8).

We also define

$$\widetilde{\mathcal{HT}}_{p,q}(\vartheta,\mathcal{W},\mathcal{V}) = \mathcal{HT}_{p,q}(\vartheta,\mathcal{W},\mathcal{V}) \bigcap \widetilde{\mathcal{H}}(p,j).$$

Example 1. *If $k = 1, \sigma = 0$, and $\rho = 1$, the class $\mathcal{HT}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ would reduce to the following subclass $\mathcal{HK}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$*

$$\frac{\mathfrak{D}_q(z\mathfrak{D}_q f(z))}{\mathfrak{D}_q(f(z))} \prec ([p]_q - \vartheta)\frac{1+\mathcal{W}z}{1+\mathcal{V}z} + \vartheta. \tag{11}$$

3. A Set of Main Results

To demonstrate the geometric properties for the class $\mathcal{HT}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$, the necessary and sufficient condition must first be proved.

Unless otherwise stated, in this paper, we suppose that $0 \le \vartheta < [p]_q, -1 \le \mathcal{V} < \mathcal{W} \le 1$ and $0 < q < 1$.

Theorem 1. *Let $f = \mathcal{F} + \overline{\mathcal{G}} \in \mathcal{H}(p, j)$ in the form (3), then $f \in \mathcal{HT}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ if the following inequality holds:*

$$\begin{aligned}\sum_{j=p+1}^{\infty} &\{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)\}\psi_q|a_j| \\ &+ \sum_{j=p}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)\}\psi_q|d_j| \\ &\le (\mathcal{V}-\mathcal{W})([p]_q - \vartheta)[p]_q,\end{aligned} \tag{12}$$

where $\psi_q = \frac{\Gamma_q(\sigma+\mu)\Gamma(\rho+jk)}{\Gamma_q(\sigma j+\mu)\Gamma(\rho+k)j!}$.

Proof. Suppose that the inequality (12) is correct, it follows from (10) that

$$\begin{aligned}&|\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z)) - [p]_q\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z)| \\ &\qquad - \left|\mathcal{V}\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z)) - [(\mathcal{W}-\mathcal{V})([p]_q - \vartheta) + \mathcal{V}[p]_q]\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z)\right| \\ &= |\sum_{j=p+1}^{\infty}[j]_q([j]_q - [p]_q)\psi_q a_j z^{j-1} + \sum_{j=p}^{\infty}[j]_q([j]_q - [p]_q)\psi_q\overline{d_j z^{j-1}}| \\ &\qquad - | - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)[p]_q z^{p-1} \\ &\qquad + \sum_{j=p+1}^{\infty}[-\mathcal{V}[j]_q([j]_q - [p]_q) + (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q a_j z^{j-1} \\ &\qquad + \sum_{j=p}^{\infty}[-\mathcal{V}[j]_q([j]_q - [p]_q) + (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q\overline{d_j z^{j-1}}| \\ &\le -(\mathcal{V}-\mathcal{W})([p]_q - \vartheta)[p]_q|z|^{p-1} \\ &\qquad + \sum_{j=p+1}^{\infty}[(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q|a_j||z|^{j-1} \\ &\qquad + \sum_{j=p}^{\infty}[(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q|d_j||z|^{j-1} \\ &< -(\mathcal{V}-\mathcal{W})([p]_q - \vartheta)[p]_q \\ &\qquad + \sum_{j=p+1}^{\infty}[(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q|a_j| \\ &\qquad + \sum_{j=p}^{\infty}[(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q|d_j|,\end{aligned}$$

thus, we observe

$$\begin{aligned}&-(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q\\&\quad+\sum_{j=p+1}^{\infty}[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q a_j\\&\quad+\sum_{j=p}^{\infty}[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q d_j\le 0.\end{aligned}$$

Consequently, utilizing the maximum modulus theorem, we obtain

$$\left|\frac{\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}{\mathfrak{D}_q(\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}-[p]_q}{\mathcal{V}\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}{\mathfrak{D}_q(\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}-\{(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)+\mathcal{V}[p]_q\}}\right|<1.$$

Therefore, $f\in\mathcal{HT}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$.

For the following harmonic function, the coefficient bound (12) is sharp

$$\begin{aligned}f(z)=&z^p+\sum_{j=p+1}^{\infty}\frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}\nu_{1,j}z^j\\&+\sum_{j=p}^{\infty}\frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}\bar{\nu}_{2,j}\bar{z}^j,\end{aligned}\tag{13}$$

with $\sum_{j=p+1}^{\infty}|\nu_{1,j}|+\sum_{j=p}^{\infty}|\nu_{2,j}|=1.$ □

When $k=1,\sigma=0,$ and $\rho=1$, Theorem 1 becomes

Corollary 1. *Let $f=\mathcal{F}+\overline{\mathcal{G}}\in\mathcal{H}(p,j)$ in the form (3), then $f\in\mathcal{HK}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ if the following inequality holds:*

$$\begin{aligned}&\sum_{j=p+1}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}|a_j|\\&\quad+\sum_{j=p}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}|d_j|\\&\quad\le(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q.\end{aligned}\tag{14}$$

Next, we prove that the inquality (12) is necessary and sufficient condition for the class $\widetilde{\mathcal{HT}}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$.

Theorem 2. *Let $f=\mathcal{F}+\overline{\mathcal{G}}\in\widetilde{\mathcal{H}}(p,j)$ in the form (4). Then the harmonic function $f\in\widetilde{\mathcal{HT}}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ if and only if the inequality condition (12) holds.*

Proof. Since $\widetilde{\mathcal{HT}}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})\subset\mathcal{HT}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$, then the sufficient condition holds by the previous Theorem 1. Now, we have to prove just the necessity condition.
Let $f\in\widetilde{\mathcal{HT}}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$, from (10) yields

$$\left|\frac{\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}{\mathfrak{D}_q(\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}-[p]_q}{\mathcal{V}\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}{\mathfrak{D}_q(\mathcal{L}_{p,q}^{\rho,\sigma,\mu}f(z))}-\{(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)+\mathcal{V}[p]_q\}}\right|=$$

$$\left|\frac{-\sum_{j=p+1}^{\infty}[j]_q([j]_q-[p]_q)\psi_q|a_j|z^{j-1}-\sum_{j=p}^{\infty}[j]_q([j]_q-[p]_q)\psi_q|d_j|\bar{z}^{j-1}}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q z^{p-1}-\sum_{j=p+1}^{\infty}\mathcal{A}_q|a_j|z^{j-1}-\sum_{j=p}^{\infty}\mathcal{A}_q|d_j|\bar{z}^{j-1}}\right|<1,$$

where $\mathcal{A}_q = [\mathcal{V}[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)]\psi_q$.

For $z = r < 1$, we deduce that

$$\frac{\sum_{j=p+1}^{\infty} [j]_q([j]_q - [p]_q)\psi_q |a_j| r^{j-1} + \sum_{j=p}^{\infty} [j]_q([j]_q - [p]_q)\psi_q |d_j| r^{j-1}}{(\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q r^{p-1} - \sum_{j=p+1}^{\infty} \mathcal{A}_q |a_j| r^{j-1} - \sum_{j=p}^{\infty} \mathcal{A}_q |d_j| r^{j-1}} < 1. \quad (15)$$

When $r \to 1$, if condition (12) is not satisfied, inequality (15) is also not satisfied. In the range $(0, 1)$, we may, thus, identify at least one $z_0 = r_0$ for which the quotient (15) is greater than 1. This conflicts with the prerequisite for $f \in \widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$, hence the proof is complete. □

In the next result, we establish the extreme points of closed convex hulls of the subclass $\widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$.

Theorem 3. *The function* $f \in \widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ *if and only if*

$$f(z) = \sum_{j=p}^{\infty} (\vartheta_j \mathcal{F}_j + \aleph_j \mathcal{G}_j), \quad (16)$$

where $\mathcal{F}_p = z^p$,

$$\mathcal{F}_j = z^p - \frac{(\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q}{[(1 + \mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)]\psi_q} z^j; \ (j = p + 1, p + 2, \ldots)$$

and

$$\mathcal{G}_j = z^p - \frac{(\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q}{[(1 + \mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)]\psi_q} \overline{z}^j; \ (j = p, p + 1, \ldots),$$

with $\sum_{j=p}^{\infty} (\vartheta_j + \aleph_j) = 1$, $\vartheta_j \geq 0$, *and* $\aleph_j \geq 0$.

Particularly, the extreme points of the subclass $\widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ *are* $\{\mathcal{F}_j\}$ *and* $\{\mathcal{G}_j\}$.

Proof. Let f be defined as below

$$\begin{aligned} f(z) = \sum_{j=p}^{\infty} (\vartheta_j \mathcal{F}_j + \aleph_j \mathcal{G}_j) &= \sum_{j=p}^{\infty} (\vartheta_j + \aleph_j) z^p \\ &- \sum_{j=p}^{\infty} \frac{(\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q}{[(1 + \mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)]\psi_q} \vartheta_j z^j \\ &- \sum_{j=p}^{\infty} \frac{(\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q}{[(1 + \mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)]\psi_q} \aleph_j \overline{z}^j. \end{aligned} \quad (17)$$

We deduce from (17) and (4) that

$$|a_j| = \frac{(\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q}{[(1 + \mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)]\psi_q} \vartheta_j$$

and

$$|d_j| = \frac{(\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q}{[(1 + \mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)]\psi_q} \aleph_j.$$

Now
$$\sum_{j=p+1}^{\infty}\frac{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}|a_j|$$
$$+\sum_{j=p}^{\infty}\frac{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}|d_j|$$
$$=\sum_{j=p}^{\infty}(\vartheta_j+\aleph_j)-\vartheta_p=1-\vartheta_p\leq 1.$$

Thus, Theorem 2 leads to the result $f\in\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$.

Conversely: Let $f\in\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$, then

$$|a_j|\leq\frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}$$

and

$$|d_j|\leq\frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}.$$

Letting

$$\vartheta_j=\frac{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}|a_j|;\quad (j=p+1,p+2,\ldots)$$

and

$$\aleph_j=\frac{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}|d_j|;\quad (j=p,p+1,\ldots),$$

with $\sum_{j=p}^{\infty}(\vartheta_j+\aleph_j)=1$.
We get the result $f(z)=\sum_{j=p}^{\infty}(\vartheta_j\mathcal{F}_j+\aleph_j\mathcal{G}_j)$, after substituting the values of $|a_j|$ and $|d_j|$ from the above relations in (4). □

Theorem 4. *The subclass $\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ is a convex set of the functions $f=\mathcal{F}+\overline{\mathcal{G}}\in\widetilde{\mathcal{H}}(p,j)$.*

Proof. Let $f_i\in\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ given by

$$f_i(z)=z^p-\sum_{j=p+1}^{\infty}|a_{i,j}|z^j-\sum_{j=p}^{\infty}\overline{|d_{i,j}|z^j},\quad (i=1,2). \tag{18}$$

Then, for $0\leq\gimel\leq 1$

$$\mathcal{J}(z)=\gimel f_{1,j}(z)+(1-\gimel)f_{2,j}(z)$$
$$=z^p-\sum_{j=p+1}^{\infty}(\gimel|a_{1,j}|+(1-\gimel)|a_{2,j}|)z^j-\sum_{j=p}^{\infty}(\gimel|d_{1,j}|+(1-\gimel)|d_{2,j}|)\overline{z}^j,$$

also belongs to the subclass $\widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$.
By the result of Theorem 2, we get

$$\begin{aligned}
&\daleth\Big[\sum_{j=p+1}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\psi_q\}|a_{1,j}| \\
&\qquad+\sum_{j=p}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\psi_q\}|d_{1,j}|\Big] \\
&+(1-\daleth)\Big[\sum_{j=p+1}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\psi_q\}|a_{2,j}| \\
&\qquad+\sum_{j=p}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\psi_q\}|d_{2,j}|\Big] \\
&\qquad\leq \daleth((\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q)+(1-\daleth)((\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q) \\
&\qquad=(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q.
\end{aligned}$$

Here the subclass $\widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ is convex set, because $\mathcal{J} \in \widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$. □

Theorem 5. *We have*

$$\mathcal{E}\widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V}) = \{\mathcal{F}_j : j = \{p, p+1, \ldots\}\} \bigcup \{\mathcal{G}_j : j = \{p+1, p+2, \ldots\}\},$$

where

$$\begin{aligned}
\mathcal{F}_p &= z^p, \\
\mathcal{F}_j &= z^p - \frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q} z^j, \\
\mathcal{G}_j &= -\frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q} \overline{z}^j.
\end{aligned} \tag{19}$$

Proof. Suppose that $0 < \daleth < 1$ and

$$\mathcal{G}_j = \daleth f_1 + (1-\daleth) f_2,$$

where $f_1, f_2 \in \widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ are defined in (18).

From (12), we obtain

$$|d_{j,1}| = |d_{j,2}| = \frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}$$

and as result, $a_{1,n} = a_{2,n} = 0$ for $n \in \{p+1, p+2, \ldots\}$ and $d_{1,n} = d_{2,n} = 0$ for $n \in \{p+1, p+2, \ldots\} \setminus \{j\}$.

Thus, it follows that $\mathcal{G}_j = f_1 = f_2$, hence $\mathcal{G}_j \in \widetilde{\mathcal{HT}}_{p,q}(\mu, \mathcal{W}, \mathcal{V})$.

Similarly, we can satisfy that the functions $\mathcal{F}_j$ in (19) are also extreme points of $\widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$.

Now, let the function f in (18) belongs to the extreme points of the class $\widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ and f is not of the form (19).

Then there exists $n \in \{p+1, p+2, \ldots\}$, such that

$$0 < |a_n| < \frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[n]_q([n]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}, \tag{20}$$

or

$$0 < |d_n| < \frac{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}{[(1+\mathcal{V})[n]_q([n]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta)]\psi_q}. \tag{21}$$

If (20) is satisfied, we have

$$\beth = \frac{[(1+\mathcal{V})[n]_q([n]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q|a_n|}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}, \ \phi = \frac{1}{1-\beth}(f - \beth\mathcal{F}_n),$$

we get $0 < \beth < 1$, $\mathcal{F}_n \neq \phi$, and $f = \beth\mathcal{F}_n + (1-\beth)\phi$. Hence $f \notin \mathcal{E}\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$.

Similarly, if (21) is satisfied, we also have

$$\beth = \frac{[(1+\mathcal{V})[n]_q([n]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta)]\psi_q|a_n|}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}, \ \nu = \frac{1}{1-\beth}(f - \beth\mathcal{G}_n),$$

thus $0 < \beth < 1$, $\mathcal{G}_n \neq \nu$, and $f = \beth\mathcal{G}_n + (1-\beth)\nu$. Hence $f \notin \mathcal{E}\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$. □

4. Hadamard Product Property

The Hadamard product and the closed under a convex linear combination of the subclass $\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$ are provided in the following results.

The Hadamard product of harmonic functions with negative coefficient is given by

$$(f * h)(z) = z^p - \sum_{j=p+1}^{\infty} |a_{1,j}a_{2,j}|z^j - \sum_{j=p}^{\infty} |d_{1,j}d_{2,j}|\overline{z}^j, \tag{22}$$

where

$$f(z) = z^p - \sum_{j=p+1}^{\infty} |a_{1,j}|z^j - \sum_{j=p}^{\infty} |d_{1,j}|\overline{z}^j \tag{23}$$

and

$$h(z) = z^p - \sum_{j=p+1}^{\infty} |a_{2,j}|z^j - \sum_{j=p}^{\infty} |d_{2,j}|\overline{z}^j. \tag{24}$$

Theorem 6. *If $f, h \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$, then $f * h \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$.*

Proof. Let $f, h \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$. Since $h \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$, we find that $|a_{2,p}| < 1$ and $|d_{2,p}| < 1$. Then from the Hadamard product $f * h$, we obtain

$$\begin{aligned}
&\sum_{j=p+1}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta))\}\psi_q|a_{1,j}||a_{2,j}| \\
&\qquad + \sum_{j=p}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta))\}\psi_q|d_{1,j}||d_{2,j}| \\
&\leq \sum_{j=p+1}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta))\}\psi_q|a_{1,j}| \\
&\qquad + \sum_{j=p}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W}-\mathcal{V})([p]_q - \vartheta))\}\psi_q|d_{1,j}| \\
&\leq (\mathcal{V}-\mathcal{W})([p]_q - \vartheta)[p]_q.
\end{aligned}$$

Hence $f * h \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$. □

Theorem 7. *Let $f_s(z) = z^p - \sum_{j=p+1}^{\infty} |a_{j,s}|z^j - \sum_{j=p+1}^{\infty} |d_{j,s}|\overline{z}^j$ $(s = 1, 2, \dots)$ be in the subclass $\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$. Then the function*

$$\mathcal{J}(z) = \sum_{s=1}^{m} \eta_s f_s(z), \quad (\eta_s \geq 0, \sum_{s=1}^{\infty} \eta_s = 1),$$

also belongs to the subclass $\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$. This means $\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ is closed under convex linear combination.

Proof. Since $f_s \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$, then

$$\sum_{j=p+1}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}\psi_q|a_{j,s}| + \sum_{j=p}^{\infty}\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}\psi_q|d_{j,s}| \leq (\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q.$$

Now,

$$\mathcal{J}(z) = z^p - \sum_{j=p+1}^{\infty}(\sum_{s=1}^{\infty}\eta_s|a_{j,s}|)z^j - \sum_{j=p}^{\infty}(\sum_{s=1}^{\infty}\eta_s|d_{j,s}|)\overline{z}^j. \tag{25}$$

From (25) and (17), we conclude that

$$\begin{aligned}&\sum_{j=p+1}^{\infty}\frac{\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}(\sum_{s=1}^{\infty}\eta_s|a_{j,s}|)\\ &+\sum_{j=p}^{\infty}\frac{\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}(\sum_{s=1}^{\infty}\eta_s|d_{j,s}|)\\ &=\sum_{s=1}^{\infty}\eta_s\{\sum_{j=p+1}^{\infty}\frac{\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}(|a_{j,s}|)\\ &+\sum_{j=p}^{\infty}\frac{\{(1+\mathcal{V})[j]_q([j]_q-[p]_q)-(\mathcal{W}-\mathcal{V})([p]_q-\vartheta))\}\psi_q}{(\mathcal{V}-\mathcal{W})([p]_q-\vartheta)[p]_q}(|d_{j,s}|)\}\\ &\leq\sum_{s=1}^{\infty}\eta_s=1.\end{aligned}$$

Hence, $\mathcal{J} \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$. □

5. Closure Property

Next, we prove the closure property of the subclass $\widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$ under the q-Bernardi integral operator for p-valent functions (see [40]), which is given by

$$\mathcal{B}^p_{\omega,q}f(z) = \frac{[p+\omega]_q}{z^\omega}\int_0^z t^{\omega-1}f(t)d_qt \ \ (\omega > -p, z\in\mathbb{U}). \tag{26}$$

Definition 4. *For $f \in \widetilde{\mathcal{H}}(p,j)$, we define the q-Bernardi integral operator for p-valent functions $\mathcal{I}^p_{\omega,q}f : \widetilde{\mathcal{H}}(p,j) \to \widetilde{\mathcal{H}}(p,j)$ as follows:*

$$\begin{aligned}\mathcal{I}^p_{\omega,q}f(z) =&\frac{[p+\omega]_q}{z^\omega}\int_0^z t^{\omega-1}[t^p-\sum_{j=p+1}^{\infty}a_jt^j-\sum_{j=p}^{\infty}\overline{d_jt^j}]d_qt\\ &=z^p-\sum_{j=p+1}^{\infty}\frac{[p+\omega]_q}{[j+\omega]_q}|a_j|z^j-\sum_{j=p}^{\infty}\frac{[p+\omega]_q}{[j+\omega]_q}|d_j|\overline{z}^j,\ (\omega>-p,z\in\mathbb{U}).\end{aligned} \tag{27}$$

Then

$$\mathcal{I}^p_{\omega,q}f(z) = \mathcal{I}^p_{\omega,q}\mathcal{F}(z) + \overline{\mathcal{I}^p_{\omega,q}\mathcal{G}(z)}. \tag{28}$$

Theorem 8. *If $f \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$, then $\mathcal{I}^p_{\omega,q}f \in \widetilde{\mathcal{H}}\mathcal{T}_{p,q}(\vartheta,\mathcal{W},\mathcal{V})$.*

Proof. Since $f \in \widetilde{\mathcal{HT}}_{p,q}(\vartheta, \mathcal{W}, \mathcal{V})$, by Theorem 2, we conclude that

$$\begin{aligned}
&\sum_{j=p+1}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)\}\psi_q |a_j| \\
&\quad + \sum_{j=p}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)\}\psi_q |d_j| \\
&\quad \leq (\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q.
\end{aligned} \tag{29}$$

We have to prove

$$\begin{aligned}
&\sum_{j=p+1}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)\}\psi_q \frac{[p+\omega]_q}{[j+\omega]_q} |a_j| \\
&\quad + \sum_{j=p}^{\infty} \{(1+\mathcal{V})[j]_q([j]_q - [p]_q) - (\mathcal{W} - \mathcal{V})([p]_q - \vartheta)\}\psi_q \frac{[p+\omega]_q}{[j+\omega]_q} |d_j| \\
&\quad \leq (\mathcal{V} - \mathcal{W})([p]_q - \vartheta)[p]_q,
\end{aligned} \tag{30}$$

and we observe that the inequality (30) is correct, if

$$\frac{[p+\omega]_q}{[j+\omega]_q} \leq 1.$$

Since $p \leq j$, then the inequality (30) is satisfied, and this yields to the result. □

6. Concluding Remarks

Recently, the q-calculus and its applications have received great attention in several fields of mathematical and physical sciences (especially quantum physics), as well as an affirmation of the importance of the Mittag–Leffler function in the structure of fractional calculus. In this paper, we have introduced the subclass of q-convex harmonic p-valent functions connected with the q-Mittag–Leffler function. For this harmonic subclass, we have obtained the necessary and sufficient condition, convex hulls, convex linear combination, extreme point, and Hadamard product. Finally, this research has investigated the closure property for this class employing the q-Bernardi integral operator for harmonic p-valent functions.

The outcomes of this study may be beneficial to investigate several different classes of univalent (or p-valent) functions connected to various fields, notably those that use the generalized q-Mittag–Leffler function. Therefore, the findings of this paper can facilitate new research works in Geometric Function Theory and related subjects, such as differential subordination notions, the upper bounds of Fekete–Szegö inequality, and Hankel determinant. For more details on the suggested works, see [33,41,42].

It should be noted that the Fox–Wright hypergeometric function ${}_q\Psi_s$ is much more general than many of the expansions of the Mittag–Leffler function. The survey of the more complicated and general case of the Srivastava–Wright operator (see [43,44]), defined by the Fox–Wright function ${}_q\Psi_s$, is a recent interesting subject in Geometric Function Theory. Many properties of the Srivastava–Wright operator can be found in several recent works (see [30,39,45,46]).

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